

Sequential Data

CMPT 419/726

Mo Chen

SFU Computing Science

Mar. 2, 2020

Bishop PRML Ch. 13

Russell and Norvig, AIMA

Outline

Hidden Markov Models

Inference for HMMs

Learning for HMMs

Outline

Hidden Markov Models

Inference for HMMs

Learning for HMMs

Temporal Models

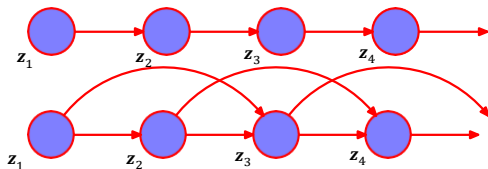
- The world changes over time
 - Explicitly model this change using Bayesian networks
 - Undirected models also exist (will not cover)
- Basic idea: copy state and evidence variables for each time step

e.g. Diabetes management

- \mathbf{z}_t is set of **unobservable state variables** at time t
 - *bloodSugar_t, stomachContents_t, ...*
- \mathbf{x}_t is set of **observable evidence variables** at time t
 - *measuredBloodSugar_t, foodEaten_t, ...*
- Assume **discrete time step**, fixed
- Notation: $\mathbf{x}_{a:b} = \mathbf{x}_a, \mathbf{x}_{a+1}, \dots, \mathbf{x}_{b-1}, \mathbf{x}_b$

Markov Chain

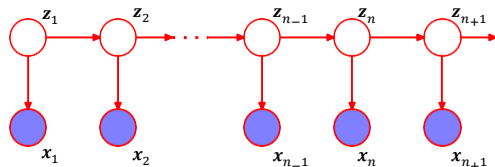
- Construct Bayesian network from these variables
 - parents? distributions? for state variables \mathbf{z}_t :
- **Markov assumption:** \mathbf{z}_t depends on **bounded** subset of $\mathbf{z}_{1:t}$
 - **First-order Markov process:** $p(\mathbf{z}_t | \mathbf{z}_{1:t-1}) = p(\mathbf{z}_t | \mathbf{z}_{t-1})$
 - **Second-order Markov process:** $p(\mathbf{z}_t | \mathbf{z}_{1:t-1}) = p(\mathbf{z}_t | \mathbf{z}_{t-2}, \mathbf{z}_{t-1})$



- **Stationary process:** $p(\mathbf{z}_t | \mathbf{z}_{t-1})$ fixed for all t

Hidden Markov Model (HMM)

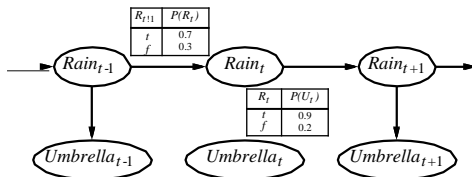
- **Sensor Markov assumption:** $p(x_t | z_{1:t}, x_{1:t-1}) = p(x_t | z_t)$
- **Stationary process:** transition model $p(z_t | z_{t-1})$ and sensor model $p(x_t | z_t)$ fixed for all t (separate $p(z_1)$)
- HMM special type of Bayesian network, z_t is a **single discrete** random variable:



- Joint distribution:

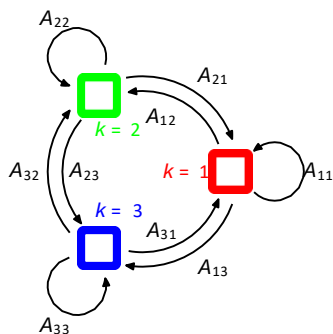
$$p(z_{1:t}, x_{1:t}) = p(z_1) \prod_{i=2:t} p(z_i | z_{i-1}) \prod_{i=1:t} p(x_i | z_i)$$

HMM Example



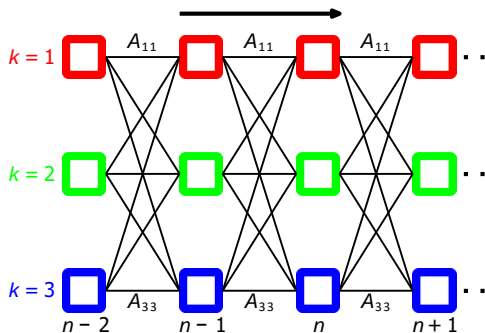
- First-order Markov assumption not true in real world
- Possible fixes:
 - **Increase order** of Markov process
 - **Augment state**, add $temp_t, pressure_t$

Transition Diagram



- z_n takes one of 3 values
- Using one-of- K coding scheme, $z_{nk} = 1$ if in state k at time n
- **Transition matrix** \mathbf{A} where $p(z_{nk} = 1 | z_{n-1,j} = 1) = \mathbf{A}_{jk}$

Lattice / Trellis Representation



- The **lattice** or **trellis** representation shows possible paths through the latent state variables z_n

