Problem 1

Given the following MRF, find its corresponding junction tree decomposition with the minimum max clique size.

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Problem 2

An order-\( k \) Markov model is a probabilistic model over variables \( X_1, \ldots, X_n \), where \( X_i \) depends on the \( k \) variables \( X_{i-k}, \ldots, X_{i-1} \).

1. What is the treewidth of an order-\( k \) Markov model?

2. Your colleague wants to do exact inference in a Markov model with \( k = 100 \). Explain why this is computationally expensive.
3. They suggest that to make inference easier while retaining long-range dependencies, they instead use a Markov-like model where $X_i$ depends only on $X_{i-k}$, thus reducing the number of edges in the network by a factor of 100. Explain why this does not meaningfully improve performance.

**Problem 3**

A random variable with density $g(y) = \sqrt{\frac{2}{\pi}} e^{-y^2/2} \mathbb{1}\{y \geq 0\}$ is to be simulated by rejection sampling. The candidate values are realizations from a $Exp(\lambda)$-distributed random variable with density $f(x) = \lambda e^{-\lambda x} \mathbb{1}\{x \geq 0\}$.

1. Determine the smallest value $c$ (with subject to $\lambda > 0$) such that $cf(y) \geq g(y)$.

2. For which value of $\lambda$ is the theoretical percentage of rejected samples minimal?

   Hint: Recall that the probability of acceptance is $\frac{Z_g}{c}$, where $Z_g = \int_x g(x) dx$. 
