Recursion

Section 3.5

The Recursion Pattern

- Recursion: when a method calls itself
- Classic example: the factorial function:

  $$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a C++method:

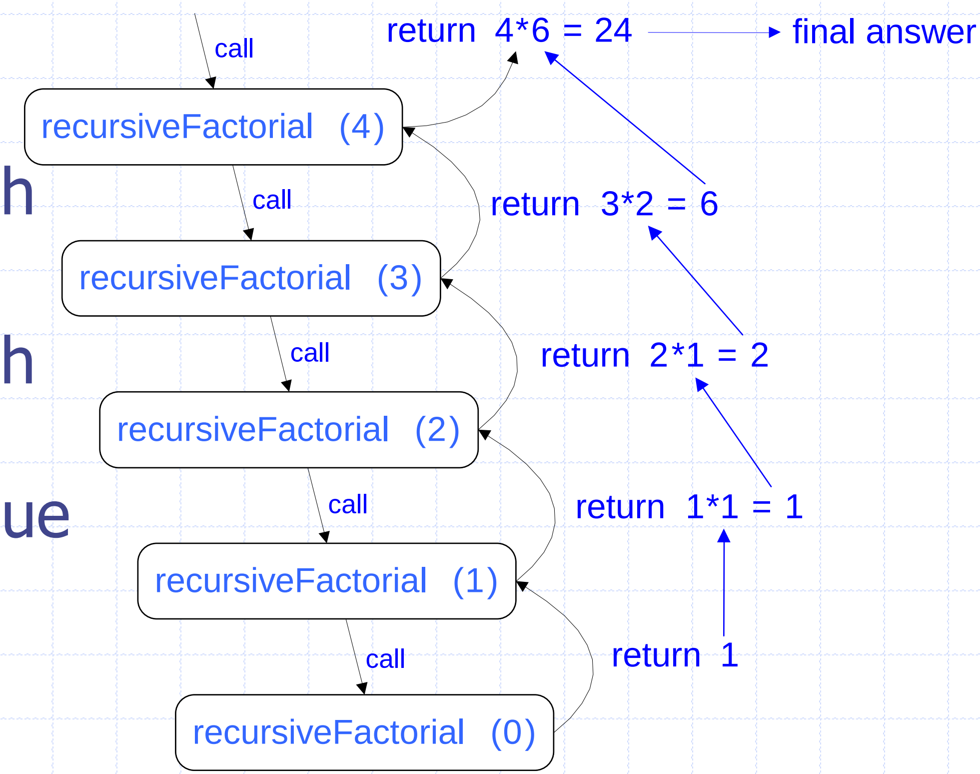
```
// recursive factorial function
int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  1;      // basis case
    else return  n  *  recursiveFactorial(n - 1); // recursive case
}
```

Content of a Recursive Method

- Base case(s)
  - Values of the input variables for which we perform no recursive calls are called base cases (there should be at least one base case).
  - Every possible chain of recursive calls must eventually reach a base case.
- Recursive calls
  - Calls to the current method.
  - Each recursive call should be defined so that it makes progress towards a base case.

Visualizing Recursion

- Recursion trace
  - A box for each recursive call
  - An arrow from each caller to callee
  - An arrow from each callee to caller showing return value
- Example



Example: English Ruler

- Print the ticks and numbers like an English ruler:

```
---- 0          ----- 0          --- 0
-               -                -
--              --               --
-               -                -
---             ---              --- 1
-               -                -
--              --               --
-               -                -
---- 1          ----             --- 2
-               -                -
--              --               --
-               -                -
---             ---              --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

Using Recursion

drawTicks(length)
  Input: length of a 'tick'
  Output: ruler with tick of the given length in the middle and smaller rulers on either side