Outline

Hidden Markov Models

Inference for HMMs

Learning for HMMs

Inference Tasks

- **Filtering:** $p(z_t | x_{1:t})$
 - Estimate current unobservable state given all observations to date
- **Prediction:** $p(z_n | x_{1:t})$ for $n > t$
 - Similar to filtering, without evidence
- **Smoothing:** $p(z_n | x_{1:t})$ for $n < t$
 - Better estimate of past states
- **Most likely explanation:** $\arg \max_{z_{1:t}} p(z_{1:t} | x_{1:t})$
 - e.g. speech recognition, decoding noisy input sequence

Filtering

- Aim: devise a **recursive** state estimation algorithm:

$$p(z_{t+1}|x_{1:t+1}) = f(x_{t+1}, p(z_t|x_{1:t}))$$

$$\begin{aligned} p(z_{t+1}|x_{1:t+1}) &= p(z_{t+1}|x_{1:t}, x_{t+1}) \\ &= \alpha p(x_{t+1}|x_{1:t}, z_{t+1})p(z_{t+1}|x_{1:t}) && \text{(Bayes rule)} \\ &= \alpha p(x_{t+1}|z_{t+1})p(z_{t+1}|x_{1:t}) && \text{(Markov assumption)} \end{aligned}$$

- i.e. **measurement + prediction**. Prediction by summing out z_t :

$$\begin{aligned} p(z_{t+1}|x_{1:t+1}) &= \alpha p(x_{t+1}|z_{t+1}) \sum_{z_t} p(z_{t+1}, z_t|x_{1:t}) && \text{(Marginalize)} \\ &= \alpha p(x_{t+1}|z_{t+1}) \sum_{z_t} p(z_{t+1}|z_t, x_{1:t})p(z_t|x_{1:t}) && \text{(Product rule)} \\ &= \alpha p(x_{t+1}|z_{t+1}) \sum_{z_t} p(z_{t+1}|z_t)p(z_t|x_{1:t}) && \text{(Markov assumption)} \end{aligned}$$

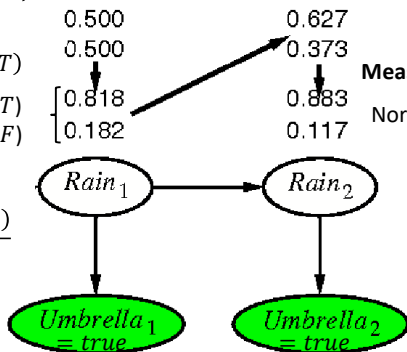
Filtering Example

Prior: $p(\text{rain}_1 = \text{true}) = 0.5$

Measurement: $p(R_1|U_1 = T)$

Normalize $0.5 \times 0.9 (R_1 = T)$
 $0.5 \times 0.2 (R_1 = F)$

$$p(R_1|U_1) = \frac{p(U_1|R_1)p(R_1)}{P(U_1)}$$



Prediction: $\sum_{R_1} p(R_2|R_1)p(R_1|U_1 = T)$
 $0.7 \times 0.818 + 0.3 \times 0.182 (R_2 = T)$
 $0.3 \times 0.818 + 0.7 \times 0.182 (R_2 = F)$

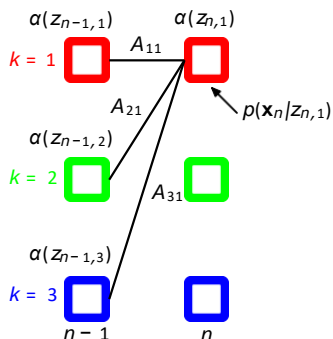
Measurement: $p(R_2|U_2 = T)$

Normalize $0.627 \times 0.9 (R_2 = T)$
 $0.117 \times 0.2 (R_2 = F)$

R_{t-1}	$P(R_t)$	R_t	$P(U_t)$
t	0.7	t	0.9
f	0.3	f	0.2

$$p(z_{t+1}|x_{1:t+1}) = \alpha p(x_{t+1}|z_{t+1}) \sum_{z_t} p(z_{t+1}|z_t)p(z_t|x_{1:t})$$

Filtering - Lattice



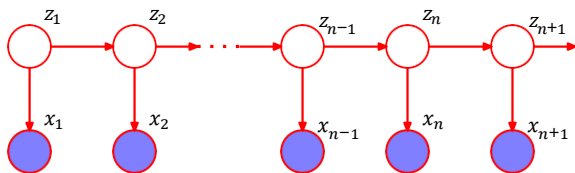
$$\underbrace{p(z_{t+1}|x_{1:t+1})}_{\substack{K\text{-dim} \\ \text{vector}}} = \alpha p(x_{t+1}|z_{t+1}) \sum_{z_t} \underbrace{p(z_{t+1}|z_t)}_{\substack{K \times K \\ \text{matrix}}} \underbrace{p(z_t|x_{1:t})}_{\substack{K\text{-dim} \\ \text{vector}}}$$

