Uncertainty

CMPT 310

CHAPTER 13

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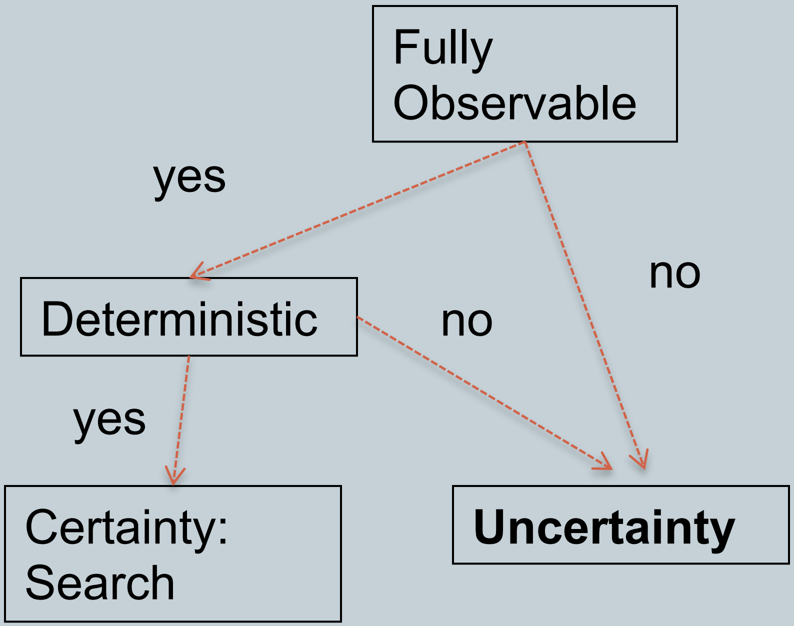
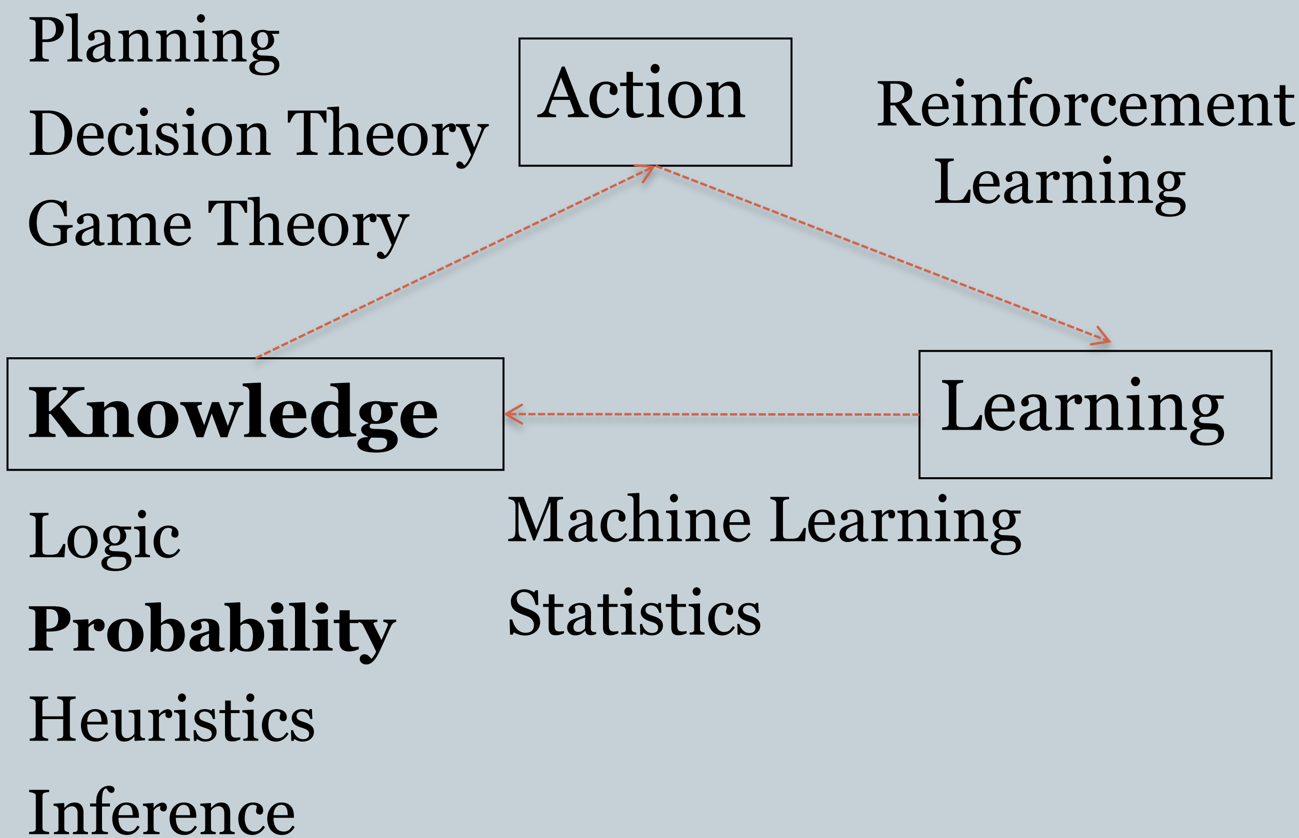
Environments with Uncertainty



