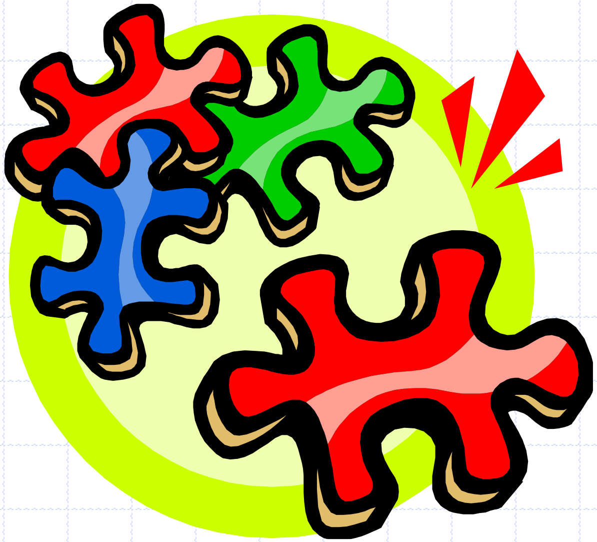
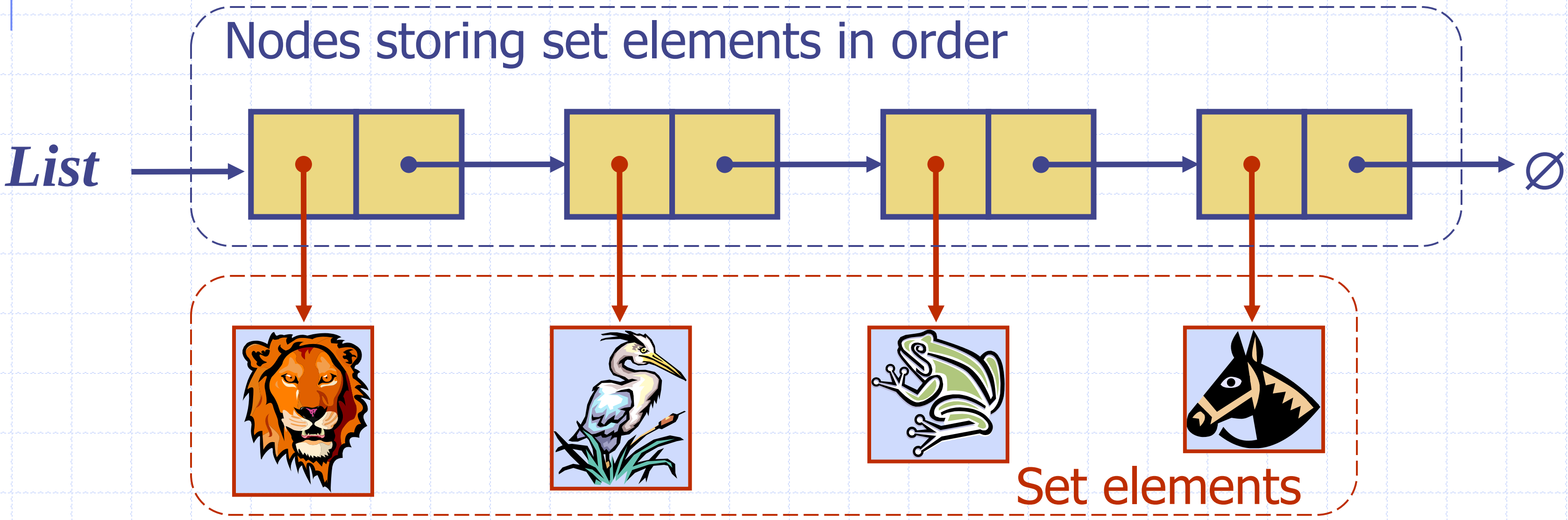
Sets and Union-Find

Section 11.4



Storing a Set in a List

- We can implement a set with a list
- Elements are stored sorted according to some canonical ordering
- The space used is $O(n)$



Generic Merging

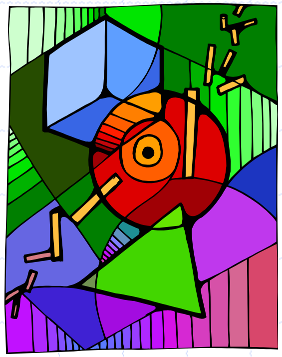
- Generalized merge of two sorted lists A and B
- Template method genericMerge
- Auxiliary methods
  - aIsLess
  - bIsLess
  - bothAreEqual
- Runs in $O(n_A + n_B)$ time provided the auxiliary methods run in $O(1)$ time

```
Algorithm genericMerge(A, B)
    S ← empty sequence
    while ¬A.empty() ∧ ¬B.empty()
        a ← A.front(); b ← B.front()
        if a < b
            aIsLess(a, S); A.eraseFront()
        else if b < a
            bIsLess(b, S); B.eraseFront()
        else { b = a }
            bothAreEqual(a, b, S)
            A.eraseFront(); B.eraseFront()
    while ¬A.empty()
        aIsLess(a, S); A.eraseFront()
    while ¬B.empty()
        bIsLess(b, S); B.eraseFront()
    return S
```