$\Rightarrow O(K^2)$ for each time step, $O(NK^2)$ for N time steps

Forward message passing: $\alpha(z_{t+1}) = p(x_{t+1}|z_{t+1}) \sum_{z_t} p(z_{t+1}|z_t) \alpha(z_t)$

- $\alpha(z_t) = p(x_{1:t}, z_t)$; previous normalization constant can be dropped
- Initial condition: $\alpha(z_1) = p(x_1, z_1) = p(x_1|z_1)p(z_1)$

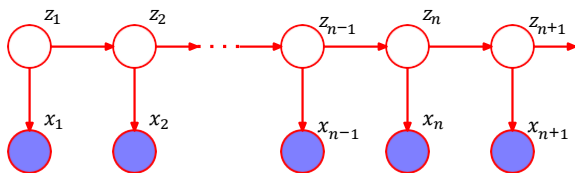
Smoothing



- Divide evidence $x_{1:t}$ into $x_{1:n}, x_{n+1:t}$

$$\begin{aligned}
 p(z_n | x_{1:t}) &= \frac{p(x_{1:t} | z_n) p(z_n)}{p(x_{1:t})} && \text{(Bayes rule)} \\
 &= \frac{p(x_{1:n} | z_n) p(x_{n+1:t} | z_n) p(z_n)}{p(x_{1:t})} && \text{(Cond. indep.)} \\
 &= \frac{p(x_{1:n}, z_n) p(x_{n+1:t} | z_n)}{p(x_{1:t})} && \text{(Product rule)} \\
 &= \frac{\alpha(z_n) \beta(z_n)}{p(x_{1:t})}
 \end{aligned}$$

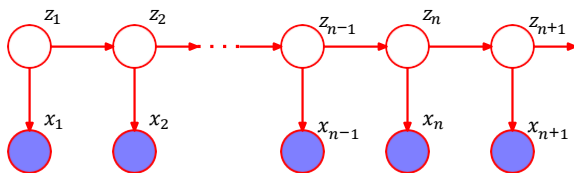
Smoothing



- Divide evidence $x_{1:t}$ into $x_{1:n}, x_{n+1:t}$, $p(z_n | x_{1:t}) = \eta \alpha(z_n) \beta(z_n)$
- Backwards message another recursion:

$$\begin{aligned}
 \underbrace{p(x_{n+1:t} | z_n)}_{\beta(z_n)} &= \sum_{z_{n+1}} p(x_{n+1:t}, z_{n+1} | z_n) && \text{(Marginalize)} \\
 &= \sum_{z_{n+1}} p(x_{n+1:t} | z_{n+1}, z_n) p(z_{n+1} | z_n) && \text{(Product rule)} \\
 &= \sum_{z_{n+1}} p(x_{n+1:t} | z_{n+1}) p(z_{n+1} | z_n) && \text{(Markov assumption)} \\
 &= \sum_{z_{n+1}} p(x_{n+1} | z_{n+1}) \underbrace{p(x_{n+2:t} | z_{n+1})}_{\beta(z_{n+1})} p(z_{n+1} | z_n) && \text{(Cond. indep.)}
 \end{aligned}$$

Smoothing



- Final condition: go back 2 slides and set $n = t$

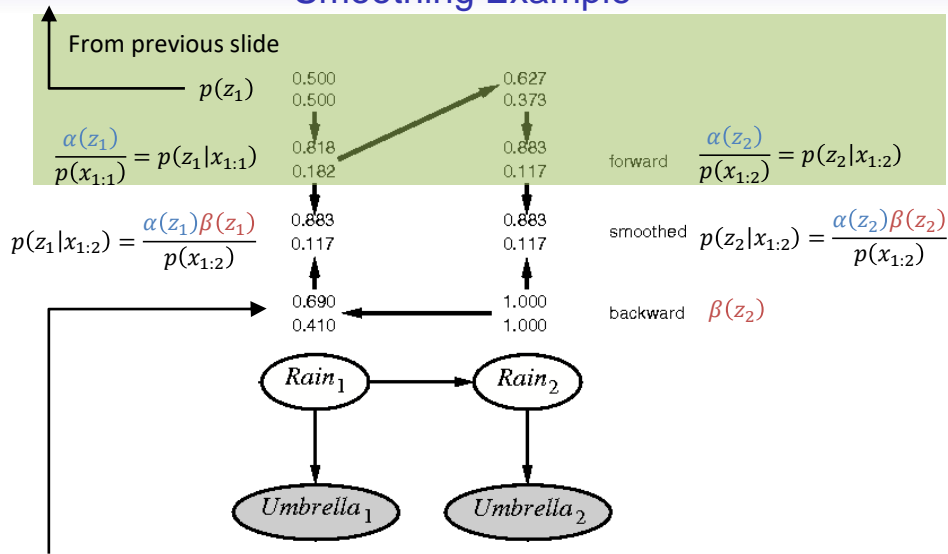
$$p(z_t | x_{1:t}) = \frac{\alpha(z_t)\beta(z_t)}{p(x_{1:t})}$$

