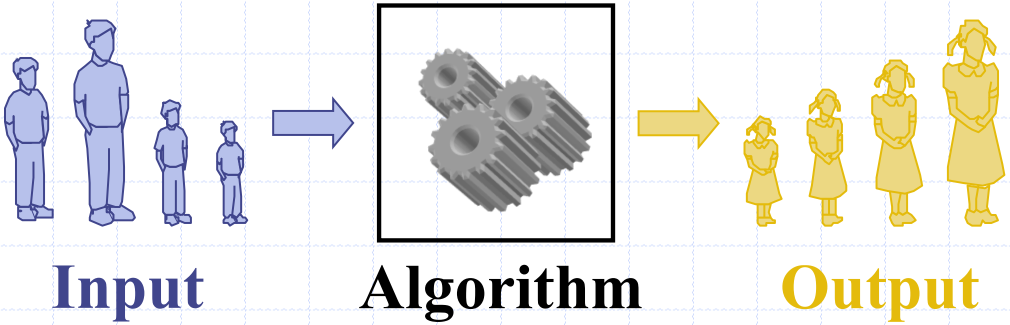
Analysis of Algorithms

Chapter 4



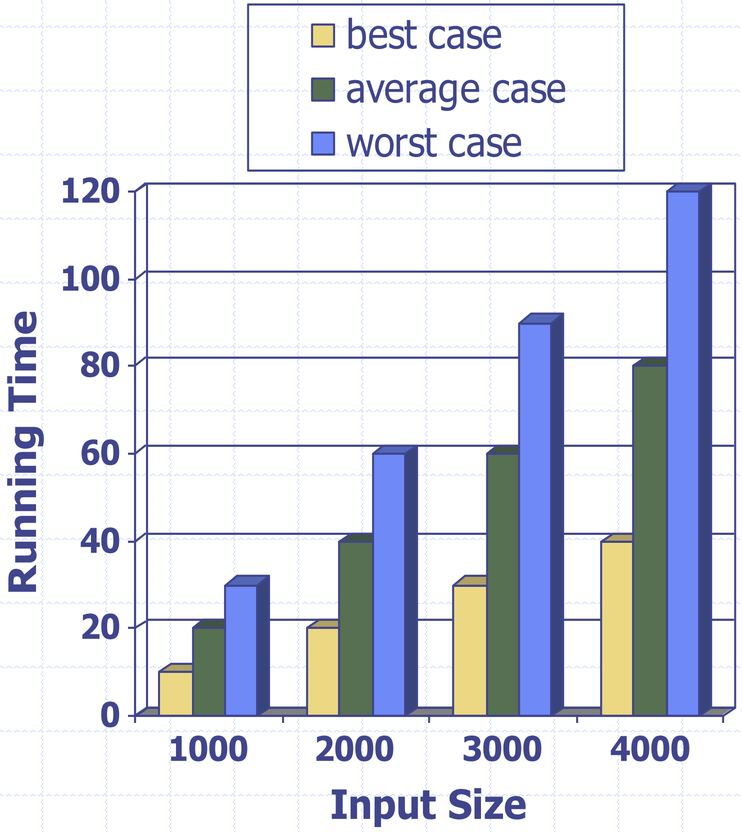
Input

Algorithm

Output

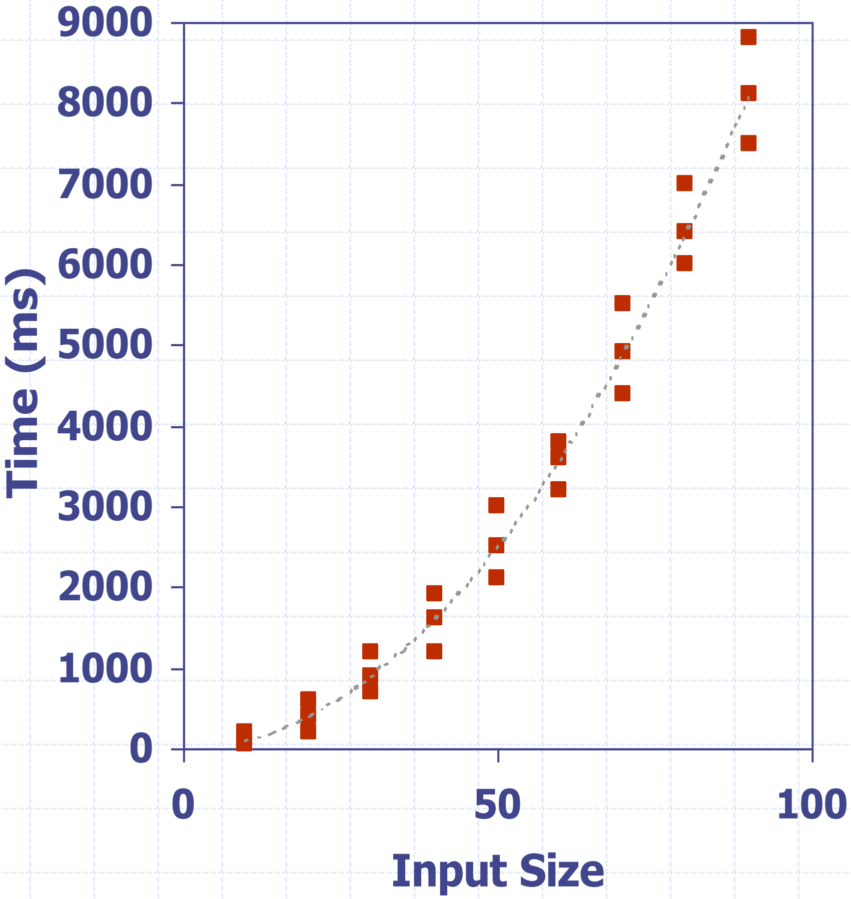
Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



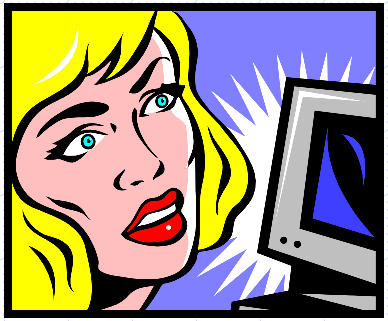
Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like clock() to get an accurate measure of the actual running time
- Plot the results

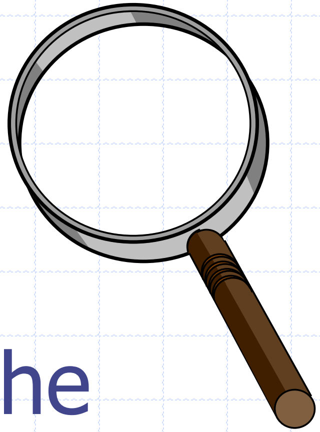


Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used, and the same amount of care in implementation.