Using Generic Merge for Set Operations

- Any of the set operations can be implemented using a generic merge
- For example:
  - For intersection: only copy elements that occur in both lists
  - For union: copy every element from both lists except for the duplicates
- All methods run in linear time

Set Operations



- We represent a set by the sorted sequence of its elements
- By specializing the auxliliary methods the generic merge algorithm can be used to perform basic set operations:
  - union
  - intersection
  - subtraction
- The running time of an operation on sets A and B should be at most $O(n_A + n_B)$

- Set union:
  - $aIsLess(a, S)$
    $S.insertBack(a)$
  - $bIsLess(b, S)$
    $S.insertBack(b)$
  - $bothAreEqual(a, b, S)$
    $S.insertBack(a)$
- Set intersection:
  - $aIsLess(a, S)$
    $\{ \text{ do nothing } \}$
  - $bIsLess(b, S)$
    $\{ \text{ do nothing } \}$
  - $bothAreEqual(a, b, S)$
    $S.\ insertBack(a)$

Template Method

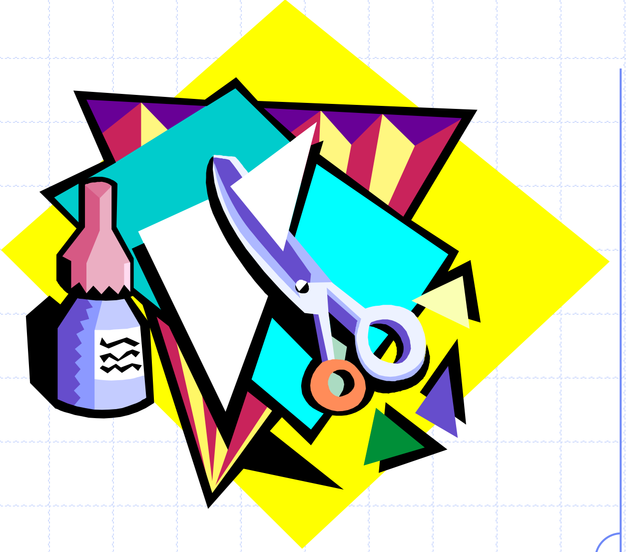
- A superclass implements an algorithm in a method using its subclasses's methods. Subclasses specialize the steps of the algorithm.

```
// E is set element type.
class Merger {
public:
    Set genericMerge(Set A, Set B);
    virtual void aIsLess(E a, Set S) = 0;
    virtual void bIsLess(E b, Set S) = 0;
    virtual void bothAreEqual(E a, E b, Set S) = 0;
}
```

```
class SetUnion: public Merger {
    virtual void aIsLess(E a, Set S) {
        S.insertBack(a);
    }
    virtual void bIsLess(E b, Set S) {
        S.insertBack(b);
    }
    virtual void bothAreEqual(E a, E b, Set S) {
        S.insertBack(a);
    }
}

// class SetIntersection would be similar.
```