$$p(z_t | x_{1:t}) = \frac{p(x_{1:t}, z_t)\beta(z_t)}{p(x_{1:t})}$$

$$\Rightarrow \beta(z_t) = 1$$

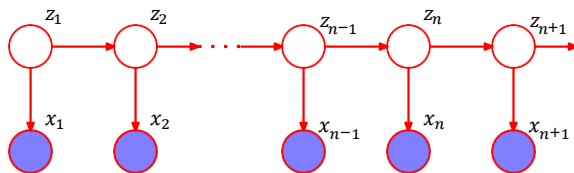
$$\alpha(z_1) = p(x_1|z_1)p(z_1)$$

Smoothing Example



$$\beta(z_1) = \sum_{z_2} p(x_2|z_2)\beta(z_2)p(z_2|z_1)$$

Smoothing

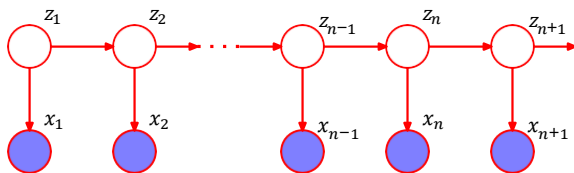


- Backwards message another recursion:

$$\underbrace{\beta(z_n)}_{\substack{K\text{-dim} \\ \text{vector}}} = \sum_{z_{n+1}} p(x_{n+1}|z_{n+1}) \underbrace{\beta(z_{n+1})}_{\substack{K\text{-dim} \\ \text{vector}}} \underbrace{p(z_{n+1}|z_n)}_{\substack{K \times K \\ \text{matrix}}}$$

$\Rightarrow O(K^2)$ for each time step, $O(NK^2)$ for N time steps

Forward-Backward Algorithm



- Filter from time 1 to N , and cache forward messages $\alpha(z_n)$
- Smooth from time N to 1, and cache backward messages $\beta(z_n)$
- Can now compute $p(z_n | x_1, x_2, \dots, x_t)$ for all n
- Total complexity $O(NK^2)$
- a.k.a Baum-Welch algorithm

Outline

Hidden Markov Models

Inference for HMMs

Learning for HMMs

HMM Parameters

- The **parameters** of an HMM:
 - **Transition matrix** A where $p(z_{nk} = 1 | z_{n-1,j} = 1) = A_{jk}$
 - **Sensor model** ϕ_k parameters to each $p(x_n | z_{nk} = 1, \phi_k)$ (e.g. ϕ_k could be mean and variance of Gaussian)
 - **Prior for initial state** z_1 , model as multinomial
 $p(z_{1k} = 1) = \pi_k$, parameters π
- Call these parameters $\theta = (A, \pi, \phi)$
- **Learning problem: given one sequence x , find best θ**
 - Extension to multiple sequences straight-forward (assume independent, log of product is sum)

Maximum Likelihood for HMMs

- We can use maximum likelihood to choose the best parameters:

$$\theta_{ML} = \arg \max p(\mathbf{x}|\theta)$$

- Unfortunately this is hard to do: we can get $p(\mathbf{x}|\theta)$ by summing out from the joint distribution:

$$\begin{aligned} p(\mathbf{x}|\theta) &= \sum_{z_1} \sum_{z_2} \cdots \sum_{z_N} p(\mathbf{x}, z_1, z_2, \dots, z_N | \theta) \\ &\equiv \sum_{\mathbf{z}} p(\mathbf{x}, \mathbf{z} | \theta) \end{aligned}$$

- But this sum has K^N terms in it
- No simple closed-form solution
- Instead, use expectation-maximization (EM)

EM for HMMs

- Start with initial guess for parameters $\theta^{old} = (\mathbf{A}, \boldsymbol{\pi}, \boldsymbol{\phi})$
- **E-step**: Calculate posterior on latent variables $p(\mathbf{z}|\mathbf{x}, \theta^{old})$

Forward-backward algorithm

$$\mathbb{E}_{z \sim p(z|\mathbf{x}, \theta^{old})} [\ln p(\mathbf{x}, \mathbf{z}|\theta)]$$

- **M-step**: Maximize $Q(\boldsymbol{\theta}, \theta^{old}) = \sum_{\mathbf{z}} p(\mathbf{z}|\mathbf{x}, \theta^{old}) \ln p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta})$ wrt $\boldsymbol{\theta}$
- Let's look at the M-step, and see how the HMM structure helps us

HMM M-step

- **M-step:** Maximize $Q(\theta, \theta^{old}) = \sum_z p(\mathbf{z}|\mathbf{x}, \theta^{old}) \ln p(\mathbf{x}, \mathbf{z}|\theta)$ wrt θ :
- The **complete data log-likelihood** factors nicely:

$$\begin{aligned} \ln p(\mathbf{x}, \mathbf{z}|\theta) &= \ln \left\{ p(z_1|\pi) \prod_{i=2}^N p(z_i|z_{i-1}, \mathbf{A}) \prod_{i=1}^N p(x_i|z_i, \phi) \right\} \\ &= \ln p(z_1|\pi) + \sum_{i=2}^N \ln p(z_i|z_{i-1}, \mathbf{A}) + \sum_{i=1}^N \ln p(x_i|z_i, \phi) \end{aligned}$$