The Big Picture: AI for Model-Based Agents



Review Example: Paying for Parking

Which option minimizes expected cost?

Acts/States	Get Caught (p)	Not get caught (1-p)
Pay	-\$3.00	-\$3.00
Don't pay	-\$100	\$0

Probabilities represent knowledge

Motivation for Uncertainty

- In many cases, our perceptions are incomplete (not enough information) or uncertain (sensors are unreliable).
- Rules about the domain are incomplete or admit exceptions.
- Probabilistic knowledge
  - Quantifies uncertainty.
  - Supports rational decision-making.

Outline

- Uncertainty and Rationality
- Probability
- Syntax and Semantics
- Inference Rules

Probabilistic Knowledge

Uncertainty vs. Logical Rules

- Cavity causes toothache.
- Cavity is detected by probe (catches).
- In logic:
  - Cavity => Toothache.
    - But not always, e.g.
      Cavity, dead nerve does not cause Toothache.
    - **Nonmonotonic rules**: *adding information changes conclusions*.
  - Cavity => CatchProbe.
    - Also not always.

Probability vs. Determinism

- Medical diagnosis is not deterministic.
  - Laziness: failure to enumerate exceptions, qualifications, etc.
  - Theoretical ignorance: domain theory is incomplete
  - Practical ignorance: Even if we know all the rules, a patient might not have done all the necessary tests.
- Probabilistic assertions summarize effects of
  - Laziness
  - Ignorance

Probability Syntax

Probability Syntax

- Basic element: variable that can be assigned a value.
  - Traditionally a factored representation

- Boolean variables
  e.g., *Cavity* (do I have a cavity?)

- Discrete variables
  e.g., *Weather* is one of <*sunny,rainy,cloudy,snow*>

- Atom = assignment of value to variable.
  - Aka atomic event. Examples:
  - *Weather = sunny*
  - *Cavity = false.*

- Sentences are Boolean combinations of atoms.
  - *Same as propositional logic. Examples:*
  - *Weather = sunny* OR *Cavity = false.*
  - *Catch = true* AND *Tootache = False.*

Probabilities and Possible Worlds

- Possible World/State: A <u>complete</u> assignment of a value to each variable.
- Removes all uncertainty.
- **Event** or **proposition** = set of possible worlds.
- Atomic event = a single possible world.

Logic	Statistics	Examples
n/a	Variable	Weather, Cavity, Probe, Toothache
Atom	Variable Assignment	Weather = sunny Cavity = false
Possible World	Atomic Event	[Weather = sunny Cavity = false Catch = false Toothache = true]

Random Variables

- A **random variable** has a probability associated with each of its values.

Variable	Value	Probability
Weather	Sunny	0.7
Weather	Rainy	0.2
Weather	Cloudy	0.08
Weather	Snow	0.02
Cavity	True	0.2
Cavity	False	0.8

Probability for Sentences

- Sentences also have probabilities assigned to them.

