Solving Recurrences

\[ T(n) \leq cn + T(n/5) + T(3n/4) \]

\[ \rightarrow T(n) \in O(n) \]
Merge-Sort Review

Merge-sort on an input sequence $S$ with $n$ elements consists of three steps:

- **Divide**: partition $S$ into two sequences $S_1$ and $S_2$ of about $n/2$ elements each
- **Recur**: recursively sort $S_1$ and $S_2$
- **Conquer**: merge $S_1$ and $S_2$ into a unique sorted sequence

**Algorithm** $\text{mergeSort}(S, C)$

- **Input** sequence $S$ with $n$ elements, comparator $C$
- **Output** sequence $S$ sorted according to $C$

  if $S.\text{size}() > 1$

  $(S_1, S_2) \leftarrow \text{partition}(S, n/2)$
  $\text{mergeSort}(S_1, C)$
  $\text{mergeSort}(S_2, C)$
  $S \leftarrow \text{merge}(S_1, S_2)$
Recurrence Equation Analysis

The conquer step of merge-sort consists of merging two sorted sequences, each with \( n/2 \) elements and implemented by means of a doubly linked list, takes at most \( bn \) steps, for some constant \( b \).

Likewise, the basis case (\( n < 2 \)) will take at \( b \) most steps.

Therefore, if we let \( T(n) \) denote the running time of merge-sort:

\[
T(n) = \begin{cases} 
b & \text{if } n < 2 \\
2T(n/2) + bn & \text{if } n \geq 2 
\end{cases}
\]

We can therefore analyze the running time of merge-sort by finding a closed form solution to the above equation.

- That is, a solution that has \( T(n) \) only on the left-hand side.
In the iterative substitution, or “plug-and-chug,” technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern:

\[ T(n) = 2T(n/2) + bn \]

\[ = 2(2T(n/2^2) + b(n/2)) + bn \]

\[ = 2^2 T(n/2^2) + 2bn \]

\[ = 2^3 T(n/2^3) + 3bn \]

\[ = 2^4 T(n/2^4) + 4bn \]

\[ = \ldots \]

\[ = 2^i T(n/2^i) + ibn \]

Note that base, \( T(n) = b \), case occurs when \( 2^i = n \). That is, \( i = \log n \).

So,

\[ T(n) = bn + bn \log n \]

Thus, \( T(n) \) is \( O(n \log n) \).
The Recursion Tree

Draw the recursion tree for the recurrence relation and look for a pattern:

\[
T(n) = \begin{cases} 
  b & \text{if } n < 2 \\
  2T(n/2) + bn & \text{if } n \geq 2 
\end{cases}
\]

<table>
<thead>
<tr>
<th>depth</th>
<th>T’s</th>
<th>size</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>n</td>
<td>bn</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>n/2</td>
<td>bn</td>
</tr>
<tr>
<td>i</td>
<td>2^i</td>
<td>n/2^i</td>
<td>bn</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Total time = \(bn + bn \log n\)

(last level plus all previous levels)
In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

\[
T(n) = \begin{cases} 
  b & \text{if } n < 2 \\
  2T(n/2) + bn\log n & \text{if } n \geq 2 
\end{cases}
\]

Guess: \( T(n) < cn \log n \).

\[
T(n) = 2T(n/2) + bn\log n \\
= 2(c(n/2)\log(n/2)) + bn\log n \\
= cn(\log n - \log 2) + bn\log n \\
= cn \log n - cn + bn\log n
\]

Wrong: we cannot make this last line be less than \( cn \log n \) for all \( n \geq \) some constant.
Guess-and-Test Method, (cont.)

- Recall the recurrence equation:
  \[ T(n) = \begin{cases} 
  b & \text{if } n < 2 \\
  2T(n/2) + bn \log n & \text{if } n \geq 2
  \end{cases} \]

- Guess #2: \( T(n) < cn \log^2 n \).
  \[
  T(n) = 2T(n/2) + bn \log n \\
  = 2(c(n/2) \log^2 (n/2)) + bn \log n \\
  = cn (\log n - \log 2)^2 + bn \log n \\
  = cn \log^2 n - 2cn \log n + cn + bn \log n \\
  \leq cn \log^2 n
  \]

  - if \( c > b \).