- To maximize Q we now have 3 separate problems, one for each parameter
 - Let's consider each in turn

Prior π

- Maximize Q wrt prior on initial state π :

$$\begin{aligned}
 Q(\pi, \theta^{old}) &= \sum_z p(z|x, \theta^{old}) \ln p(z_1|\pi) \\
 &= \sum_z p(z|x, \theta^{old}) \ln \prod_{k=1}^K \pi_k^{z_{1k}} \\
 &= \sum_z p(z|x, \theta^{old}) \sum_{k=1}^K z_{1k} \ln \pi_k \\
 &= \sum_{k=1}^K \ln \pi_k \sum_z p(z|x, \theta^{old}) z_{1k} \\
 &= \sum_{k=1}^K p(z_{1k} = 1 | x, \theta^{old}) \ln \pi_k
 \end{aligned}$$

- i.e. smoothed value for z_1 being in state k

$$Q(\pi, \theta^{old}) = \sum_{k=1}^K p(z_{1k} = 1 | \mathbf{x}, \theta^{old}) \ln \pi_k$$

- Can solve for best π
- Use Lagrange multiplier to enforce constraint $\sum_k \pi_k = 1$

$$\pi_k = \frac{p(z_{1k} = 1 | \mathbf{x}, \theta^{old})}{\sum_{j=1}^K p(z_{1j} = 1 | \mathbf{x}, \theta^{old})}$$

- Intuitively sensible result: new π_k is smoothed probability of being in state k at time 1 using old parameters
- E-step needs to calculate smoothed $p(z_{1k} = 1 | \mathbf{x}, \theta^{old})$; this is fast $O(NK^2)$

Transition Matrix \mathbf{A}

- Maximize Q wrt transition matrix \mathbf{A} :

$$\begin{aligned}
 Q(\mathbf{A}, \theta^{old}) &= \sum_z p(z|x, \theta^{old}) \sum_{i=2}^N \ln p(z_i|z_{i-1}, \mathbf{A}) \\
 &= \sum_z p(z|x, \theta^{old}) \sum_{i=2}^N \ln \prod_{k=1}^N \prod_{j=1}^K \mathbf{A}_{jk}^{z_{i-1}, j z_{i,k}} \\
 &= \sum_z p(z|x, \theta^{old}) \sum_{i=2}^N \sum_{k=1}^K \sum_{j=1}^K z_{i-1, j} z_{i, k} \ln \mathbf{A}_{jk} \\
 &= \sum_{\substack{K \\ k=1}} \sum_{\substack{K \\ j=1}} \ln \mathbf{A}_{jk} \sum_{\substack{N \\ i=2}} \sum_z p(z|x, \theta^{old}) z_{i-1, j} z_{i, k} \\
 &= \sum_{k=1}^K \sum_{j=1}^K \ln \mathbf{A}_{jk} \sum_{i=2}^N p(z_{i-1} = j, z_i = k | x, \theta^{old})
 \end{aligned}$$

- E-step needs to calculate $p(z_{i-1} = j, z_i = k | x, \theta^{old})$; can be done quickly using forward and backward messages

$$Q(\mathbf{A}, \boldsymbol{\theta}^{old}) = \sum_{k=1}^K \sum_{j=1}^K \ln A_{jk} \sum_{i=2}^N p(z_{i-1} = j, z_i = k | \mathbf{x}, \boldsymbol{\theta}^{old})$$

- Can solve for best \mathbf{A}
- Again use Lagrange multipliers to enforce constraint $\sum_k A_{jk} = 1$

$$A_{jk} = \frac{\sum_{n=2}^N p(z_{n-1} = j, z_n = k | \mathbf{x}, \boldsymbol{\theta}^{old})}{\sum_{l=1}^K \sum_{n=2}^N p(z_{n-1} = j, z_n = l | \mathbf{x}, \boldsymbol{\theta}^{old})}$$

- Again sensible result: A_{jk} set to expected number of times we transition from state j to k using the smoothed results from old parameters

Sensor Model

- Similar derivation for sensor model parameters ϕ
- Again end up with weighted parameter estimated based on expected values of states given smoothed estimates

HMM EM Summary

- Start with initial guess for parameters $\theta^{old} = (\mathbf{A}, \boldsymbol{\pi}, \boldsymbol{\phi})$
- Run forward-backward algorithm to get all messages $\alpha(z_n), \beta(z_n)$ (E-step)
 - $O(NK^2)$ time complexity
 - Can use these to compute any smoothed posterior $p(z_{nk} = 1 | \mathbf{x}, \theta^{old})$
 - Also can compute any $p(z_{nk} = 1, z_{n,k} = 1 | \mathbf{x}, \theta^{old})$
 - Using these, update values for parameters (M-step)
 - $\boldsymbol{\pi}_k$ is smoothed probability of being in state k at time 1
 - \mathbf{A}_{jk} is smoothed probability of transitioning from state j to k averaged over all time steps
 - $\boldsymbol{\phi}$ is weighted sensor parameters using smoothed probabilities (e.g. similar to mixture of Gaussians)
- Repeat until convergence