Union-Find Partition Structures

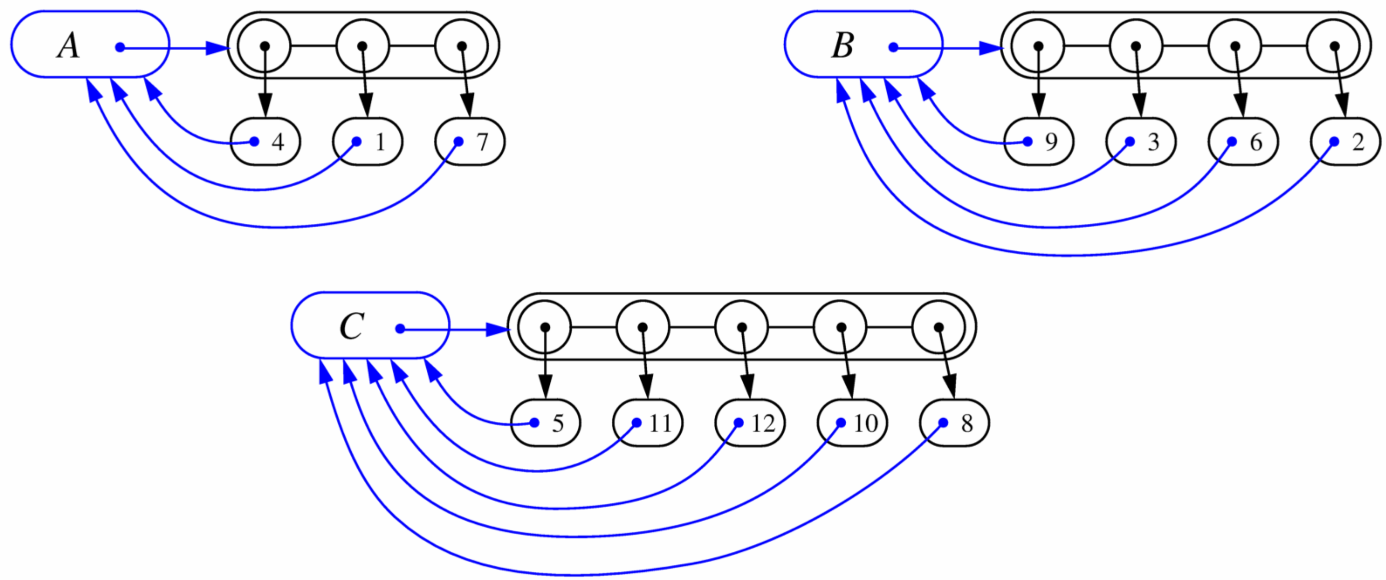


Partitions with Union-Find Operations

- makeSet(x): Create a singleton set containing the element x and return the position storing x in this set
- union(A, B): Return the set A U B, destroying the old A and B
- find(p): Return the set containing the element at position p

List-based Implementation

- Each set is stored in a sequence represented with a linked-list
- Each node should store an object containing the element and a reference to the set name

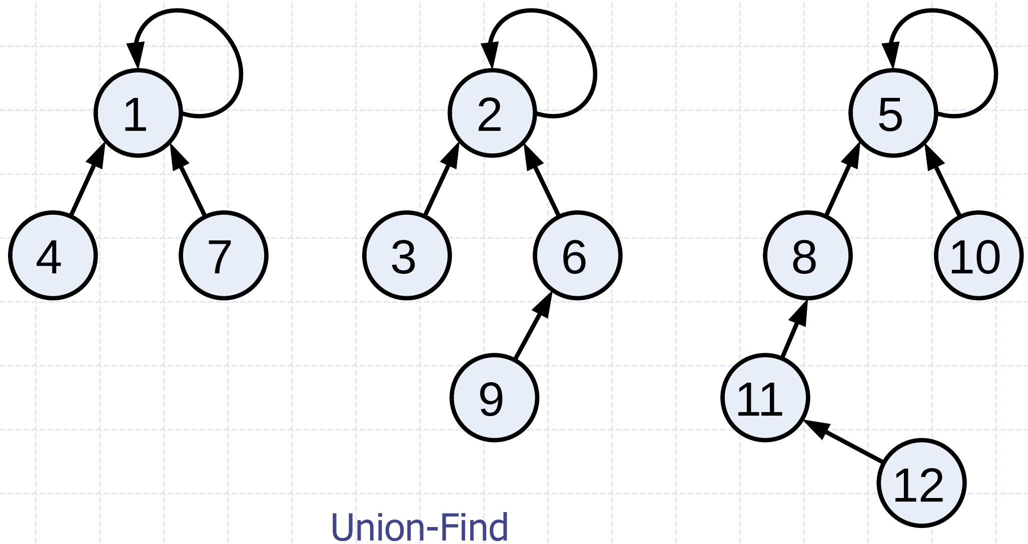


Analysis of List-based Representation

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and finds is O(n log n).