```
---- 0          ----- 0          --- 0
-               -                -
--              --               --
-               -                -
---             ---              --- 1
-               -                -
--              --               --
-               -                -
---- 1          ----             --- 2
-               -                -
--              --               --
-               -                -
---             ---              --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

drawTicks(length)

if(length > 0) then

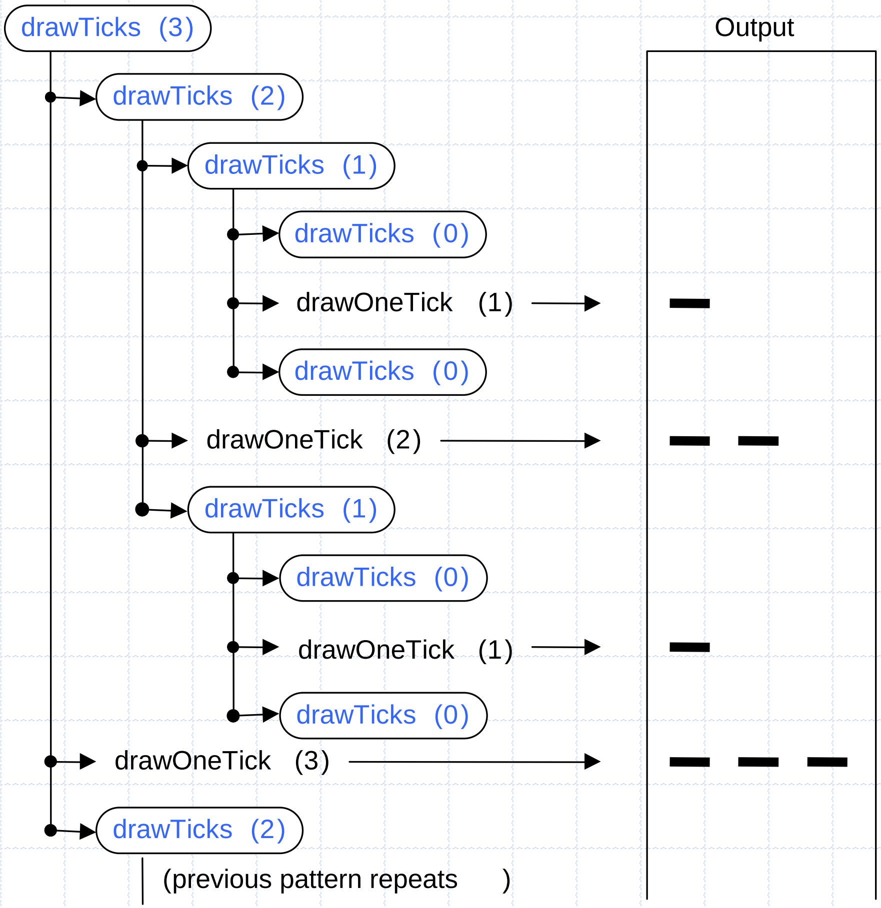
drawTicks(length – 1)

draw tick of the given length

drawTicks(length – 1)

Recursive Drawing Method

- The drawing method is based on the following recursive definition
- An interval with a central tick length L ≥ 1 consists of:
  - An interval with a central tick length L–1
  - An single tick of length L
  - An interval with a central tick length L–1



C++ Implementation (1)

```
// draw ruler
void drawRuler(int nInches, int majorLength) {
   drawOneTick(majorLength, 0);              // draw tick 0 and its label
   for (int i = 1; i <= nInches; i++){
      drawTicks(majorLength - 1);            // draw ticks for this inch
      drawOneTick(majorLength, i);           // draw tick i and its label
   }
}

// draw ticks of given length
void drawTicks(int tickLength) {
   if (tickLength > 0) {                     // stop when length drops to 0
      drawTicks(tickLength - 1);             // recursively draw left ticks
      drawOneTick(tickLength);               // draw center tick
      drawTicks(tickLength - 1);             // recursively draw right ticks
   }
}
```

C++ Implementation (2)

```
// draw a tick with no label
void drawOneTick(int tickLength) {
    drawOneTick(tickLength, - 1);
}

// draw one tick
void drawOneTick(int tickLength, int tickLabel) {
   for (int i = 0; i < tickLength; i++) {
       cout << "-";
   }

   if (tickLabel >= 0) {
       cout << " " << tickLabel;
   }

   cout << "\n";
}
```

