Programming Assignment 2

In an image restoration problem, you are given an image corrupted by noise X and you want to recover the original image Y. You can apply Gibbs sampling to a simple Ising Markov random field model to solve this task.

In this assignment, you will implement Gibbs sampling for the image restoration problem.

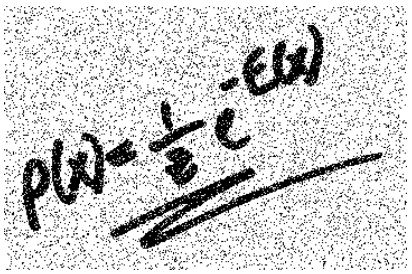


Figure 1: Noisy Image

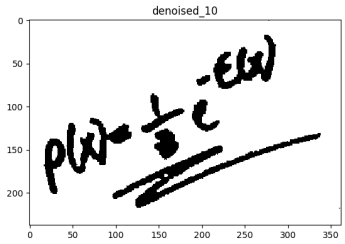
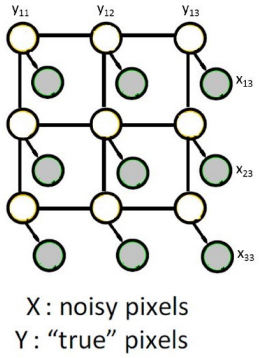


Figure 2: Denoised Image



Graphical model structure, for $N = M = 3$.

Figure 3: Graphical model

W use $x = \{x_{ij}\}$ to denote the observed image, with $x_{ij} \in \{-1, +1\}$ representing the pixel at row i and column j. Assume a black-and-white image, with -1 corresponding to white and $+1$ to black. The image has dimensions $N \times M$, so that $1 \leq i \leq N$ and $1 \leq j \leq M$. Assume a set of (unobserved) variables $\mathbf{y} = \{y_{ij}\}$ representing the true (unknown) image, with $y_{ij} \in \{-1, +1\}$ indicating the value of x_{ij} before noise was added. Each (internal) y_{ij} is linked with four immediate neighbors, $y_{i-1,j}$, $y_{i+1,j}$, $y_{i,j-1}$, and $y_{i,j+1}$ which together are denoted $y_{Nbr(i,j)}$. Pixels at the borders of the image (with $i \in \{1, N\}$ or $j \in \{1, M\}$) also have neighbors denoted $y_{Nbr(i,j)}$, but these sets are reduced in the obvious way. We denote E the corresponding set of edges.

The joint probability of y and x can be written (with no prior preference for black or white):

$$p(\mathbf{y}, \mathbf{x}) = \frac{1}{Z}\left(\prod_{i=1}^{N}\prod_{j=1}^{M}\exp^{\eta y_{ij}x_{ij}}\right) \times \left(\prod_{((i,j),(i',j'))\in E}\exp^{\beta y_{ij}y_{i'j'}}\right)$$

$$= \frac{1}{Z}\exp\left(\eta\sum_{i=1}^{N}\sum_{j=1}^{M} y_{ij}x_{ij} + \beta\sum_{((i,j),(i',j'))\in E} y_{ij}y_{i'j'}\right)$$

where

$$Z = \sum_{\mathbf{y},\mathbf{x}}\exp\left(\eta\sum_{i=1}^{N}\sum_{j=1}^{M} y_{ij}x_{ij} + \beta\sum_{((i,j),(i',j'))\in E} y_{ij}y_{i'j'}\right)$$

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(Notice in particular that each pair of neighbors, y_{ij} and $y_{i'j'}$, factors into the formula only once, despite that each variable is a neighbor of the other. Failing to account for this will lead to double counting of β values.) This is equivalent to a Boltzmann (sometimes called Gibbs) distribution with "energy":

$$E(\mathbf{y}, \mathbf{x}) = -\eta \sum_{i=1}^{N} \sum_{j=1}^{M} y_{ij} x_{ij} - \beta \sum_{((i,j),(i',j')) \in E} y_{ij} y_{i'j'}$$

The system will have lower energy, and hence higher probability, in states in which neighboring y_{ij} variables, and neighboring y_{ij} and x_{ij} variables, tend to have the same value (assuming η and β are positive). This captures the fact that each noisy pixel x_{ij} is likely to be similar to the corresponding "true" pixel y_{ij} , and that images tend to be "smooth".