Inference Tasks

- **Filtering:** $p(z_t | x_{1:t})$
 - Estimate current unobservable state given all observations to date
- **Prediction:** $p(z_n | x_{1:t})$ for $n > t$
 - Similar to filtering, without evidence
- **Smoothing:** $p(z_n | x_{1:t})$ for $n < t$
 - Better estimate of past states
- **Most likely explanation:** $\arg \max_{z_{1:t}} p(z_{1:t} | x_{1:t})$
 - e.g. speech recognition, decoding noisy input sequence

Sequence of Most Likely States

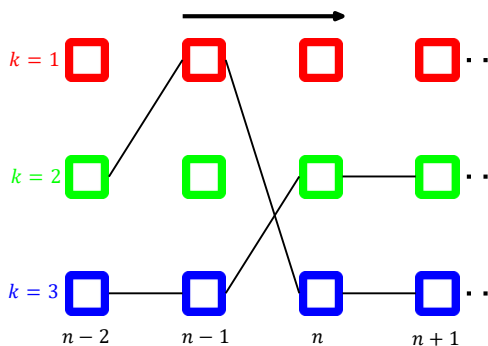
- Most likely sequence is not same as sequence of most likely states:

$$\arg \max_{z_{1:N}} p(z_{1:N} | x_{1:N})$$

versus

$$\left(\arg \max_{z_1} p(z_1 | x_{1:N}), \dots, \arg \max_{z_N} p(z_N | x_{1:N}) \right)$$

Paths Through HMM

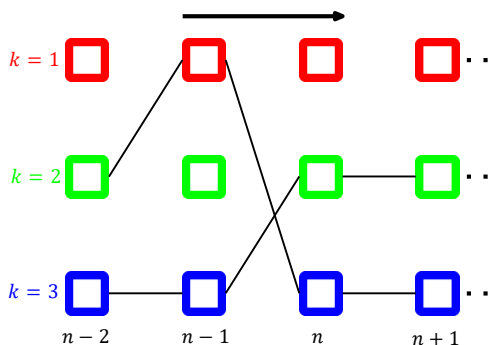


- There are K^N paths to consider through the HMM for computing

$$\arg \max_{z_{1:N}} p(z_{1:N} | x_{1:N})$$

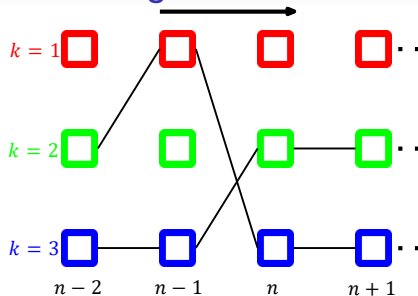
- Need a faster method

Viterbi Algorithm



- Insight: for any value k for z_n , the best path $(z_1, z_2, \dots, z_n = k)$ ending in $z_n = k$ consists of the best path $(z_1, z_2, \dots, z_{n-1} = j)$ for some j , plus one more step
 - Don't need to consider exponentially many paths, just K at each time step
 - Dynamic programming algorithm – [Viterbi algorithm](#)

Viterbi Algorithm - Math



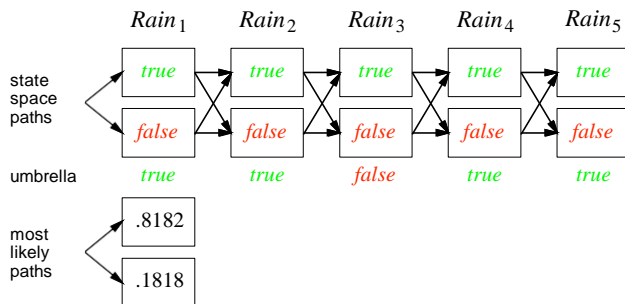
- Define message

$$w(n, k) = \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k)$$

- From factorization of joint distribution:

$$\begin{aligned} w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_{n-1}, z_1, \dots, z_{n-1}) p(x_n | z_n = k) p(z_n = k | z_{n-1}) \\ &= \max_{z_{n-1}} \max_{z_1, \dots, z_{n-2}} p(x_{1:n-1}, z_{1:n-1}) p(x_n | z_n = k) p(z_n = k | z_{n-1}) \\ &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j) \end{aligned}$$