Recursion Examples

- Example 3.2 in the text: Programming languages are often defined in a recursive way. We can define an argument list in C++ as follows:

argument-list: ε
nonempty-argument-list

nonempty-argument-list: *argument*
nonempty-argument-list , *argument*

- That is, an argument list consists of either (i) the empty string, (ii) an argument, or (iii) a nonempty argument list followed by a comma and an argument.

foo(); bar(14); bletch(23.1, ‘a’, 14);

Example of Linear Recursion

Algorithm LinearSum(*A, n*):

Input:

A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements

Output:

The sum of the first *n* integers in *A*

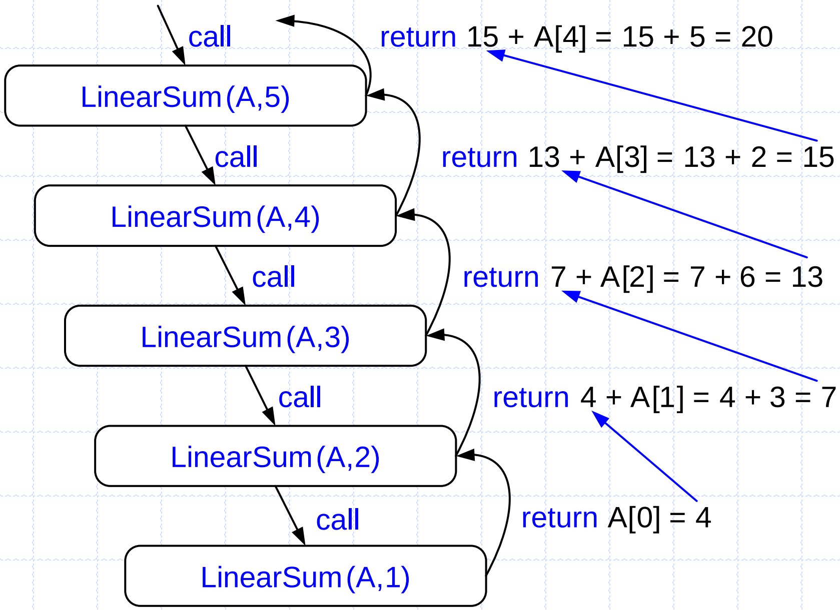
if *n* = 1 **then**

return *A*[0]

else

return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional parameters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(A, i, j), not ReverseArray(A).

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } n > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } n > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$

$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$

$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$

$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$$

Recursive Squaring Method

Algorithm Power(x, n):

Input: A number x and integer $n = 0$

Output: The value x^n

if $n = 0$ **then**

  return 1

if n is odd **then**

  $y = \text{Power}(x, (n - 1)/2)$

  return $x \cdot y \cdot y$

else

  $y = \text{Power}(x, n/2)$

  return $y \cdot y$

Analysis

Algorithm Power(x, n):
 Input: A number x and integer $n = 0$
 Output: The value x^n
 if $n = 0$ **then**
  return 1
 if n is odd **then**
  y = Power(x, $(n - 1)/2$)
  return $x \cdot y \cdot y$
 else
  y = Power(x, $n/2$)
  return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):

    Input: An array A and nonnegative integer indices i and j

    Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

    Swap $A[i]$ and $A[j]$

    $i = i + 1$

    $j = j - 1$

return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.

```
---- 0          ----- 0          --- 0
-               -                -
--              --               --
-               -                -
---             ---              --- 1
-               -                -
--              --               --
-               -                -
---- 1          ----             --- 2
-               -                -
--              --               --
-               -                -
---             ---              --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

A Binary Recursive Method for Drawing Ticks