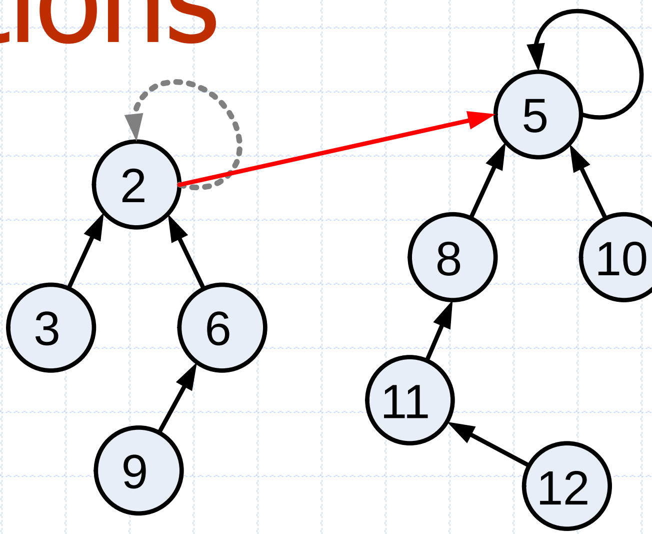
Tree-based Implementation

- Each element is stored in a node, which contains a "parent" pointer
- A node v whose parent pointer points back to v is called a set name
- Each set is a tree, rooted at a node with a self-referencing parent pointer
- For example: The sets "1", "2", and "5":

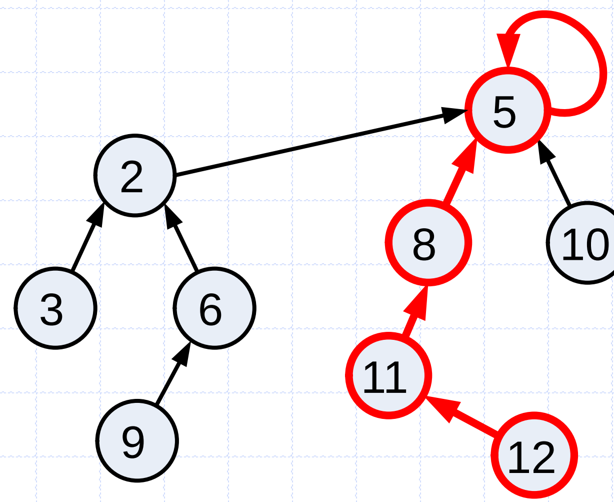


Union-Find Operations

- To do a union, simply make the root of one tree point to the root of the other

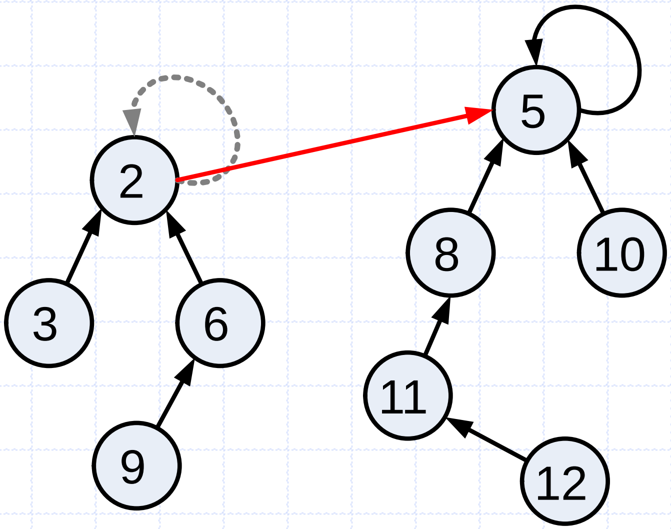


- To do a find, follow parent pointers from the starting node until reaching a node whose parent pointer refers back to itself



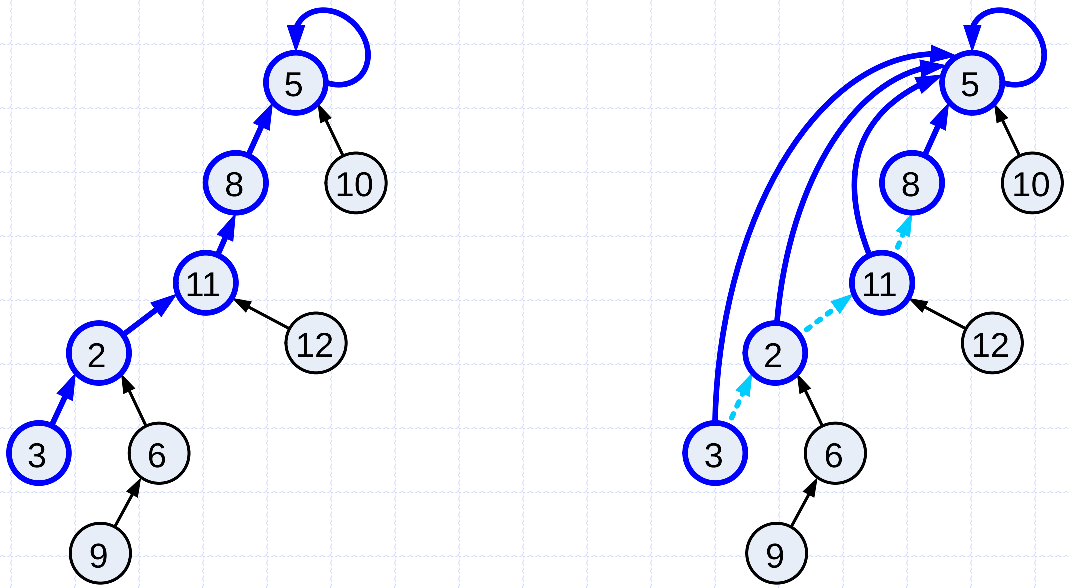
Union-Find Heuristic 1

- Union by size:
  - When performing a union, make the root of smaller tree point to the root of the larger
- Implies O(n log n) time for performing n union-find operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most O(log n) pointers for any find.