- So, \( T(n) \) is \( O(n \log^2 n) \).

- In general, to use this method, you need to have a good guess and you need to be good at induction proofs.
Master Method (Section 4.3)

Many divide-and-conquer recurrence equations have the form:

\[ T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d
\end{cases} \]

The Master Theorem (case 2 different from text)

1. if \( f(n) \) is \( O(n^{\log_b a - \varepsilon}) \), then \( T(n) = \Theta(n^{\log_b a}) \)
2. if \( f(n) \) is \( \Theta(n^{\log_b a \log^k n}) \), then \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
3. if \( f(n) \) is \( \Omega(n^{\log_b a + \varepsilon}) \), then \( T(n) = \Theta(f(n)) \),
   provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).
Master Method, Example 1

The form:

\[ T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases} \]

The Master Theorem:

1. If \( f(n) = O(n^{\log_b a - \varepsilon}) \), then \( T(n) = \Theta(n^{\log_b a}) \)
2. If \( f(n) = \Theta(n^{\log_b a} \log^k n) \), then \( T(n) = \Theta(n^{\log_b a} \log^{k+1} n) \)
3. If \( f(n) = \Omega(n^{\log_b a + \varepsilon}) \), then \( T(n) = \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[ T(n) = 4T(n/2) + n \]

Solution: \( \log_b a = 2 \), so case 1 says \( T(n) \) is \( O(n^2) \).
Master Method, Example 2

The form:

\[ T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases} \]

The Master Theorem:

1. if \( f(n) \) is \( O(n^{\log_b a - \epsilon}) \), then \( T(n) \) is \( \Theta(n^{\log_b a}) \)
2. if \( f(n) \) is \( \Theta(n^{\log_b a} \log^k n) \), then \( T(n) \) is \( \Theta(n^{\log_b a} \log^{k+1} n) \)
3. if \( f(n) \) is \( \Omega(n^{\log_b a + \epsilon}) \), then \( T(n) \) is \( \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[ T(n) = 2T(n/2) + n \log n \]

Solution: \( \log_b a = 1 \), so case 2 says \( T(n) \) is \( O(n \log^2 n) \).
Master Method, Example 3

The form:

\[ T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases} \]

The Master Theorem:

1. if \( f(n) \) is \( O(n^{\log_b a - \varepsilon}) \), then \( T(n) \) is \( \Theta(n^{\log_b a}) \)
2. if \( f(n) \) is \( \Theta(n^{\log_b a} \log^k n) \), then \( T(n) \) is \( \Theta(n^{\log_b a} \log^{k+1} n) \)
3. if \( f(n) \) is \( \Omega(n^{\log_b a + \varepsilon}) \), then \( T(n) \) is \( \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[ T(n) = T(n/3) + n \log n \]

Solution: \( \log_b a = 0 \), so case 3 says \( T(n) \) is \( O(n \log n) \).
Master Method, Example 4

The form:

\[
T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases}
\]

The Master Theorem:

1. if \( f(n) \) is \( O(n^{\log_b a - \epsilon}) \), then \( T(n) \) is \( \Theta(n^{\log_b a}) \)
2. if \( f(n) \) is \( \Theta(n^{\log_b a \log^k n}) \), then \( T(n) \) is \( \Theta(n^{\log_b a \log^{k+1} n}) \)
3. if \( f(n) \) is \( \Omega(n^{\log_b a + \epsilon}) \), then \( T(n) \) is \( \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[
T(n) = 8T(n/2) + n^2
\]

Solution: \( \log_b a = 3 \), so case 1 says \( T(n) \) is \( O(n^3) \).
Master Method, Example 5

The form:

\[
T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases}
\]

The Master Theorem:

1. if \( f(n) \) is \( O(n^{\log_b a - \epsilon}) \), then \( T(n) \) is \( \Theta(n^{\log_b a}) \)

2. if \( f(n) \) is \( \Theta(n^{\log_b a \log^k n}) \), then \( T(n) \) is \( \Theta(n^{\log_b a \log^{k+1} n}) \)

3. if \( f(n) \) is \( \Omega(n^{\log_b a + \epsilon}) \), then \( T(n) \) is \( \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[
T(n) = 9T(n/3) + n^3
\]

Solution: \( \log_b a = 2 \), so case 3 says \( T(n) \) is \( O(n^3) \).
Master Method, Example 6

The form: 

\[ T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases} \]

The Master Theorem:

1. if \( f(n) \) is \( O(n^{\log_b a - \epsilon}) \), then \( T(n) \) is \( \Theta(n^{\log_b a}) \)
2. if \( f(n) \) is \( \Theta(n^{\log_b a \log^k n}) \), then \( T(n) \) is \( \Theta(n^{\log_b a \log^{k+1} n}) \)
3. if \( f(n) \) is \( \Omega(n^{\log_b a + \epsilon}) \), then \( T(n) \) is \( \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[ T(n) = T(n/2) + 1 \] (binary search)

Solution: \( \log_b a = 0 \), so case 2 says \( T(n) \) is \( O(\log n) \).
Master Method, Example 7

The form:  
\[ T(n) = \begin{cases} 
  c & \text{if } n < d \\
  aT(n/b) + f(n) & \text{if } n \geq d 
\end{cases} \]

The Master Theorem:

1. if \( f(n) = O(n^{\log_b a - \epsilon}) \), then \( T(n) = \Theta(n^{\log_b a}) \)
2. if \( f(n) = \Theta(n^{\log_b a \log^k n}) \), then \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \)
3. if \( f(n) = \Omega(n^{\log_b a + \epsilon}) \), then \( T(n) = \Theta(f(n)) \), provided \( af(n/b) \leq \delta f(n) \) for some \( \delta < 1 \).

Example:

\[ T(n) = 2T(n/2) + \log n \] (heap construction)

Solution: \( \log_b a = 1 \), so case 1 says \( T(n) = O(n) \).
Iterative “Proof” of the Master Theorem

Using iterative substitution, let us see if we can find a pattern:

\[ T(n) = aT(n/b) + f(n) \]

\[ = a(aT(n/b^2)) + f(n/b) + bn \]

\[ = a^2T(n/b^2) + af(n/b) + f(n) \]

\[ = a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \]

\[ = \ldots \]

\[ = a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \]

\[ = n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \]

We then distinguish the three cases as:

- The first term is dominant
- Each term in the summation is the same
- The summation is a geometric series with decreasing terms
Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.
- Divide step: Split I and J into high-order and low-order bits
  \[ I = I_h 2^{n/2} + I_l \]
  \[ J = J_h 2^{n/2} + J_l \]
- We can then define \( I*J \) by multiplying the parts and adding:
  \[ I * J = \left( I_h 2^{n/2} + I_l \right) \left( J_h 2^{n/2} + J_l \right) \]
  \[ = I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \]
- So, \( T(n) = 4T(n/2) + n \), which implies \( T(n) \) is \( O(n^2) \).
- But that is no better than the algorithm we learned in grade school.
An Improved Integer Multiplication Algorithm

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits
  \[ I = I_h 2^{n/2} + I_l \]
  \[ J = J_h 2^{n/2} + J_l \]

- Observe that there is a different way to multiply parts:
  \[
  I \times J = I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l]2^{n/2} + I_l J_l
  
  = I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_l) + I_h J_h + I_l J_l]2^{n/2} + I_l J_l
  
  = I_h J_h 2^n + (I_h J_l + I_l J_h)2^{n/2} + I_l J_l
  
- So, \( T(n) = 3T(n/2) + n \), which implies \( T(n) \) is \( O(n^{\log_2 3}) \), by the Master Theorem.
- Thus, \( T(n) \) is \( O(n^{1.585}) \).