The Ford-Fulkerson Method

Chapter 26

Lecture Overview

- Ford-Fulkerson Overview
- Residual Networks
- Augmenting Paths
- Cuts in Flow Networks
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- Edmonds-Karp Algorithm

The Ford-Fulkerson Method

It's more than an algorithm. It's a general scheme with several different implementations.

The Ford-Fulkerson method iteratively increases the value of a flow in a flow network, starting with the everywhere-zero flow. At each iteration, we have a flow and a residual network. We then find an augmenting path and increase the flow along it. Then we repeat.

FORD-FULKERSON-METHOD(G, s, t)

1. initialize flow f to 0
2. **while** there is an augmenting path p in residual network G_f
3. augment flow f along p
4. **return** f

Residual Networks

- Given a flow network G with a flow f on it, the residual network G_f consists of edges with capacities that represent how we can change the flow on edges of G and still respect the original capacities. An edge of the flow network can admit an amount of additional flow equal to the edge's capacity minus the flow on that edge.

- If flow on an edge is positive, we place that edge into G_f with a residual capacity of $c_f(u, v) = c(u, v) - f(u, v)$. 0-capacity edges are not included in G_f.

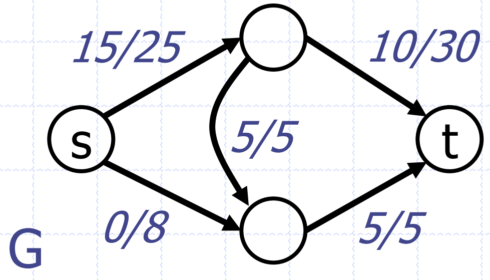
- In order to represent a possible decrease of a positive flow along edge (u, v) in G, we place an edge (v, u) into G_f with residual capacity $c_f(v, u) = f(u, v)$.

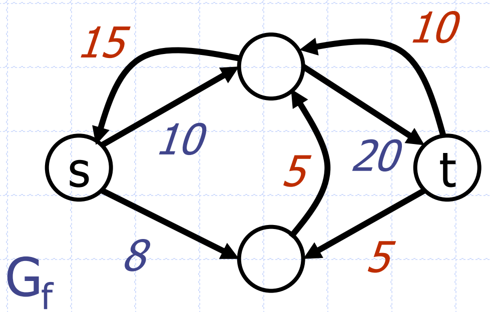
Residual Networks

To summarize,

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

And the residual network of G induced by f is $G_f = (V, E_f)$ where $E_f = \{(u, v) : c_f(u, v) > 0\}$.





Flows in Residual Networks

A flow in a residual network is a roadmap for adding flow to the original flow network. If f is a flow in G and f' is a flow in G_f, we define $f \uparrow f'$, the augmentation of flow f by f', to be defined by

$$(f \uparrow f')(\mathrm{u}, \mathrm{v}) = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

In other words, we increase the flow on (u, v) by f'(u, v), but also decrease it by f'(v, u), because pushing flow on the reverse edge in the residual network signifies decreasing or cancelling the flow in the original network.

Flows in Residual Networks

Lemma. If f is a flow in $G = (V, E)$ and f' is a flow in G_f, then $f \uparrow f'$ is a flow in G with value $|f \uparrow f'| = |f| + |f'|$.

Proof. We first verify that $f \uparrow f'$ obeys the capacity constraints and flow conservation.

If (u, v) is in E, then $c_f(v, u) = f(u, v)$. Therefore, we have $f'(v, u) \leq c_f(v, u) = f(u, v)$ and

$$
\begin{aligned}
(f \uparrow f')(u, v) &= f(u, v) + f'(u, v) - f'(v, u) && \text{by definition} \\
&\geq f(u, v) + f'(u, v) - f(u, v) \\
&= f'(u, v) \\
&\geq 0.
\end{aligned}
$$