Theoretical Analysis



- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

```
Algorithm arrayMax($A$, $n$)
  Input array $A$ of $n$ integers
  Output maximum element of $A$

  $currentMax \leftarrow A[0]$
  for $i \leftarrow 1$ to $n - 1$ do
    if $A[i] > currentMax$ then
      $currentMax \leftarrow A[i]$
  return $currentMax$
```

Book Pseudocode
(not mandatory rules)

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

```
Algorithm method (arg [, arg…])
    Input …
    Output …
```

- Method call

  var.method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions

  ← Assignment (like = in C++)

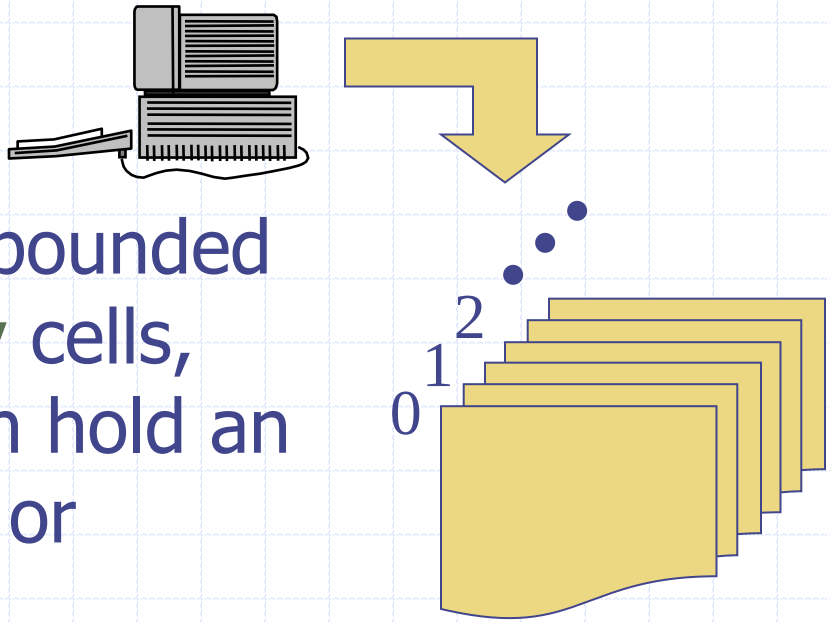
  = Equality testing (like == in C++)

  n^2 Superscripts and other mathematical formatting allowed

The Random Access Machine (RAM) Model

- A **CPU**

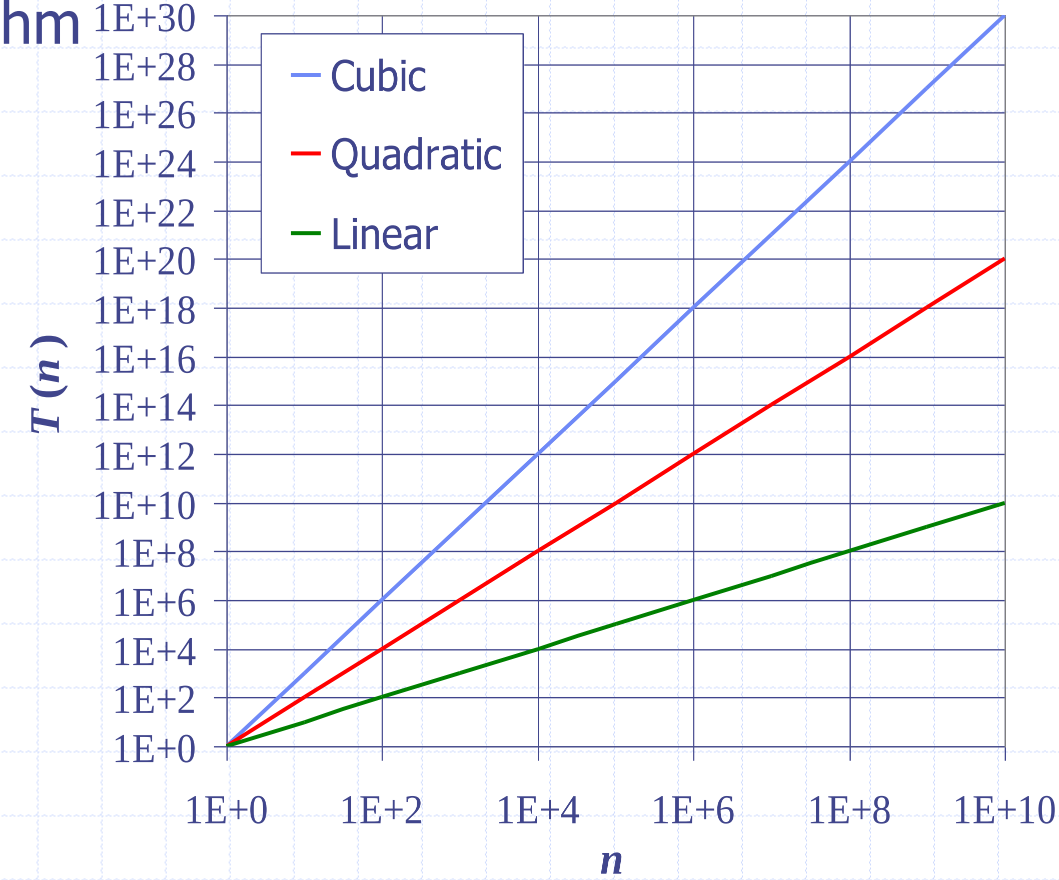
- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character



- Memory cells are numbered and accessing any cell in memory takes unit time.

Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ $\boldsymbol{1}$
  - Logarithmic ≈ log $\boldsymbol{n}$
  - Linear ≈ $\boldsymbol{n}$
  - N-Log-N ≈ $\boldsymbol{n}$ log $\boldsymbol{n}$
  - Quadratic ≈ $\boldsymbol{n}^2$
  - Cubic ≈ $\boldsymbol{n}^3$
  - Exponential ≈ 2^n



- In a log-log chart, the slope of the line corresponds to the growth rate

Functions Graphed Using "Normal" Scale

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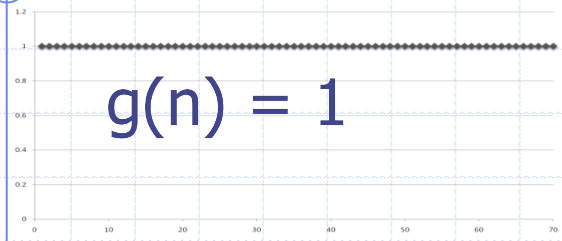
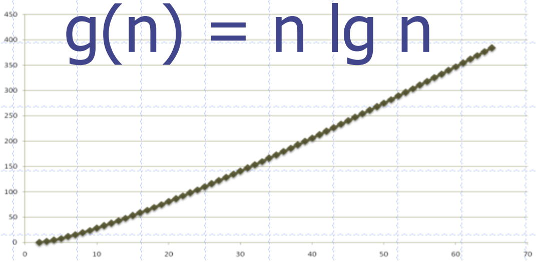
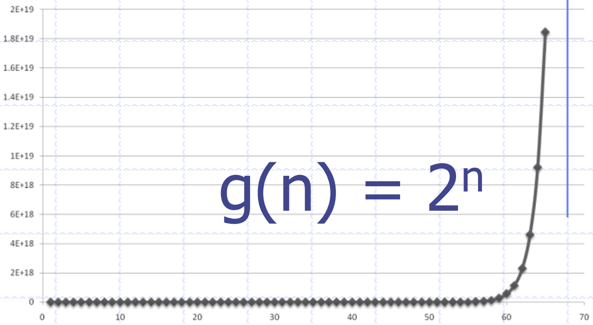
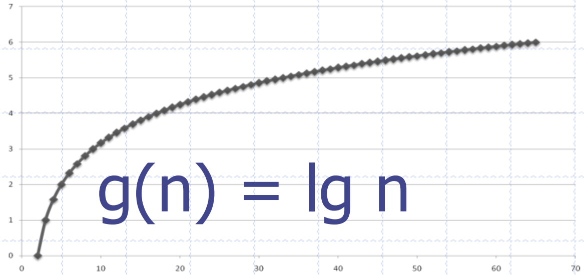
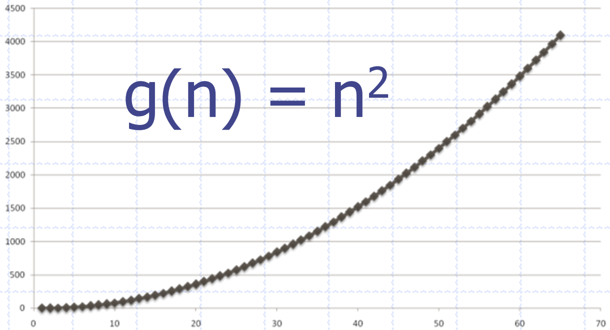
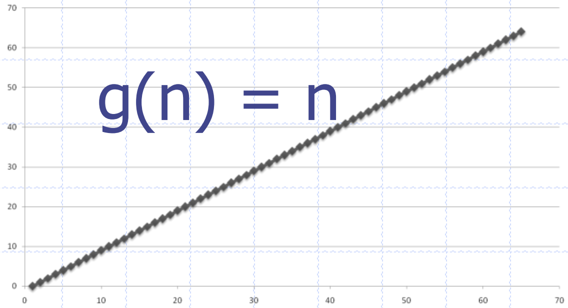

g(n) = 1

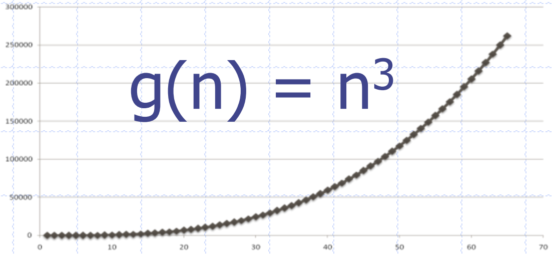

g(n) = n lg n


g(n) = 2^n


g(n) = lg n


g(n) = n^2


g(n) = n


g(n) = n^3

Primitive Operations

- ❑ Basic computations performed by an algorithm
- ❑ Identifiable in pseudocode
- ❑ Largely independent from the programming language
- ❑ Exact definition not important (we will see why later)
- ❑ Assumed to take a constant amount of time in the RAM model

- ❑ Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

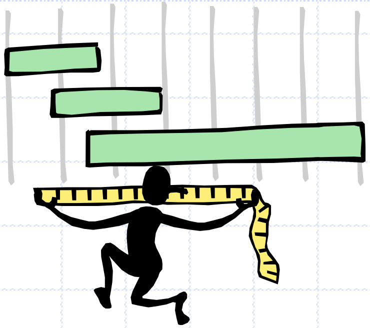
Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

```
Algorithm $arrayMax(A, n)$                          # operations
    $currentMax \leftarrow A[0]$                          $2$
    for $i \leftarrow 1$ to $n - 1$ do                         $2n$
        if $A[i] > currentMax$ then                    $3(n - 1)$
            $currentMax \leftarrow A[i]$                 $2(n - 1)$
        { increment counter $i$ }                      $2(n - 1)$
        { jump to top of loop }                        $n - 1$
    return $currentMax$                                   $1$

                                             Total    $10n - 5$
```

Estimating Running Time



- Algorithm ***arrayMax*** executes $10\boldsymbol{n} - 5$ primitive operations in the worst case. Define:

  $\boldsymbol{a}$ = Time taken by the fastest primitive operation

  $\boldsymbol{b}$ = Time taken by the slowest primitive operation

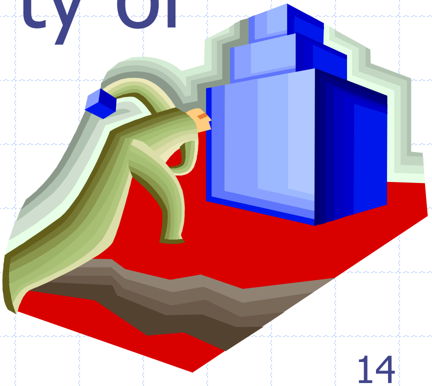
- Let $\boldsymbol{T}(\boldsymbol{n})$ be worst-case time of ***arrayMax***. Then

$$\boldsymbol{a}(10\boldsymbol{n} - 5) \le \boldsymbol{T}(\boldsymbol{n}) \le \boldsymbol{b}(10\boldsymbol{n} - 5)$$

- Hence, the running time $\boldsymbol{T}(\boldsymbol{n})$ is bounded by two linear functions

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***



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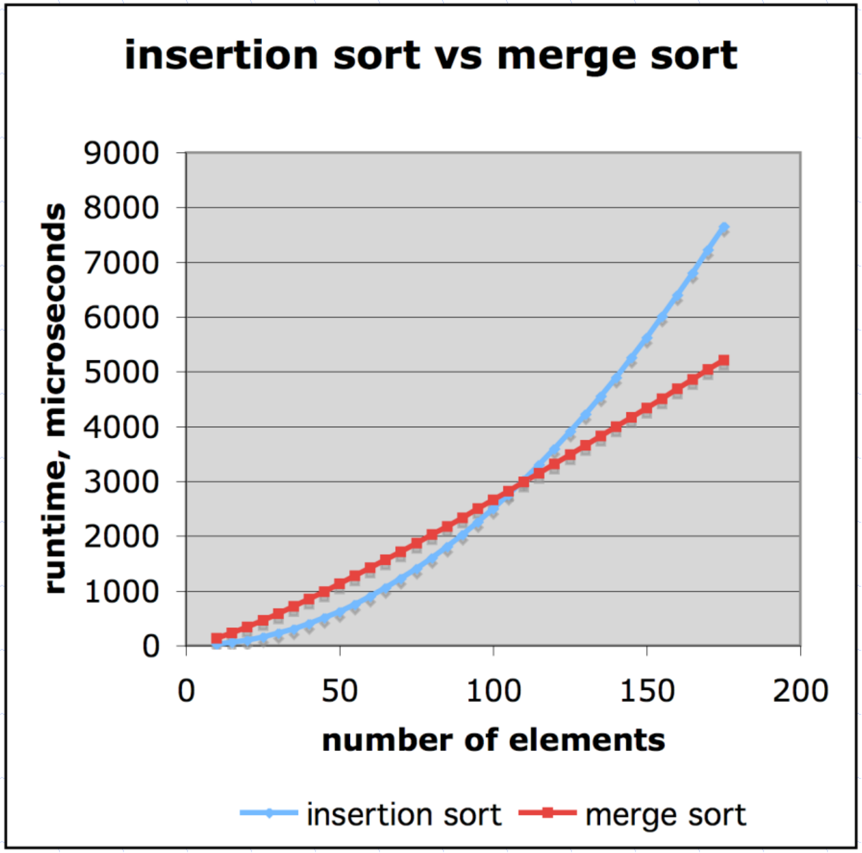
Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
$c \lg n$	$c \lg (n + 1)$	$c (\lg n + 1)$	$c(\lg n + 2)$
$c\,n$	$c\,(n + 1)$	$2c\,n$	$4c\,n$
$c\,n \lg n$	$\sim c\,n \lg n + c\,n$	$2c\,n \lg n + 2cn$	$4c\,n \lg n + 4cn$
$c\,n^2$	$\sim c\,n^2 + 2c\,n$	$\mathbf{4c\,n^2}$	$16c\,n^2$
$c\,n^3$	$\sim c\,n^3 + 3c\,n^2$	$8c\,n^3$	$64c\,n^3$
$c\,2^n$	$c\,2^{\,n+1}$	$c\,2^{\,2n}$	$c\,2^{\,4n}$

runtime quadruples when problem size doubles

Comparison of Two Algorithms



insertion sort is

$n^2 / 4$

merge sort is

2 n lg n

sort a million items?