Sentence	Probability
P(Cavity = false AND Toothache = false)	0.72
P(Cavity = true OR Toothache = false)	0.08

Probability Notation

- Often probability theorists write A,B instead of $A \wedge B$
- If the intended random variables are known, they are often not mentioned.
- If a statement is true for all values, the values are often not mentioned.

Shorthand	Full Notation
P(Cavity = false,Toothache = false)	P(Cavity = false $\wedge$Toothache = false)
P(false, false)	P(Cavity = false $\wedge$Toothache = false)
P(C)>0	P(C=true)>0 and P(C=false)>0

Joint Probabilities

Assigning Probabilities to Sentences

- The **joint probability distribution** specifies the probability of each possible world (atomic event).

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- <u>A joint distribution determines an probability for every sentence.</u>
- How? Spot the pattern.

Probabilities for Sentences: Spot the Pattern

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

Sentence	**Probability**
P(Cavity = false AND Toothache = false)	0.72
P(Cavity = true OR Toothache = false)	0.92
P(Toothache = false)	0.8

Inference by enumeration

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

Inference by enumeration

- Marginalization: For any sentence *A*, sum the atomic events (possible worlds) where *A* is true.
- P(*Toothache=T)* = 0.108 + 0.012 + 0.016 + 0.064 = 0.2.

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

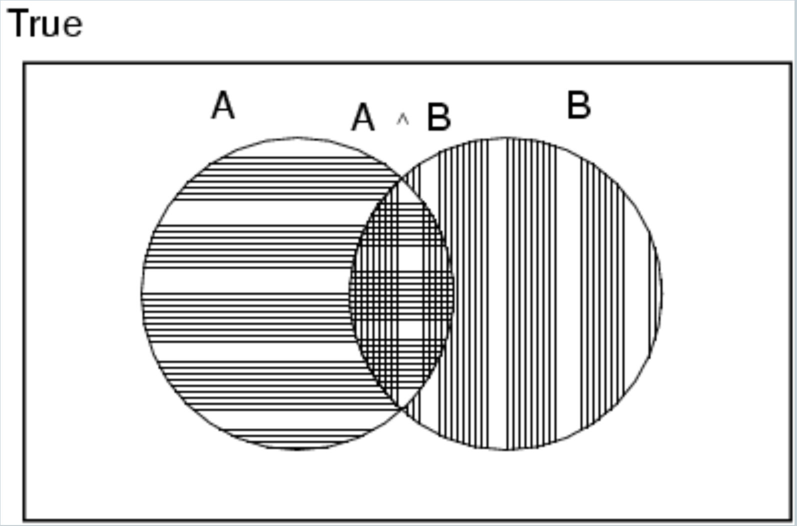
Probabilistic Inference Rules

Axioms of probability

For any sentence A, B

$0 \le P(A) \le 1$

- $P(\mathit{true}) = 1$ and $P(\mathit{false}) = 0$
- P(A) = P(B) if A and B are logically equivalent.
- $P(A \lor B) =$ $P(A) + P(B) - P(A \land B)$



sentences considered as sets of possible worlds.

Rule 1: Logical Equivalence

P(NOT (NOT Cavity=T))	P(Cavity=T)
0.2	0.2

P(NOT (Cavity = T AND Toothache = T))	P(Cavity = F OR Toothache = F)
0.88	0.88

P(NOT (Cavity = T OR Toothache = T))	P(Cavity = F AND Toothache = F)
0.72	0.72

The Logical Equivalence Pattern

P(NOT (NOT Cavity=T))	=	P(Cavity=T)
0.2		0.2

Rule 1: Logically equivalent expressions have the same probability.

P(NOT (Cavity =T AND Toothache =T))	=	P(Cavity = F OR Toothache = F)
0.88		0.88

P(NOT (Cavity=T OR Toothache=T)	=	P(Cavity = F AND Toothache = F)
0.72		0.72

Psychology: Probability Judgements

Consider the following famous experiment.

Two groups of subjects.

1. Would you opt for surgery if the survival rate is 90 percent?
2. Would you opt for surgery if the mortatility rate is 10 percent?

Which would you prefer?