Viterbi Algorithm - Example



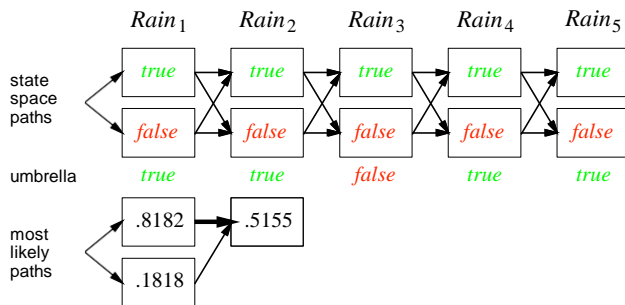
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



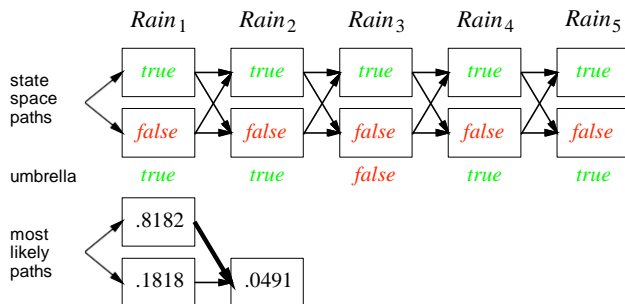
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



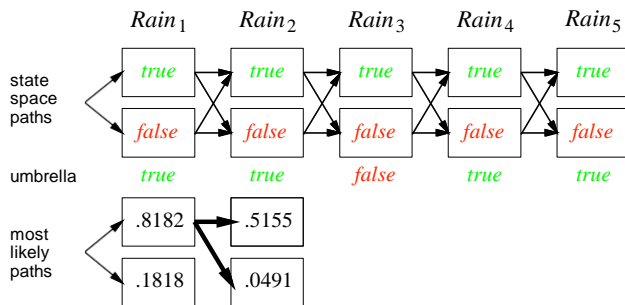
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



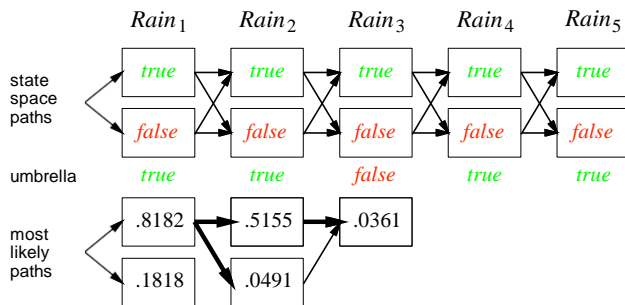
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



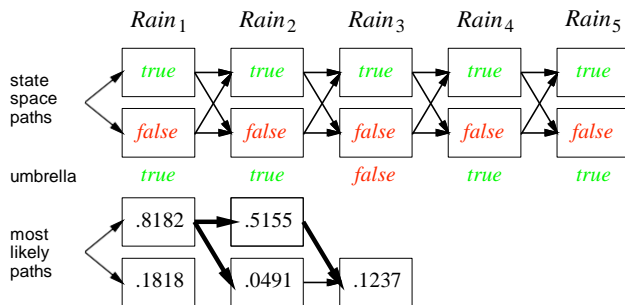
R _{t-1}	P(R _t)
t	0.7
f	0.3

R _t	P(U _t)
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



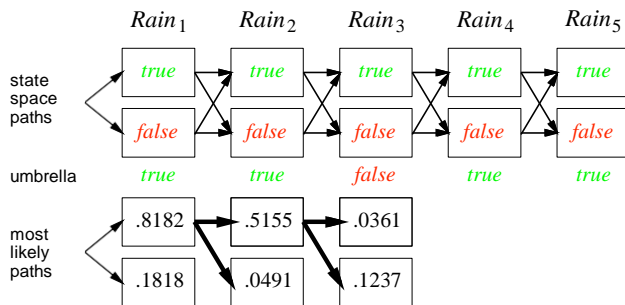
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



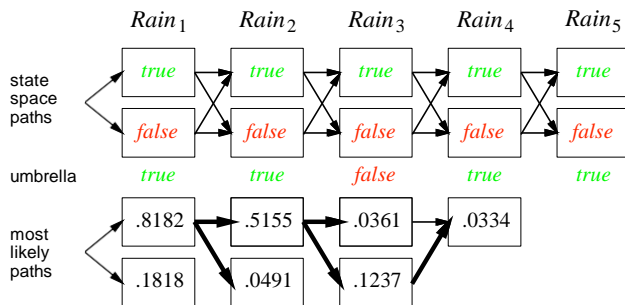
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



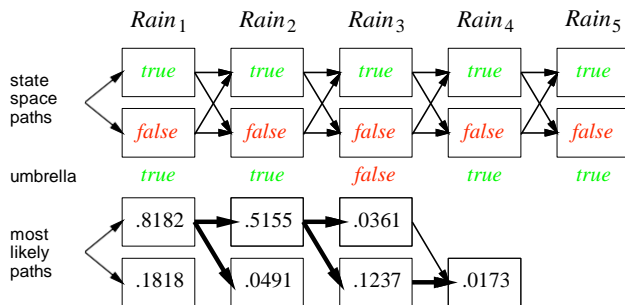
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