Flows in Residual Networks

also,

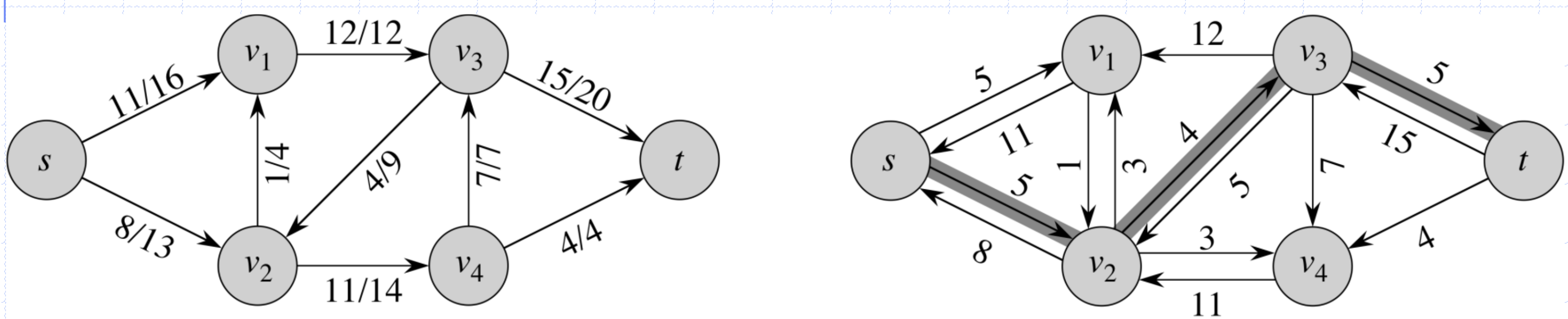
$$
\begin{aligned}
(f \uparrow f')(u, v) &= f(u, v) + f'(u, v) - f'(v, u) && \text{by definition} \\
&\leq f(u, v) + f'(u, v) \\
&\leq f(u, v) + c_f(u, v) && \text{(capacity constraint)} \\
&= f(u, v) + c(u, v) - f(u, v) && \text{(defn. of } c_f\text{)} \\
&= c(u, v).
\end{aligned}
$$

So the $f \uparrow f'$ meets the capacity constraints.

For flow conservation of $f \uparrow f'$, we appeal to the flow conservation of f and the flow conservation of f'. A formal proof is in the text. ■

Augmenting Paths

Given a flow network $G = (V, E)$ and a flow f, an augmenting path is a simple path from s to t in the residual network G_f. We may increase the flow on an edge (u, v) of an augmenting path by up to $c_f(u, v)$ without violating the capacity constraint on whichever of (u, v) and (v, u) is in the original graph.



Augmenting Paths

Let p be an augmenting path, and define the capacity of p to be the minimum capacity of the edges in p:

$$c_f(p) = \min \{c_f(u, v) : (u, v) \text{ is on } p\}.$$

We can then define a flow f_p along p in residual graph G_f:

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \\ 0 & \text{otherwise} \end{cases}$$

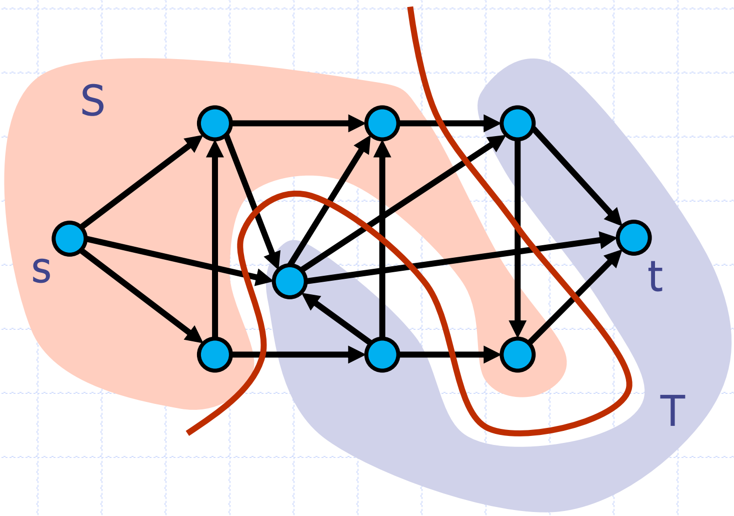
Lemma. f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Corollary. $f \uparrow f_p$ is a flow in G with value $|f \uparrow f_p| = |f| + |f_p| > |f|$

Cuts in Flow Networks

To be certain that our algorithm terminates correctly, we need to show that Ford-Fulkerson finds a maximum flow. To prove this, we will need to explore cuts in flow networks.

A cut (S, T) of a flow network G = (V, E) is a partition of V into S and T=V – S such that $s \in S$ and $t \in T$.