Rule 2: Marginalization

P(Cavity=T, Toothache=T)	P(Cavity=T, Toothache = F)	P(Cavity=T)
0.12	0.08	0.2

P(Cavity = F, Toothache=T)	P(Cavity = F, Toothache = F)	P(Cavity = F)
0.08	0.72	0.8

P(Cavity=T, Toothache=T)	P(Cavity = F, Toothache=T)	P(Toothache=T)
0.12	0.08	0.2

The Marginalization Pattern

P(Cavity=T, Toothache=T)	+	P(Cavity=T, Toothache = F)	=	P(Cavity=T)
0.12		0.08		0.2

P(Cavity = F, Toothache=T)	+	P(Cavity = F, Toothache = F)	=	P(Cavity = F)
0.08		0.72		0.8

P(Cavity=T, Toothache=T)	+	P(Cavity = F, Toothache=T)	=	P(Toothache=T)
0.12		0.08		0.2

Prove the Pattern: Marginalization

Theorem. P(A) = P(A,B) + P(A, not B)

- Proof.

1. A is logically equivalent to [A and B) ∨ (A and not B)].
2. P(A) = P([A and B) ∨ (A and not B)]) = P(A and B) + P(A and not B) – P([(A and B) ∧ (A and not B)]). Disjunction Rule.
3. [A and B) ∧ (A and not B)] is logically equivalent to **false**, so P([(A and B) ∧ (A and not B)]) =0.
4. So 2. implies P(A) = P(A and B) + P(A and not B).

Conditional Probabilities

Conditional Probabilities: Intro

- Given (A) that a die comes up with an odd number, what is the probability that (B) the number is
  1. a 2
  2. a 3
- Answer: the number of cases that satisfy both A and B, out of the number of cases that satisfy A.
- Examples:
  1. #faces with (odd and 2)/#faces with odd
     = 0 / 3 = 0.
  2. #faces with (odd and 3)/#faces with odd
     = 1 / 3.

Conditional Probs ctd.

- Suppose that 50 students are taking 310 and 30 are women. Given (A) that a student is taking 310, what is the probability that (B) they are a woman?
- Answer:
  #students who take 310 and are a woman/
  #students in 310
  = 30/50 = 3/5.
- **Notation**: P(B|A)

Conditional Ratios: Spot the Pattern

- Spot the Pattern

P(die comes up with 3)	P(die comes up with odd number)	P(3|odd number)
1/6	1/2	1/3

P(Student takes 310 and is woman)	P(Student takes 310)	P(Student is woman|Student takes 310)
30/15,000	50/15,000	3/5

Conditional Probs: The Ratio Pattern

- Spot the Pattern

P(die comes up with 3)	/	P(die comes up with odd number)	=	P(3|odd number)
1/6		1/2		1/3

P(Student takes 310 and is woman)	/	P(Student takes 310)	=	P(Student is woman|Student takes 310)
30/15,000		=50/15,000		3/5

P(A|B) = P(A and B)/ P(B) Important!

Conditional Probabilities: Motivation

- From logic: much knowledge can be represented as implications
  $B_1,..,B_k \Rightarrow A$.
- Conditional probabilities are a probabilistic version of reasoning about what follows from conditions.
- Cognitive Science: Our minds store implicational knowledge.
- Key for understanding Bayes nets.

Conjunctivitis

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations.

Here are some possibilities for what Linda is doing now; please rank them according to likelihood.

a. Linda is a bank teller.

b. Linda works for a book publisher.

c. Linda is a bankteller who is active in the feminist movement.

Modus Ponens and the Product Rule

Classical Logic	Probability
If A, then B	$P(B\|A) = x$
A	$P(A) = y$
Therefore: A and B	Therefore: $P(A,B)=?$

The Product Rule: Spot the Pattern

P(Cavity=T)	P(Toothache=T|Cavity=T)	P(Cavity=T,Toothache=T)
0.2	0.6	0.12

P(Toothache=T)	P(Cavity=T|Toothache=T)	P(Cavity=T,Toothache=T)
0.2	0.6	0.12

P(Cavity =F)	P(Toothache=T|Cavity = F)	P(Toothache=T,Cavity =F)
0.8	0.1	0.08

The Product Rule Pattern

P(Cavity=T)	x	P(Toothache|Cavity=T)	=	P(Cavity=T, Toothache=T)
0.2		0.6		0.12

P(Toothache=T)	x	P(Cavity| Toothache=T)	=	P(Cavity=T, Toothache=T)
0.2		0.6		0.12