insertion sort takes roughly 70 hours

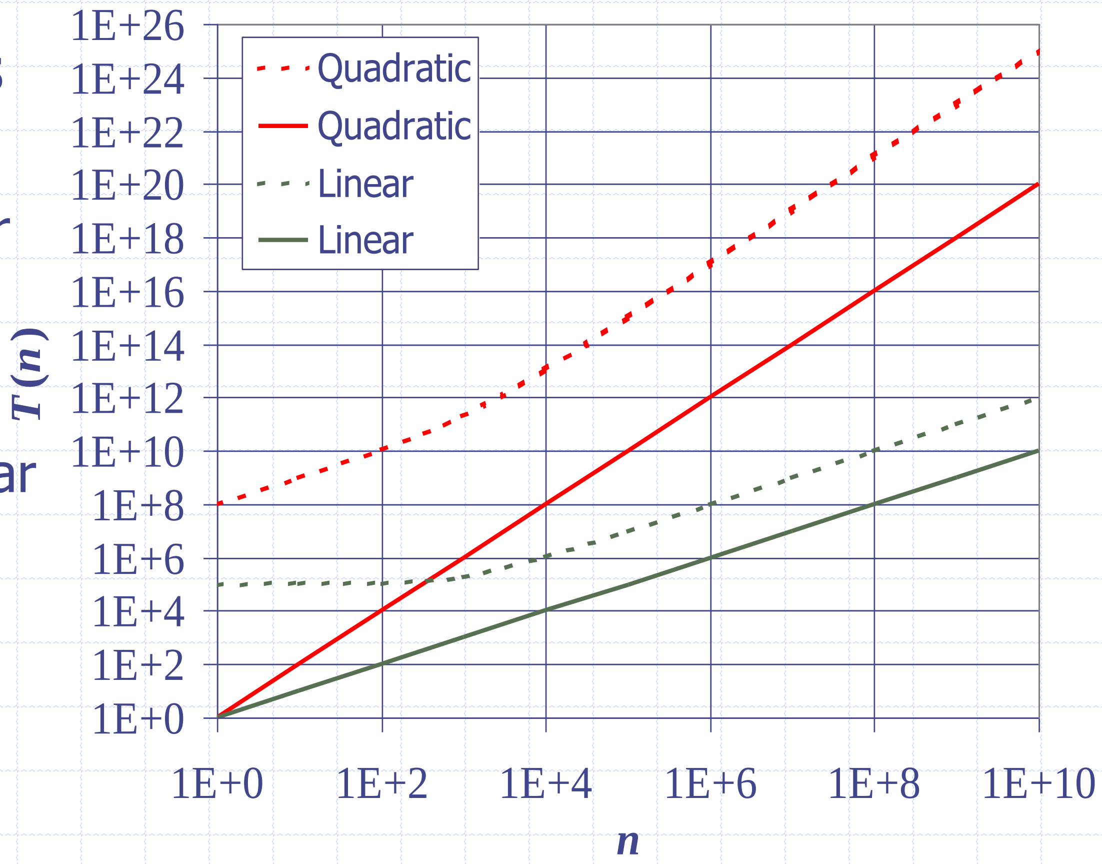
while

merge sort takes roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function

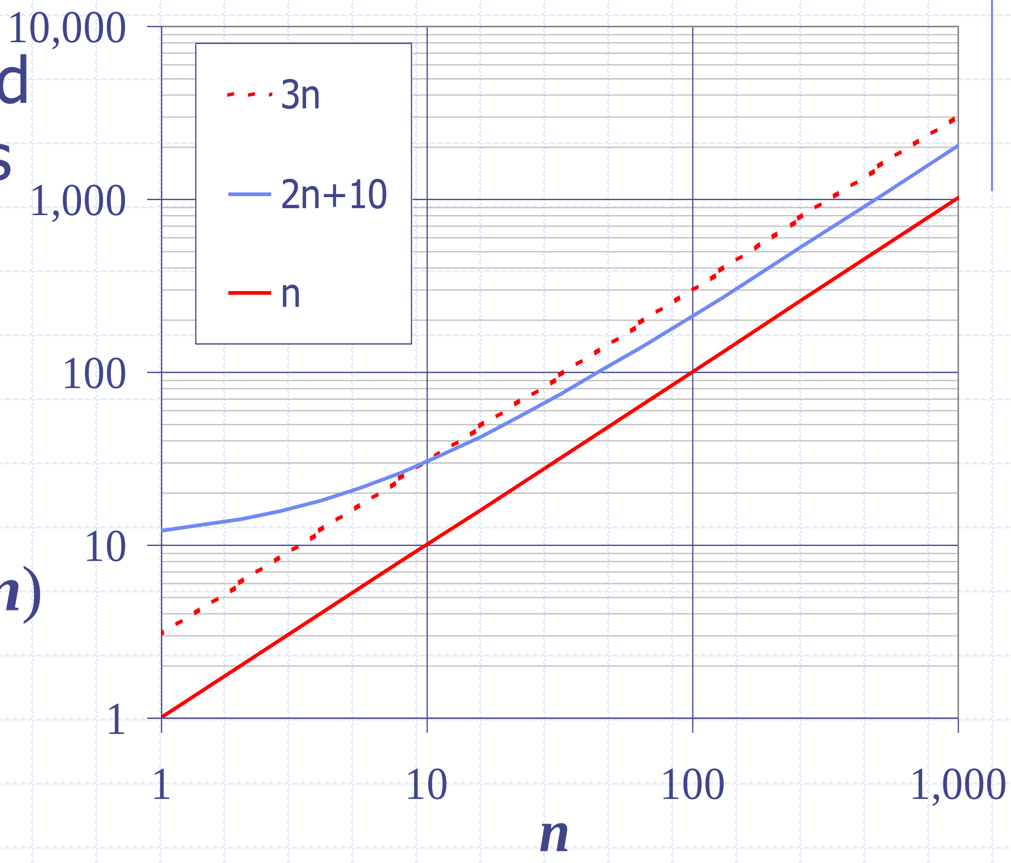


Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

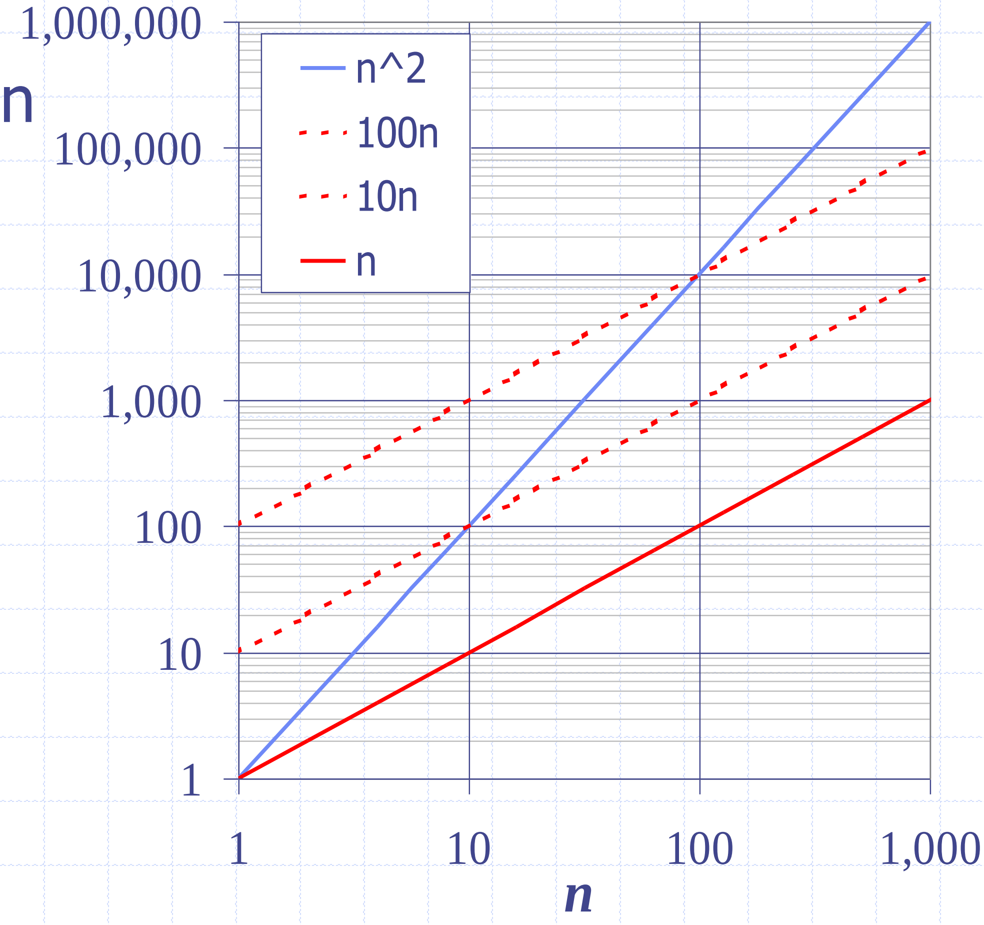
  $f(n) \leq cg(n)$ for $n \geq n_0$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$