If f is a flow, the net flow f(S, T) across the cut (S, T) is:

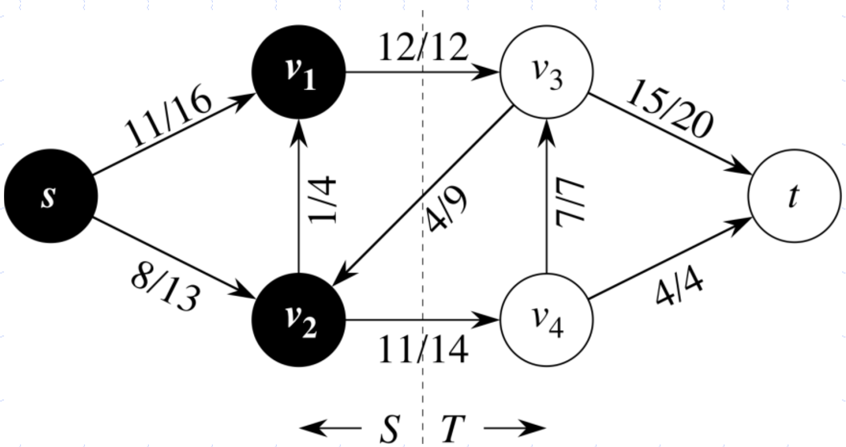
$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

Cuts in Flow Networks

The capacity of the cut (S, T) is

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

A minimum cut of a flow network is a cut whose capacity is minimum over all cuts of the network.



f(S, T) = 12 + 11 – 4 = 19

c(S, T) = 12 + 14 = 26

Cuts in Flow Networks

Lemma. Let f be a flow in a network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is f(S, T) = |f|.

Proof. In the text. Basically it follows from flow-conservation. ■

Corollary. The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.

Proof. Let (S, T) be any cut and f be any flow.

$$|f| = f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

$$\leq \sum_{u \in S} \sum_{v \in T} f(u,v) \leq \sum_{u \in S} \sum_{v \in T} c(u,v) = c(S,T)$$

Max-flow min-cut theorem

Theorem. Let f be a flow in a network G = (V, E) with source s and sink t. The following conditions are equivalent.

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. |f| = c(S, T) for some cut (S, T) of G.

Proof. (1) ⇒ (2) Suppose f is a maximum flow in G but G_f contains an augmenting path p. Then $f \uparrow f_p$ is a flow in G with value greater than |f|, a contradiction.

(2) ⇒ (3) Suppose G_f contains no augmenting path (path from s to t). Let S = {v | there is a path from s to v in G_f} and T = V – S.

Max-flow min-cut theorem

Consider (u, v) where $u \in S$ and $v \in T$. If $(u, v) \in E$, we must have $f(u, v) = c(u, v)$. If $(v, u) \in E$, we must have $f(v, u) = 0$.

Thus $f(S, T) = c(S, T)$. But $|f| = f(S, T)$.

(3) $\Rightarrow$ (1) $|f| \le c(S, T)$ for all cuts (S, T). $|f| = c(S, T)$ implies $|f|$ is a maximum flow. ■

Ford-Fulkerson

FORD-FULKERSON(G, s, t)

```
for each edge e in G.edges
    e.flow = 0
while there is a path p from s to t in residual network G_f
    c_f(p) = min {c_f(u, v) : (u, v) is in p}
    for each edge (u, v) in p
        if (u, v) is in G.edges
            (u, v).flow = (u, v).flow + c_f(p)
        else
            (v, u).flow = (v, u).flow – c_f(p)
```

Ford-Fulkerson Analysis

The running time of FORD-FULKERSON depends on the way the augmenting path p is chosen.

If p is not chosen well, FORD-FULKERSON may not even terminate (for the case when irrational capacities are allowed).

If capacities are restricted to be rational, then we can convert the problem to an integer capacity one by multiplying all capacities by their common denominator. But integer capacities are by far the most common form of the problem in practice.

For integer capacities, FORD-FULKERSON runs in time O(E |f*|), where f* is the maximum flow.