P(Cavity =F)	x	P(Toothache=T|Cavity = F)	=	P(Toothache=T, Cavity =F)
0.8		0.08		0.1

Exercise: Conditional Probability

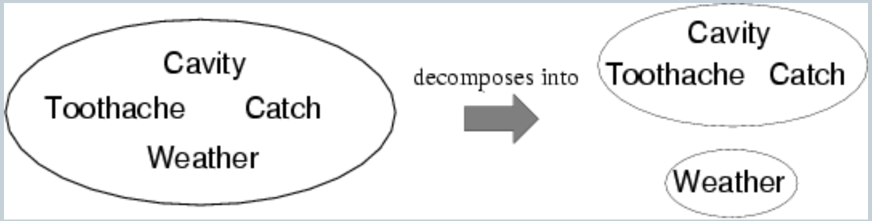
- Prove the **product rule** $P(A,B) = P(A|B) \times P(B)$.
  - Marginal x conditional → joint.
- Two sentences A,B are **independent** if $P(A|B) = P(A)$. Prove that the following conditions are equivalent if $P(A) > 0$, $P(B) > 0$.

1. $P(A|B) = P(A)$.
2. $P(B|A) = P(B)$.
3. $P(A,B) = P(A) \times P(B)$.

Independence

Independence

- Suppose that Weather is independent of the Cavity Scenario. Then the joint distribution decomposes:



P(*Toothache, Catch, Cavity, Weather*)
= **P**(*Toothache, Catch, Cavity*) **P**(*Weather*)

- Absolute independence powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
-

  (1) $\mathbf{P}(catch \mid toothache, cavity) = \mathbf{P}(catch \mid cavity)$
- The same independence holds if I haven't got a cavity:

  (2) $\mathbf{P}(catch \mid toothache, \neg cavity) = \mathbf{P}(catch \mid \neg cavity)$
- *Catch* is conditionally independent of *Toothache* given *Cavity*:

  $\mathbf{P}(Catch \mid Toothache, Cavity) = \mathbf{P}(Catch \mid Cavity)$
- The equivalences for independence also holds for conditional independence, e.g.:

  $\mathbf{P}(Toothache \mid Catch, Cavity) = \mathbf{P}(Toothache \mid Cavity)$

  $\mathbf{P}(Toothache, Catch \mid Cavity) = \mathbf{P}(Toothache \mid Cavity)\, \mathbf{P}(Catch \mid Cavity)$

Conditional Independence Conditions

- The same equivalences hold for both conditional and unconditional independence.
- **Theorem.** The following conditions are equivalent if $P(A|C) > 0$, $P(B|C) > 0$.

1. $P(A|B,C) = P(A|C)$.
2. $P(B|A,C) = P(B|C)$.
3. $P(A,B|C) = P(A|C) \times P(B|C)$.

Conditional independence and the Joint Distribution

- Write out full joint distribution using product rule:

  $\mathbf{P}(Toothache, Catch, Cavity)$

  $= \mathbf{P}(Toothache \mid Catch, Cavity)\, \mathbf{P}(Catch, Cavity)$

  $= \mathbf{P}(Toothache \mid Catch, Cavity)\, \mathbf{P}(Catch \mid Cavity)\, \mathbf{P}(Cavity)$

  $= \mathbf{P}(Toothache \mid Cavity)\, \mathbf{P}(Catch|Cavity)\, \mathbf{P}(\text{Cavity})$

  I.e., 2 + 2 + 1 = 5 independent numbers

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.
- Conditional independence is our most basic and robust form of knowledge about uncertain environments

Summary

- Probability is a rigorous formalism for uncertain knowledge.
- Joint probability distribution specifies probability of every atomic event (possible world).
- Queries can be answered by summing over atomic events.
- For nontrivial domains, we must find a way to compactly represent the joint distribution.
- Independence and conditional independence provide the tools.