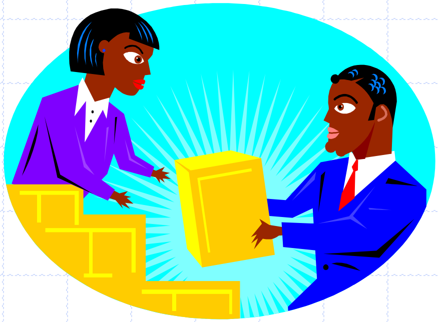


Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples



- 7n-2

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \cdot n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \cdot n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- 3 log n + 5

  3 log n + 5 is O(log n)

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \cdot \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

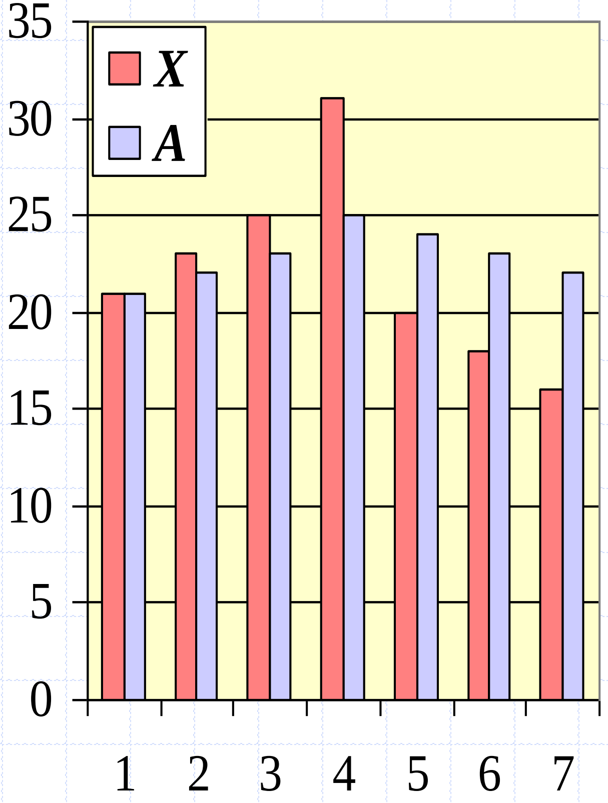
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $10n - 5$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i+1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



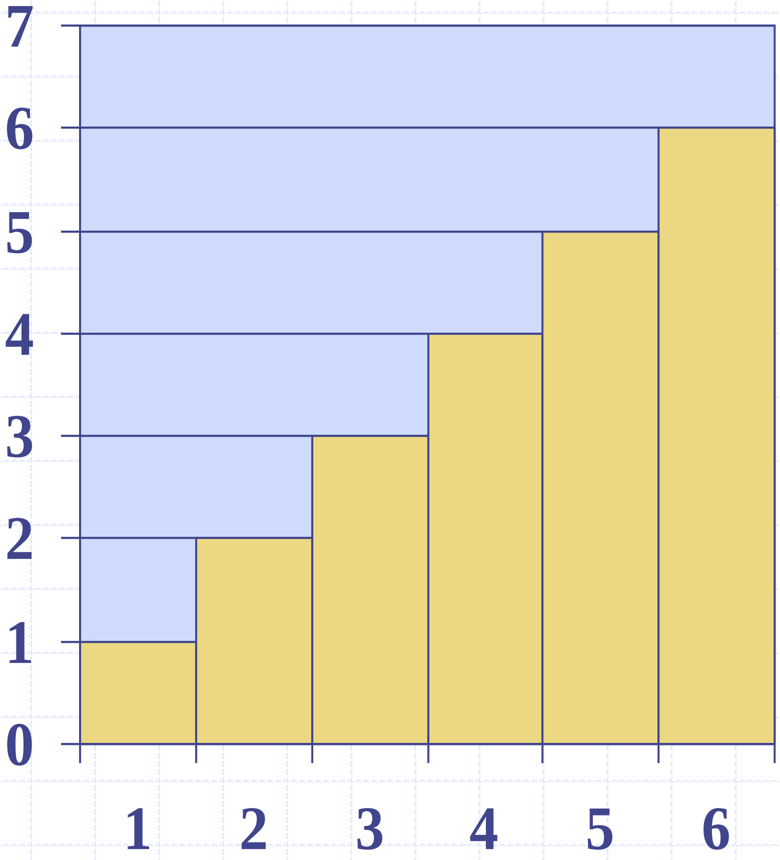
Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1($X$, $n$)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$        #operations
    $A \leftarrow$ new array of $n$ integers                   $O(n)$
    for $i \leftarrow 0$ to $n - 1$ do                          $O(n)$
        $s \leftarrow X[0]$                                     $O(n)$
        for $j \leftarrow 1$ to $i$ do                $O(1 + 2 + \ldots + (n - 1))$
            $s \leftarrow s + X[j]$                   $O(1 + 2 + \ldots + (n - 1))$
        $A[i] \leftarrow s / (i + 1)$                           $O(n)$
    return $A$                                                  $O(1)$
```

Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
    - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



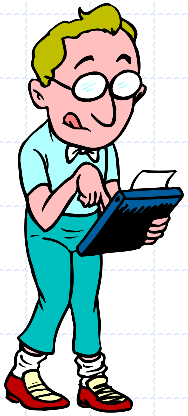
Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