Ford-Fulkerson Analysis

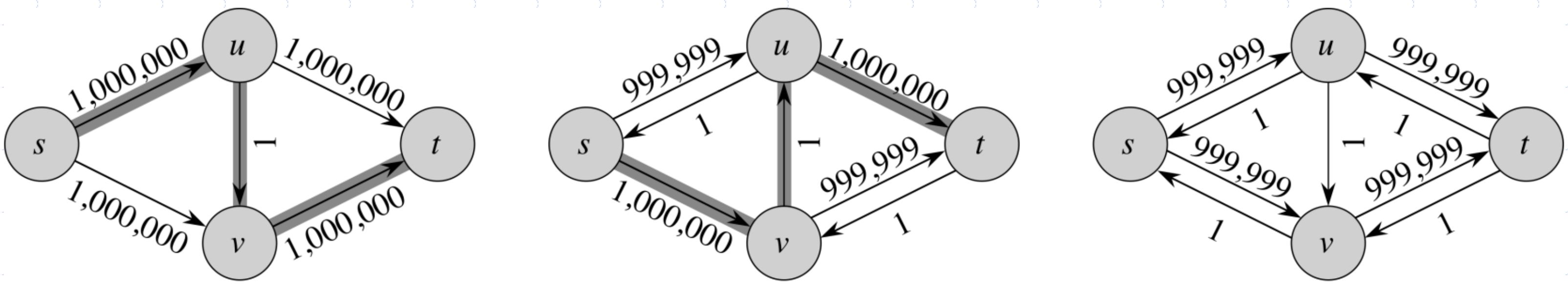
- For integer capacities, FORD-FULKERSON runs in time $O(E\,|f^*|)$, where f^* is the maximum flow.
- The loop of lines 1 – 2 takes time $O(E)$.
- The loop body of lines 4 – 9 takes time
$$O(\text{size of } p) \subseteq O(V) \subseteq O(E).$$
- Finding a path in line 3 takes time $O(E)$ (by BFS or DFS, e.g.).
- The number of iterations of the loop in lines 3 – 9 is at most $|f^*|$, as each iteration increases the flow by an integer amount.
- Therefore lines 3 – 9 take a total of $O(E\,|f^*|)$ time.

output-sensitive

Ford-Fulkerson Analysis

FORD-FULKERSON is good when capacities are integers and the optimal flow value |f*| is small. But it can be bad when |f*| is large.

This flow network could take 2 million path augmentations if we choose the augmenting paths poorly:



(a) (b) (c)

Edmonds-Karp

We can avoid this poor behaviour by always choosing p as a shortest path from s to t in the residual network; we can do this by breadth-first search. The Ford-Fulkerson method with this method of choosing p is known as the Edmonds-Karp algorithm. Edmonds-Karp runs in $O(VE^2)$ time.

The analysis depends on distances to vertices in the residual network G_f. We use $\delta_f(u, v)$ for the shortest-path distance from u to v in G_f where each edge has unit distance.

Lemma. If Edmonds-Karp is run on G, then for all vertices v in $V - \{s, t\}$, the shortest-path distance $\delta_f(s, v)$ increases monotonically with each flow augmentation.

Edmonds-Karp

Proof (sketch). By contradiction. Suppose there is an augmentation that causes $\delta_f(s, v)$ to decrease. Let f be the flow just before the first such augmentation, and f' be the flow just afterward. Let v be the vertex with minimum $\delta_{f'}(s, v)$ whose distance was decreased; $\delta_{f'}(s, v) < \delta_f(s, v)$. Let p be a shortest path from s to v in $G_{f'}$ and u the predecessor of v on this path. (u, v) is in $E_{f'}$ and $\delta_{f'}(s, u) = \delta_{f'}(s, v) - 1$.

The distance of u from s did not decrease: $\delta_{f'}(s, u) \geq \delta_f(s, u)$.

Now, (u, v) is not in E_f. But it is in $E_{f'}$. So it must be that the augmentation increased the flow from v to u. This means there was a shortest path from s to u in G_f ending with (v, u). Thus $\delta_f(s, v) = \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 2$. ■

Edmonds-Karp

Theorem. If Edmonds-Karp is run on G = (V, E), the total number of flow augmentations it performs is O(VE).

Proof. Call an edge (u, v) on p critical if $c_f(p) = c_f(u, v)$. After we have augmented with augmenting path p, any critical edge on p disappears from the network. At least one edge on any p is critical. We show that each of the edges can become critical at most O(V) times.

Let (u, v) be in E. When (u, v) is critical, we have $\delta_f(s, v) = \delta_f(s, u) + 1$. Once the flow is augmented, (u, v) disappears from the residual network. It cannot reappear until after (v, u) appears on an augmenting path. If f' is the flow when this happens, then

Edmonds-Karp

$\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \geq \delta_f(s, v) + 1 = \delta_f(s, u) + 2$.

I.e. from one time (u, v) goes critical to the next time it can go critical, u's distance from s has increased by 2. Since distances can be at most |V| – 1, and at the start the distance from s to u is at least 0, any one edge can go critical at most O(V) times.

Since there are O(E) edges, and each can go critical O(V) times, there can be no more augmentations after O(EV) of them, as each augmentation has at least one critical edge. ■

Edmonds-Karp Analysis

The analysis of Edmonds-Karp is now straightforward.

As we stated when talking about FORD-FULKERSON, the loop in lines 1-2 is $O(E)$. The loop in lines 3-9 is $O(E)$ per iteration. The theorem establishes that there are at most $O(VE)$ iterations. Therefore the total time is $O(E) + O(VE^2) = O(VE^2)$.