A Puzzle For You

Problem: Write a program to output the first N cubes, but without using multiplication (only addition/subtraction).

```
N = 10

Output:

0
1
8
27
64
125
216
343
512
729
```

Historically, CPUs are relatively slow at multiplication vs addition/subtraction.

- The differences can be small (3x) or large (20x).

The Correctness of Algorithms and Programs

CMPT 125
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Lecture 13

Today:

- Assertions and Invariants
- Good Invariants and Post-Conditions
- Proving Programs Correct

Puzzle Solution

```
int main () {
    int N = 10;
    for (int i = 0; i < N; i++) {
        //  Compute square = i*i
        int square = 0;
        for (int j = 0; j < i; j++) {
            //  Assertion:  square == j*i
            square += i;
        }

        //  Compute cube = i*i*i
        int cube = 0;
        for (int j = 0; j < i; j++) {
            cube += square;
        }
        printf("%d\n", cube);
    }
}
```

The Algorithm (Pseudocode):

For each i from 0 to N - 1:

- Compute the i^{th} square by adding i to itself i times.
- Compute the i^{th} cube by adding the i^{th} square to itself i times.
- Output the i^{th} cube.

Do you believe that at the end of this loop, the value of `square` will equal `i*i`?

Q. What's a good assertion?

Good assertions, also called *loop invariants*, are usually related to the post-condition.

What makes a *good* loop invariant?

A loop invariant is a statement that is true every loop.

- usually asserted at the beginning of the loop
- usually parametrized by the loop index (`j` in this case)

```
//  Post:  square == i*i
int square = 0;
for (int j = 0; j < i; j++) {
    //  Assertion:  square == j*i
    square += i;
}
```

A good loop invariant should indicate the progress of the algorithm

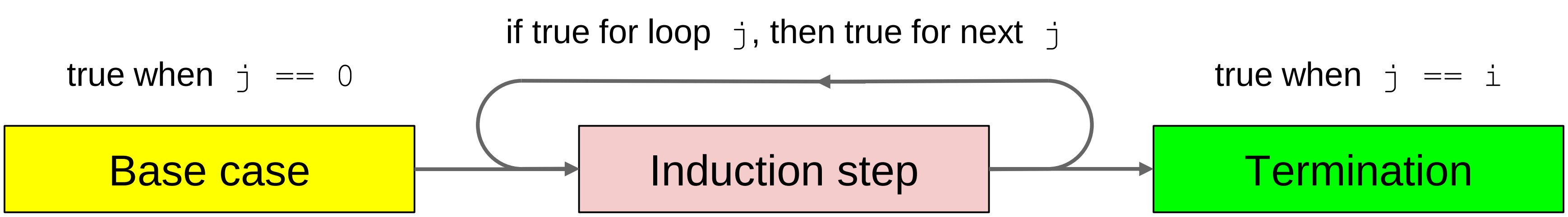
- the invariant should carry all state information, loop to loop.
- the invariant should imply the post-condition (the goal of the algorithm) at the end of the last loop.

Proving Correctness

```
//  Post:  square == i*i
int square = 0;
for (int j = 0; j < i; j++) {
    //  Assertion:  square == j*i
    square += i;
}
```

Use mathematical reasoning to capture the behaviour of an algorithm:

- State *invariants* at various *checkpoints*.
- Show that the invariant holds:
  - at the first checkpoint
  - during execution between checkpoints
- Conclude that the post-condition holds
  - the invariant holds at / after the last checkpoint



Proof

```
//  Post:  square == i*i
int square = 0;
for (int j = 0; j < i; j++) {
    //  Assertion:  square == j*i
    square += i;
}
```

Base case:

- Is the invariant true on the first loop?
  - When `j == 0`, `square` has been initialized to 0. These values satisfy `square == j*i`.

Induction step:

- If the invariant holds at the beginning of loop `j`, does it also hold for the beginning of loop `j+1`?
  - At the beginning of loop `j`, `square == j*i`.
  - After running the loop, `square == j*i + i == (j+1)*i`, which is the invariant of the next loop.

Termination:

- Since the invariant holds for all `j`, it holds after the last loop.
  - Therefore, when `j == i`, `square == i*i`.

Yay - we proved it! So what?

We honestly won't care whether or not you can do a proof of correctness 5 years from now in your job. And neither will:

- your boss
- your co-workers
- ♥♥♥ your secret crush ♥♥♥

You learn to do proofs to get better at reasoning about code.

The more practiced you are at thinking about invariants:

- the better your resulting code will be
- the easier it will be to figure out other people's code

A computer won't be able to verify your programs for you

- in general, this is an impossible problem.