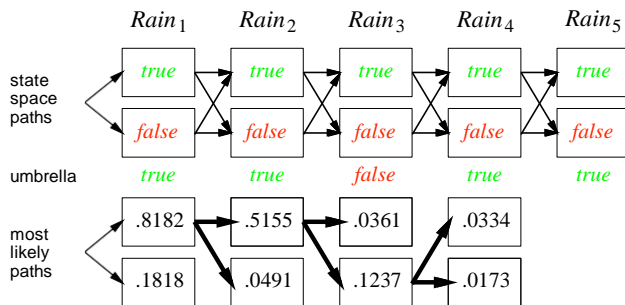
R _{t-1}	P(R _t)
t	0.7
f	0.3

R _t	P(U _t)
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



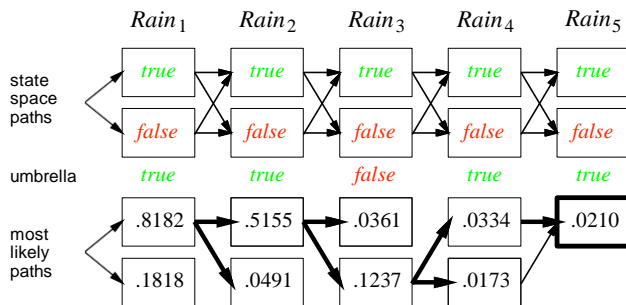
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



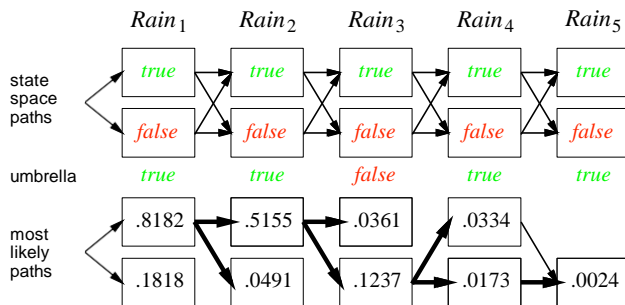
R _{t-1}	P(R _t)
t	0.7
f	0.3

R _t	P(U _t)
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



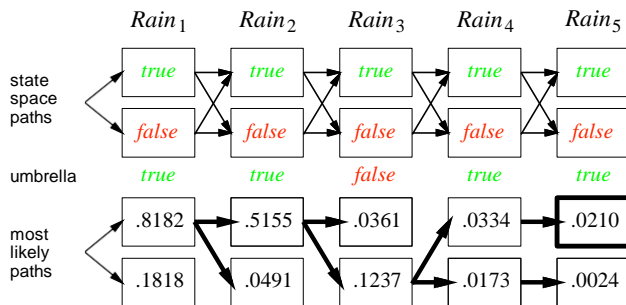
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



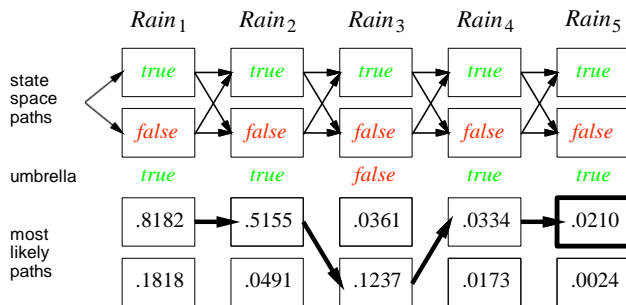
R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Example



R _{t-1}	P(R _t)
t	0.7
f	0.3

R _t	P(U _t)
t	0.9
f	0.2

$$p(\text{rain}_1 = \text{true}) = 0.5$$

$$\begin{aligned}
 w(n, k) &= \max_{z_1, \dots, z_{n-1}} p(x_1, \dots, x_n, z_1, \dots, z_n = k) \\
 &= \max_j w(n-1, j) p(x_n | z_n = k) p(z_n = k | z_{n-1} = j)
 \end{aligned}$$

Viterbi Algorithm - Complexity

- Each step of the algorithm takes $O(K^2)$ work
- With N time steps, $O(NK^2)$ complexity to find most likely sequence
- Much better than naive algorithm evaluating all K^N possible paths

Conclusion

- Readings: Ch. 13.2, 13.2.1, 13.2.2, 13.2.5
- HMM - Probabilistic model of temporal data
 - Discrete hidden (unobserved, latent) state variable at each time
 - Observation (can be discrete / continuous) at each time
 - Conditional independence assumptions (Markov)
 - Assumptions on distributions (stationary)
- Inference
 - Filtering
 - Smoothing
 - Most likely sequence (next)
- Maximum likelihood learning
 - EM – efficient computation $O(NK^2)$ time using forward-backward smoothing
- Most likely sequence in HMM
 - Viterbi algorithm – $O(NK^2)$ time, dynamic programming algorithm