Cryptography and quantum computing

Cryptography

- Cryptography relies on the **assumption of consistently hard problems**
  - NP-completeness is not suitable for cryptography
  - Average complexity is not good enough for cryptography
  - We cannot prove that these problems are hard
- Hard problems can give rise to **trapdoor functions**
  - Given *x*, finding *f(x)* is easy
  - Given *f(x)*, finding *x* is hard
  - **Given f(x) and a secret k, finding x is easy**
- Three main hard problems in cryptography: integer factorization, discrete logarithms, and elliptic curve discrete logarithms

Hard problem #1: Integer factorization

- Given pq, find p and q (large primes)
- *Not* believed to be NP-hard
- Best algorithm: Number sieves
  - Exponential time, but much faster than brute force
  - Very large primes are needed to maintain security
- Used to create RSA

Hard problem #1: Integer factorization -> RSA

- Public parameters: $n = pq$ (but not p and q)
- Alice generates e and d such that

$$ed \equiv 1 \ (mod \ (p-1)(q-1))$$

- e is the public key. Encryption of message m:

$$Enc(m) = m^e \ mod \ n$$

- Decryption of ciphertext x:

$$Dec(x) = x^d \ mod \ n$$

- This works because of Euler's theorem:

$$m^{ed} \equiv m \ (mod \ n)$$

If it was easy to find p and q, it would be easy to find d given e*!*

Trapdoor: only easy if given d

Hard problem #2: Discrete logarithm

- Given $m^x \bmod p$, p and x, find m
- Computing logarithm is easy normally
- However, it is (assumed) hard in modulo space
- Similarly, not believed to be NP-hard
- Used to create Diffie-Hellman Key Exchange and ElGamal Encryption

Integer factorization -> Order finding

- Euclid's algorithm: given a and b, it is easy to find gcd(a, b)
- A trick for integer factorization of N:
  - Start with guess **a**; determine if it is coprime with N (using Euclid's algorithm, fast)
  - If **(a, N)** is coprime, then find its order, which is r where:

$$a^r \equiv 1 \ (mod \ N)$$

    - If **a** is not coprime, we cannot do this as there will be no order
  - Furthermore if **r** is even (else try another **a**), we have

$$N \mid (a^{\frac{r}{2}} - 1)\,(a^{\frac{r}{2}} + 1)$$

  - Use Euclid's algorithm to compute $\gcd(N, a^{\frac{r}{2}} + 1)$; if it is N, restart
    - If not, we have a factor of N!

Quantum computers

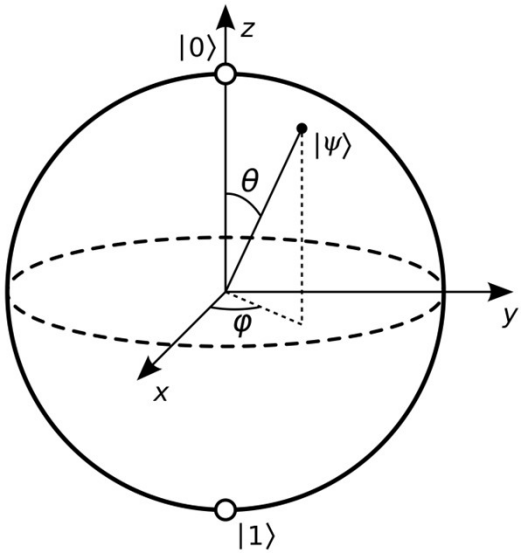
- Let's start with some quantum computing basics
- A qubit is a superposition of states 0 and 1, and can be represented as a point on Bloch's sphere:

$$|\psi\rangle = \cos\theta|0\rangle + e^{i\varphi}\sin\theta\,|1\rangle$$

- Two qubits can be entangled, e.g.:

$$\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$$

- If we measure the second qubit, it will collapse the first qubit
  - These two qubits, once measured, will only ever be 00 or 11



Incredible properties of quantum computers

- There is a Hadamard gate that can transform a zero qubit into the superposition of all possibilities:

$$\frac{1}{\sqrt{q}}\sum_{x=0}^{q}|x\rangle$$

- Quantum computers can implement all classical functions, so we can use a set of qubits to superimpose all possible outputs of a function
  - Measuring will still only give us one solution
- We can implement multiplication modulo N as a controlled-U gate:

$$U(|x\rangle) = |ax\ mod\ N\rangle\ if\ 0 \leq x \leq N, |x\rangle \text{ otherwise}$$

Shor's algorithm

- Continuing here:

$$U(|x\rangle) = |ax \; mod \; N\rangle \; if \; 0 \leq x \leq N, |x\rangle \text{ otherwise}$$