What does it do?

```
int main () {
    int N = 10;
    int a = 6;
    int b = 1;
    int c = 0;

    for (int i = 0; i < N; i++) {
        printf("%d\n", c);
        c += b;
        b += a;
        a += 6;
    }
}
```

Two ways to get started:

1. Simulate the execution on paper.
2. Key in the program and run it!

```
Output:

0
1
8
27
64
125
216
343
512
729
```

Q. Why is this program significant?

What are the invariants?

```
int main () {
    int N = 10;
    int a = 6;
    int b = 1;
    int c = 0;

    for (int i = 0; i < N; i++) {
        //  Assertion: a == 6*(i+1)
        //  Assertion: b == 3*i*(i+1) + 1
        //  Assertion: c == i*i*i
        printf("%d\n", c);
        c += b;
        b += a;
        a += 6;
    }
}
```

Base case: When `i` = 0, the assertions are:

- `a` $= 6(0+1) = 6$
- `b` $= 3\cdot 0\cdot(0+1) + 1 = 1$
- `c` $= (0)^3 = 0$

which are the 3 initial values for `a`, `b`, `c`.

Induction step: At the start of loop `i`, the assertions are:

- `a` $= 6(i + 1)$
- `b` $= 3i(i + 1) + 1$
- `c` $= i^3$

After `c += b;` the value of `c` changes to

- `c` $= i^3 + 3i(i + 1) + 1$
  $= i^3 + 3i^2 + 3i + 1$
  $= (i + 1)^3$

After `b += a;` the value of `b` changes to

- `b` $= 3i(i + 1) + 1 + 6(i + 1)$
  $= (i + 1)(3i + 6) + 1$
  $= 3(i + 1)(i + 2) + 1$

After `a += 6;` the value of `a` changes to

- `a` $= 6(i + 1) + 6$
  $= 6(i + 2)$

which are the values for `a`, `b`, `c` on loop $i + 1$.

Since the assertion `c` $= i^3$ holds on every loop, the algorithm is correct.

What are the invariants?

```
int main () {
    int N = 10;
    int a = 6;
    int b = 1;
    int c = 0;

    for (int i = 0; i < N; i++) {
        //  Assertion: a = 6(i + 1)
        //  Assertion: b = 3i(i + 1) + 1
        //  Assertion: c = i^3
        printf("%d\n", c);
        c += b;
        b += a;
        a += 6;
    }
}
```

Since the assertion c = i^3 holds on every loop, the algorithm is correct.

Base case: When i = 0, the assertions are:

- a = 6(0+1) = 6
- b = 3·0·(0+1) + 1 = 1
- c = $(0)^3$ = 0

which are the 3 initial values for a, b, c.

Induction step: At the start of loop i, the assertions are:

- a = 6(i + 1)
- b = 3i(i + 1) + 1
- c = i^3

After `c += b;` the value of c changes to

- c = i^3 + 3i(i + 1) + 1
  = $i^3 + 3i^2 + 3i + 1$
  = $(i + 1)^3$

After `b += a;` the value of b changes to

- b = 3i(i + 1) + 1 + 6(i + 1)
  = (i + 1)(3i + 6) + 1
  = 3(i + 1)(i + 2) + 1

After `a += 6;` the value of a changes to

- a = 6(i + 1) + 6
  = 6(i + 2)

which are the values for a, b, c on loop i + 1.

Example: Reversing a String S

reverse(“stressed”) returns “desserts”

head(“stressed”) = “s”

tail(“stressed”) = “tressed”

certainly for any nonempty S:

S = head(S) + tail(S)

X ← S; Y ← “”

while X not empty do:

    Y ← head(X) + Y

    X ← tail(X)

return Y

Adding Invariants & Checkpoints

What’s a good invariant?

```
X <- S; Y <- “”
while X not empty do:
        Y <- head(X) + Y
        X <- tail(X)
return Y
```

checkpoint 1: before the first loop

S = reverse(Y) + X

checkpoint 2: at the end of each loop

Proving the Invariant

```
X <- S; Y <- “”
chkpt 1: // inv:  S = reverse(Y) + X
while X not empty do:
        Y <- head(X) + Y
        X <- tail(X)
        chkpt 2: // inv:  S = reverse(Y) + X
return Y
```

Is it true at checkpoint 1?

Yes, because reverse(Y) + X = reverse(“”) + X = “” + X = X

Is it true at checkpoint 2?

Execute checkpoint to checkpoint: If the invariant is true at the beginning of the loop, then is it true at the end of the loop.

Proving the Invariant

Suppose that S = reverse(Y) + X at the beginning of some loop. Then after running the next loop Y' = head(X) + Y and X' = last(X)

So, just need to show that S = reverse(Y') + X'.

reverse(Y') + X' = reverse(head(X) + Y) + last(X) = reverse(Y) + head(X) + last(X) = reverse(Y) + X = S.

One Last Detail

So, the invariant holds at the beginning of the first loops, and at the end of every successive loop. Including the last loop!

- When X = “”, the loop terminates and S = reverse(Y) + X = reverse(Y). Thus Y = reverse(S).

But does the loop terminate?

- Another invariant: that |X| is natural and decreasing.