```
    // draw a tick with no label
public static void drawOneTick(int tickLength) { drawOneTick(tickLength, - 1); }
    // draw one tick
public static void drawOneTick(int tickLength, int tickLabel) {
  for (int i = 0; i < tickLength; i++)
      System.out.print("-");
  if (tickLabel >= 0) System.out.print(" " + tickLabel);
  System.out.print("\n");
}
public static void drawTicks(int tickLength) { // draw ticks of given length
  if (tickLength > 0) {              // stop when length drops to 0
      drawTicks(tickLength- 1);      // recursively draw left ticks
      drawOneTick(tickLength);       // draw center tick
      drawTicks(tickLength- 1);      // recursively draw right ticks
  }
}
public static void drawRuler(int nInches, int majorLength) { // draw ruler
  drawOneTick(majorLength, 0);       // draw tick 0 and its label
  for (int i = 1; i <= nInches; i++) {
      drawTicks(majorLength- 1);     // draw ticks for this inch
      drawOneTick(majorLength, i);   // draw tick i and its label
  }
}
```

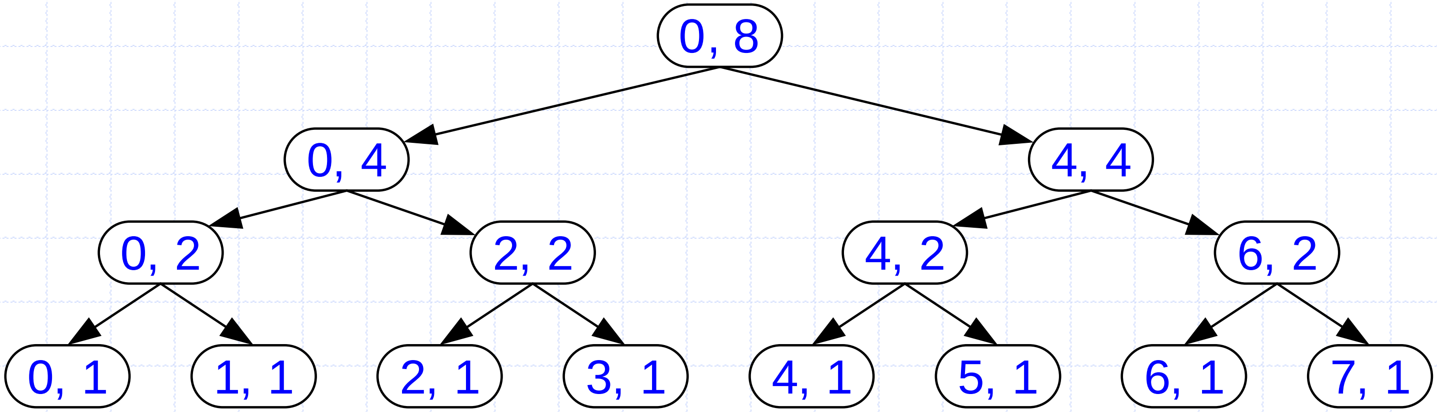
Note the two recursive calls

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

  Algorithm BinarySum(A, i, n):
  Input: An array A and integers i and n
  Output: The sum of the n integers in A starting at index i
  if $n = 1$ **then**
  return $A[i]$
  return BinarySum(A, i, $\lfloor n/2 \rfloor$) + BinarySum(A, $i + \lfloor n/2 \rfloor$, $\lceil n/2 \rceil$)

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):
  Input: Nonnegative integer k
  Output: The kth Fibonacci number F_k
  if $k = 1$ **then**
    return k
  else
    return BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