- $\mathrm{U^r = I}$, which shows that U has eigenvalues $e^{-2\pi \mathrm{i} \; /\mathrm{r}}$ for k from 1 to $r - 1$ (the r-th roots of unity)
  - Also, its eigenvectors would cancel out under a Quantum Fourier Transform (Handwavy step)
- Finding the eigenvalue of U is done by using Quantum Phase Estimation (which uses a Quantum Fourier Transform),
- This gives us j/r for some j

Final mathy bit

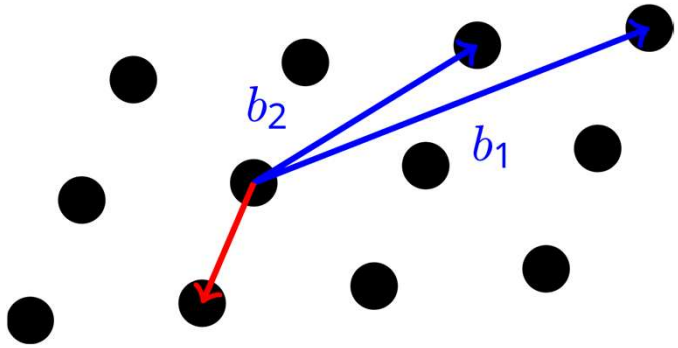
- j and r may not be coprime, e.g. j=8, r=12; we will get 0.666… which doesn’t tell us the order r=12
- Simple solution: re-run the routine, which randomizes j again.
  - Even if r is such that coprime j is rare, we can always take the lcm of results to obtain r within a few runs
  - e.g. j=8, r=12 -> 2/3 -> r?=3
  - j=3, r=12 -> 1/4 -> r?=4
  - lcm(3, 4) = 12

Shor's algorithm

- Shor's algorithm takes trivial time and requires 2n+3 logical qubits (where n is the number of bits in the key)
  - But Gidney and Eker (2021) estimate it needs about 20 million noisy "physical qubits"
- A slight alteration allows it to also break the discrete log problem as well as the elliptic curve problem
- Basically all of public key cryptography would break

Post-quantum cryptography

- There are other hard problems which have no quantum solution at all
  - This is no guarantee for the future, though experts would be surprised to see a quantum solution
- Shortest vector problem: Given an n-dimensional lattice, find the shortest vector in that lattice
- This hard problem can be used to build another trapdoor, creating cryptography based on Learning With Errors (not related to AI)



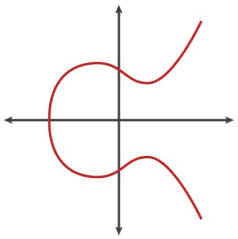
Hard problem #2: Discrete logarithm -> DH

- Diffie-Hellman key exchange:
- Public parameter prime p, generator g
  - Generator means that $\{g \bmod p\,, g^2 \bmod p\,, \ldots, g^{p-1} \bmod p\}$ are all different numbers
- Alice chooses random integer $A < p$, sends $g^A \bmod p$
- Bob chooses random integer $B < p$, sends $g^B \bmod p$
- Now both of them can compute a shared secret $g^{AB} \bmod p$
  - No one else can compute it

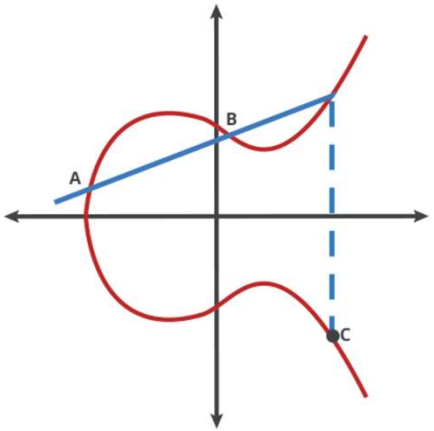
If it was easy to find A given $g^A \bmod p$, an attacker could compute the shared secret too

Hard problem #3: Elliptic Curves

- We can define new “addition” operations on elliptic curves:



- “Addition” is now:
  - Draw a line from A to B
  - Reflect it along the curve
  - The result is C = A+B
  - What is A + A?
  - What is nA?



Nick Sullivan, CloudFlare blog

Hard problem #3: Elliptic Curves

- Everything is done on finite fields instead of real numbers
  - Easy example of finite fields: integer modulo p
  - Maths on board
- Hard problem (Elliptic Curve Discrete Logarithm): Given A and B, find nA = B
  - Brute force = keep adding A until it becomes B
  - Baby step giant step algorithms: For $k = \sqrt{n}$, there must exist n_1 and n_2 such that

$$n = n_1 + n_2 k$$

  - We can **compute and store** all values of n_1 and then test all possible values for n_2 to get $\sqrt{n}$ computation time

Hard problem #3: Elliptic Curves -> ECDH

- We can rebuild the Diffie-Hellman key exchange using ECC
- Public parameters: elliptic curve, point P on curve
- Alice chooses a, sends aP
- Bob chooses b, sends bP
- Shared secret is abP