```
Algorithm prefixAverages2(X, n)
    Input array X of n integers
    Output array A of prefix averages of X          #operations
    $A \leftarrow$ new array of $n$ integers                   $O(n)$
    $s \leftarrow 0$                                              $O(1)$
    for $i \leftarrow 0$ to $n - 1$ do                               $O(n)$
        $s \leftarrow s + X[i]$                                   $O(n)$
        $A[i] \leftarrow s / (i + 1)$                              $O(n)$
    return A                                         $O(1)$
```

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

Math you need to Review



- Summations
- Logarithms and Exponents

- **properties of logarithms:**

  $\log_b(xy) = \log_b x + \log_b y$

  $\log_b (x/y) = \log_b x - \log_b y$

  $\log_b x^a = a\log_b x$

  $\log_b a = \log_x a/\log_x b$

- **properties of exponentials**:

  $a^{(b+c)} = a^b a^{\ c}$

  $a^{bc} = (a^b)^c$

  $a^b / a^c = a^{(b-c)}$

  $b = a^{\ \log_a b}$

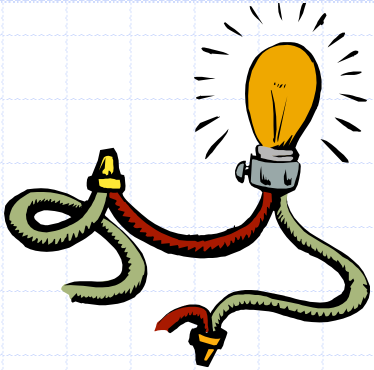
  $b^c = a^{\ c*\log_a b}$

- Proof techniques
- Basic probability

Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n)$ for $n \geq n_0$

Intuition for Asymptotic Notation



Big-Oh

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

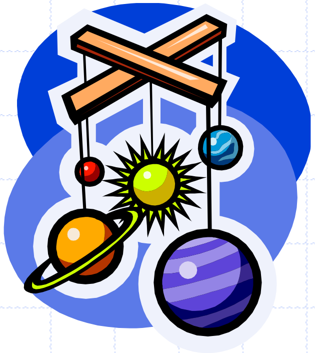
big-Omega

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

big-Theta

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

Example Uses of the Relatives of Big-Oh



- **$5n^2$ is $\Omega(n^2)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \geq c \cdot g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\Theta(n^2)$**

  $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$. We have already seen the former, for the latter recall that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \leq c \cdot g(n)$ for $n \geq n_0$

  Let $c = 5$ and $n_0 = 1$

Algorithm Analysis – Tom’s Rules of Thumb

- Start by defining a function that represents the time for the thing you are trying to analyze.
  - Often T(n)
  - Be sure to state what n is.
  - Time is worst-case if not specified.

```
quicksort(A, i, j) {
    …
}
```

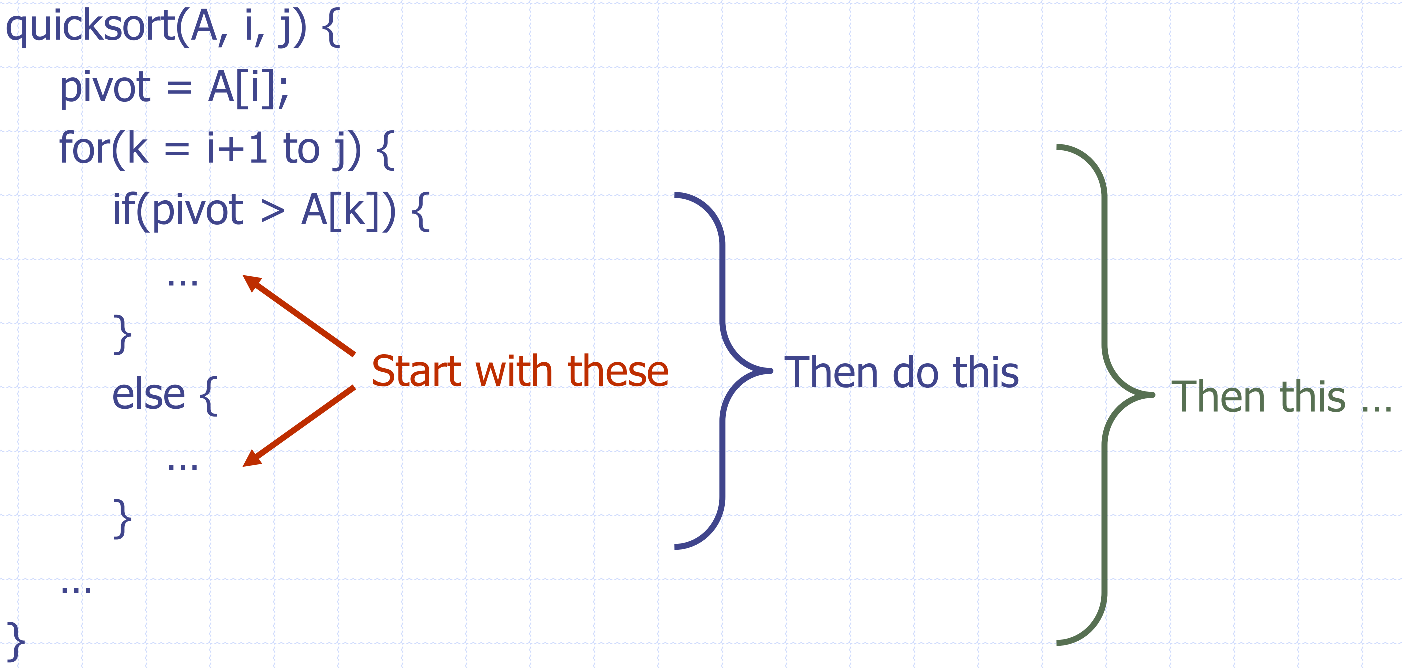
Let T(n) be the time to complete quicksort on n array elements, where n = j-i+1.