```
Algorithm LinearFibonacci(k):
  Input: A nonnegative integer k
  Output: Pair of Fibonacci numbers (F_k , F_{k−1})
 if k ≤ 1 then
     return (k, 0)
 else
     (i, j) = LinearFibonacci(k − 1)
     return (i +j, i)
```

- LinearFibonacci makes $k-1$ recursive calls

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot + pan = bib* — *321 + 375 = 696*
    - *dog + cat = pig* — *167 + 380 = 547*
    - *boy + girl = baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

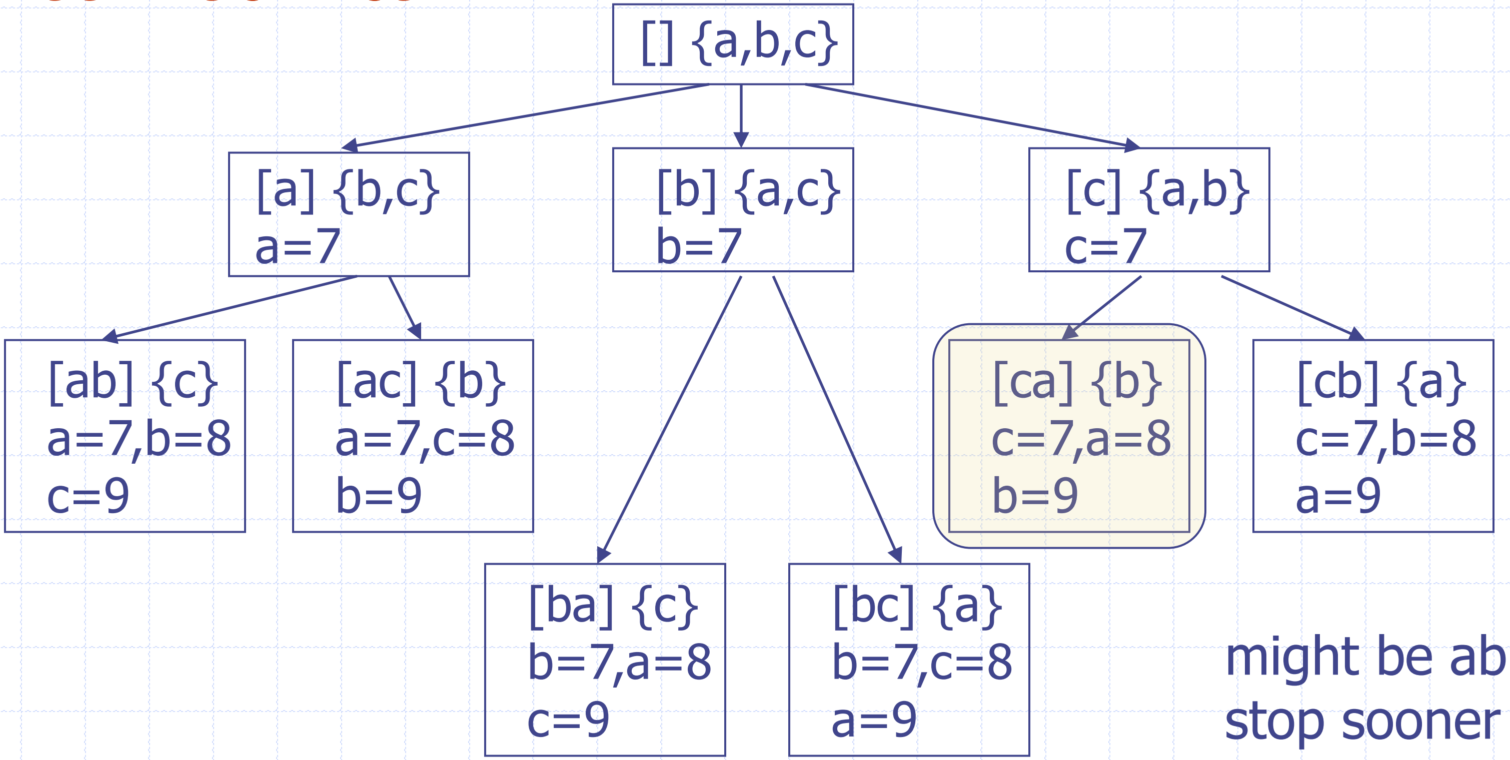
```
Algorithm PuzzleSolve(S,U):
 Input: Sequence S, and set U (universe of elements to test)
 Output:  Solution to problem encoded as a sequence