We have derived the conditional probability of the y_{ij} is black given its Markov Blanket[1], where we use the logistic function $\sigma(x) = \frac{e^x}{1+e^x}$:

$$p(y_{ij} = 1 \mid y_{Nbr(i,j)}, x_{ij}) = \sigma(2\beta[\sum_{y_{x,y} \in y_{\mathrm{Nbr}}(i,j)} y_{x,y}] + 2\eta x_{ij})$$

For example:

$$\begin{aligned} p(y_{11} = 1 \mid y_{Nbr(1,1)}, x_{11}) &= p(y_{11} = 1 \mid y_{21}, y_{12}, x_{11}) \\ &= \sigma(2\beta[y_{21} + y_{12}] + 2\eta x_{11}) \end{aligned}$$

.

You will apply your implementation to two small, black-and-white images that have been made available with the problem set. These two noisy images—and the original, undistorted image from which they derive—are available both in PNG format and in a simple text format that lists each coordinate pair (i, j) and the corresponding value of x_{ij} . You may find it useful to convert between this text representation and a viewable image format.

What to submit

Please submit the following two files to CourSYS:

- gibbs.py – Your completed implementation.
- report.pdf – A pdf file answering all the questions in this assignment.

[1] It's a good practice to try to derive the formula yourself

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No late submission is accepted.

Collaboration policy

You may freely discuss this coding assignments with other students. All writing must be your own; it is not acceptable to copy/paste or verbatim transcribe others' text, code or LaTeX source.

(4 points) Question 1

Outline a Gibbs sampling algorithm (in pseudocode) that iterates over the pixels in the image and samples each y_{ij} given its Markov blanket. Use the simple approach of sweeping across the image in row-major fashion on every iteration of the algorithm. Thus, an "iteration" will generate a complete new sample of y. Allow for a burn-in of B iterations, followed by draws of S samples.You may assume η and β are fixed constants.

(10 points) Question 2

Implement your algorithm and apply it to the noise image A (a_noise10.png.txt). Use values of $\eta = 1$, $\beta = 1$, B = 50, and S = 200. On each iteration of your algorithm, compute the energy $E(\mathbf{y}, \mathbf{x})$ using the formula below for the current sample of y and output it to a log file, keeping track of which values correspond to the burn-in. Run your algorithm with three different initializations - one in which each y_{ij} is initialized to x_{ij} , one in which each y_{ij} is initialized to $-x_{ij}$ and one in which the y_{ij} are set to 1 or +1 at random. Plot the energy of the model as a function of the iteration number for all three chains and visually inspect these traces for signs of convergence.

$$E(\mathbf{y}, \mathbf{x}) = -\eta \sum_{i=1}^{N} \sum_{j=1}^{M} y_{ij} x_{ij} - \beta \sum_{((i,j),(i',j')) \in E} y_{ij} y_{i'j'}$$

(Notice in particular that each pair of neighbors, y_{ij} and $y_{i'j'}$, factors into the formula only once, despite that each variable is a neighbor of the other. Failing to account for this will lead to double counting of β values.)

- Do all three seem to be converging to the same general region of the posterior, or are some obviously suboptimal?
- Does the burn-in seem to be adequate in length?
- Is there substantial fluctuation from iteration to iteration, indicating that the chain is mixing well, or does it become stuck at particular energies for several iterations at a time?

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This question requires about 6 minutes of computing time. Please plan ahead. You may want to use a portion of the image for testing.

(4 points) Question 3

Have your program output a restored image after completing its sampling iterations, by thresholding the estimated posterior probabilities for the y_{ij} variables at 0.5 — i.e., by estimating the "true" color of each pixel (i, j) as:

$$\hat{y}_{ij} = \begin{cases} +1, & \text{if } p(y_{ij} = 1|\mathbf{x}) > 0.5 \\ -1, & \text{otherwise} \end{cases}$$

To estimate the required posterior probabilities, store a running count c_{ij} of the number of (retained) samples for which each $y_{ij} = 1$, and then use the Monte Carlo estimate:

$$p(y_{ij} = 1|\mathbf{x}) \approx \frac{1}{S} \sum_t 1(y_{ij}^{(t)} = 1) = \frac{c_{ij}}{S}$$

where $y_{ij}^{(t)}$ represents the t^{th} sample of y_{ij}. Restore both the noise images in this way, using the same values of η, β, B, and S as above and y_{ij} initialized to x_{ij}. Evaluate the quality of the restoration by computing the fraction of all pixels that differ between the restored images (a_noise10.png.txt and b_noise10.png.txt) and the original image.

- Prepare a figure for each the the two images, showing the original, the noisy version, and the restoration side by side.
- Report the restoration error for each image.

This question requires about 10 minutes of computing time. Please plan ahead. If you have implemented your algorithm correctly, your restored images should be quite close to the original.

Question 4

How many hours did you spend on this assignment?
Please provide your answers in your report.

Thanks to Stephano Ermon for giving us permission to adapt this assignment from Stanford CS 228.