```
bubblesort(A) {
    …
}
```

Let S(n) be the time to complete bubblesort on an array of n elements.

Algorithm Analysis – Tom’s Rules of Thumb

- Work from inner blocks of (pseudo-)code to outer blocks.



```
quicksort(A, i, j) {
    pivot = A[i];
    for(k = i+1 to j) {
        if(pivot > A[k]) {
            …
        }
        else {
            …
        }
    …
}
```

Start with these

Then do this

Then this …

Assignments and Function Calls

- An assignment with no function calls is O(1).
- A function call to a known algorithm (not the one you are analyzing) takes the known time for that algorithm.

```
foo(A, n) {
    …
    k = (j+91)/3;              O(1)
    m = max(A, n);             O(n)     (finding maximum of an array takes linear time)
    p = (j – 17) + max(B, n)   O(1) + O(n) = O(n)
    …
}
```

Recursive Function Calls

- A function call to the algorithm you are analyzing takes (the function you defined)(some function of n) time.
  - For example, T(n/2)

```
foo(A, n) {                 Define S(n) to be time taken by foo with second
                            parameter n
    …
    foo(A, n-2);            S(n-2)
}

bar(A, i, j) {              Define T(n) to be time taken by bar with n = j-i+1
    m = (i + j) / 2;
    bar(A, i, m);           T(n/2)
    …
}
```

Conditionals

- Add up the time in each branch of the conditional.
- The conditional takes the time taken by the condition, plus the maximum of the branch times.

```
foo(A, n) {
    …
    if( p < A[i] ) {        $O(1)$
        …                   $O(n)$
    }
    else {
        …                   $O(n^2)$
    }
}
```

The whole if statement takes time
$O(1) + \max(O(n), O(n^2))$
$= O(n^2)$

Loops

- Add up the time in the body of the loop.
- Determine how many times t the loop will be executed, as a function of your n. Use worst-case estimate.
- The time for the loop is t * (time for body)

```
for(i = 1 to n ) {   n iterations
   …                              O(n)
}
```

The whole for loop takes time
$n * O(n) = O(n^2)$

```
i = 0;
while(p < A[i]) {  n iterations
   …
   i++;
}
```

Triangular Loops

- Triangular loops have an inner index that counts up to the outer index.
- Assume the inner loops have the same number of iterations as the outer loop.

```
for(i = 1 to n ) {       n iterations
    …
    for(j = 1 to i) {      n iterations
        …
    }
}
```

□ This also works for more than two nested loops

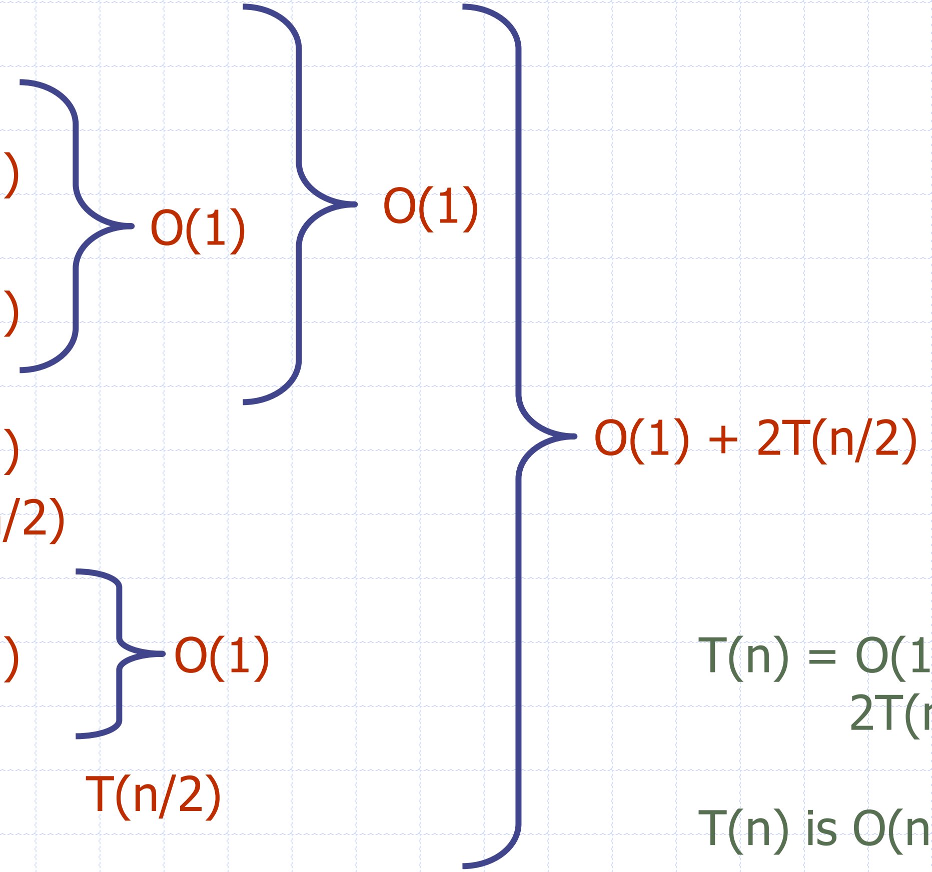
At the end

- Set (the function of n you defined) = the summed-up cost of the entire algorithm.
- Reminder: Along the way, don’t absorb T(…) factors into the big-Oh notation.
- If you end up with a recurrence, solve it.

At the end

Let T(n) be the time for find where n = j-i+1.

```
int find(A, i, j, target) {
    if( i == j) {
        if(A[i] == target)
            return i               O(1)
        else
            return -1              O(1)
    }
    m = (i + j) / 2                O(1)
    f = find(A, i, m, target)      T(n/2)
    if(f > 0) {
        return f                   O(1)
    }
    return find(A, m+1, j, target)      T(n/2)
}
```



T(n) = O(1) + 2T(n/2)

T(n) is O(n)