 for all e in U do
     Remove e from U          {e is now being used}
     Add e to the end of S
     if U is empty then
         if S solves the puzzle then
             return S         {solution found}
     else
         S' = PuzzleSolve(S,U)
         if S' ≠ Ø then
             return S'
     Add e back to U          {e is now unused}
     Remove e from the end of S
     return Ø                 {solution not found}
```

Example

cbb + ba = abc

799 + 98 = 897

a,b,c stand for 7,8,9; not necessarily in that order



might be able to stop sooner

Recursion as a Black Box

- In solving a problem of some size n, it is often helpful to think of recursion as a “black box” that solves smaller instances of the problem.
- In this method, one imagines a smaller problem(s) of the same type that would help solve the problem of size n.
- By the magic of recursion, you can assume that the smaller problem(s) are solved, and use their solution.
- You must also verify that you can solve the problem some other way for small n (like n = 0 or 1 or 2).

Black Box Recursion: Insertion Sort

- Our problem is to sort a sequence of n integers.
- If we had the first n-1 of them sorted, then we could simply insert the last one at the appropriate place in this order.
- By recursion, we can magically sort the first n-1 numbers.
- We verify that we can sort a sequence of length 1 by simply leaving it alone.

```
insertionSort(A, n)
  if(n==1)
    return
  insertionSort(A, n-1)
  insert(A[n], A, n)
```

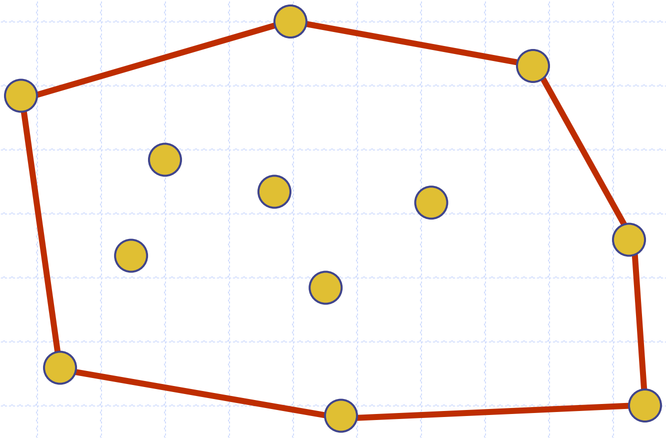
Black Box Recursion: Merge Sort

- Our problem is to sort a sequence of n integers.
- If we divide the input into two subsequences, and had both of these subsequences sorted, we could simply merge the two subsequences.
- By recursion, we can magically sort the two subsequences.
- We can sort a sequence of length 1 by simply leaving it alone.

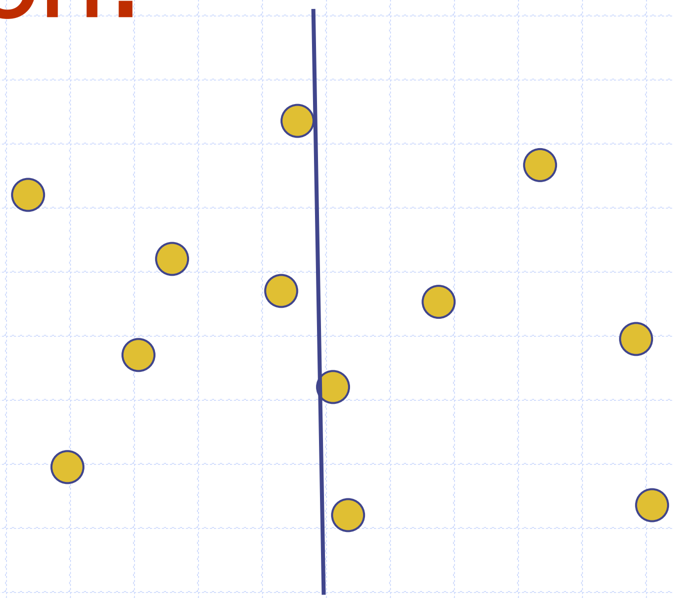
```
mergeSort(T)
  if(n==1)
     return T
  T1, T2 = divide(T)
  S1 = mergeSort(T1)
  S2 = mergeSort(T2)
  return merge(S1, S2)
```

Black Box Recursion: Convex Hulls

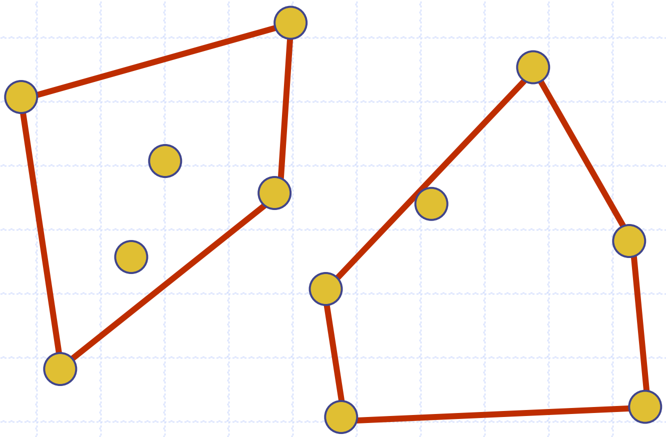
- Our problem is to find the convex hull of n points. This is the smallest convex polygon that contains the points: the "boundary" of the points. It is an ordered list of points.



Black Box Recursion: Convex Hulls



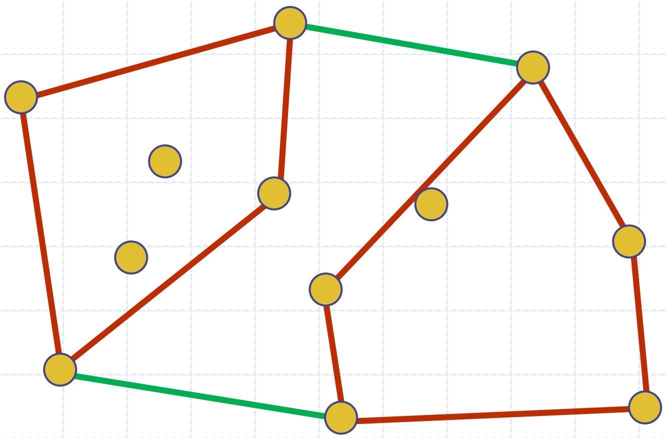
- Idea: divide the points into a left half and a right half.



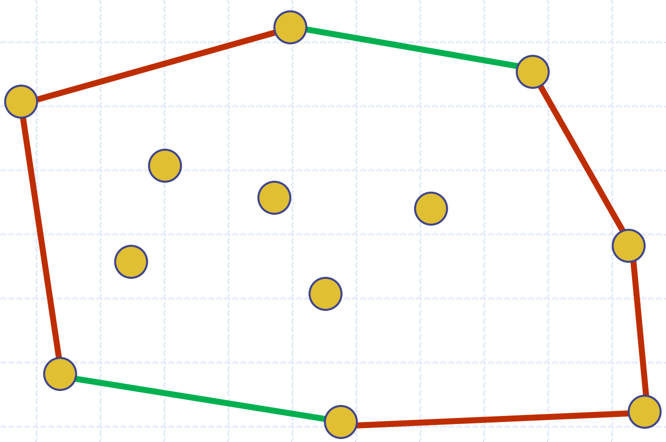
- Then compute the convex hulls of both halves.

Black Box Recursion: Convex Hulls

- Merge the two hulls by finding a top bridge edge and a bottom bridge edge.



- Then remove the parts in the middle.



Black Box Recursion: Convex Hulls

```
convexHull(P)
    if(|P|≤ 3)
        return ccw(P)
    P1, P2 = divide(P)
    C1 = convexHull(P1)
    C2 = convexHull(P2)
    return merge(C1, C2)
```

```
mergeSort(T)
    if(n==1)
        return T
    T1, T2 = divide(T)
    S1 = mergeSort(T1)
    S2 = mergeSort(T2)
    return merge(S1, S2)
```

- The base case, divide, and merge steps are more complicated for convexHull, but it's the same basic recursion technique as mergeSort.
- This is a pattern called divide-and-conquer.