Union-Find Heuristic 2

- Path compression:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root



- Implies $O(n \log^* n)$ time for performing n union-find operations:
  - Proof is somewhat involved… (and not in the book)

Proof of log* n Amortized Time

- For each node v that is a root
  - define n(v) to be the size of the subtree rooted at v (including v)
  - identify a set with the root of its associated tree.
- We update the size field n(v) only when a set is unioned into v. Thus, if v is not a root, then n(v) is the largest the subtree rooted at v can be, which occurs just before we union v into some other node whose size is at least as large as v 's.
- For any node v, then, define the rank of v, which we denote as r(v), as $r(v) = \lfloor \log n(v) \rfloor$.
- Thus, $n(v) \geq 2^{r(v)}$.
- Also, since there are at most n nodes in the tree of v, $r(v) \leq \lfloor \log n \rfloor$, for each node v.

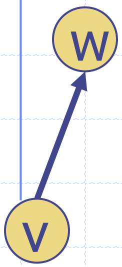
Proof of log* n Amortized Time (2)

- For each node v with parent w:
  - $r(v) < r(w)$
- Claim: There are at most $n/ 2^s$ nodes of rank s.
- Proof:
  - Since $r(v) < r(w)$, for any node v with parent w, ranks are monotonically increasing as we follow parent pointers up any tree.
  - Thus, if $r(v) = r(w)$ for two nodes v and w, then the nodes counted in $n(v)$ must be separate and distinct from the nodes counted in $n(w)$.
  - If a node v is of rank s, then $n(v) \geq 2^s$.
  - Therefore, since there are at most n nodes total, there can be at most $n/2^s$ that are of rank s.

Proof of log* n Amortized Time (3)

- Definition: Tower of two's function:
  - $t(i) = 2^{t(i-1)}$, $t(0) = 1$
- Definition: log*(n)
  - $\log^*(n) = t^{-1}(n)$, the number of successive logs needed to take n to 1.
  - $\log(\log(\log(16))) = 1$, so $\log^*(16) = 3$.
  - $\log(\log(\log(\log(65536)))) = 1$, so $\log^*(65536) = 4$.
- Nodes v and u are in the same rank group g if
  - $g = \log^*(r(v)) = \log^*(r(u))$:
- Since the largest rank is log n, the largest rank group is
  - $\log^*(\log n) = (\log^* n) - 1$

Proof of log* n Amortized Time (4)



- Charge 1 cyber-dollar per pointer hop during a find:
  - If w is the root or if w is in a different rank group than v, then charge the find operation one cyber-dollar.
  - Otherwise (w is not a root and v and w are in the same rank group), charge the node v one cyber-dollar.
- Since there are most (log* n)-1 rank groups, this rule guarantees that any find operation is charged at most log* n cyber-dollars.

Proof of log* n Amortized Time (5)

- After we charge a node v then v will get a new parent, which is a node higher up in v 's tree.
- The rank of v 's new parent will be greater than the rank of v 's old parent w.
- Thus, any node v can be charged at most the number of different ranks that are in v 's rank group.
- If v is in rank group g > 0, then v can be charged at most t(g)-t(g-1) times before v has a parent in a higher rank group (and from that point on, v will never be charged again). In other words, the total number, C, of cyber-dollars that can ever be charged to nodes can be bounded by

$$C \leq \sum_{g=1}^{\log^* n - 1} n(g) \cdot (t(g) - t(g-1))$$

Proof of log* n Amortized Time (end)

- Bounding $n(g)$:

$$n(g) \le \sum_{s=t(g-1)+1}^{t(g)} \frac{n}{2^s}$$

$$= \frac{n}{2^{t(g-1)+1}} \sum_{s=0}^{t(g)-t(g-1)-1} \frac{1}{2^s}$$

$$< \frac{n}{2^{t(g-1)+1}} \cdot 2$$

$$= \frac{n}{2^{t(g-1)}}$$

$$= \frac{n}{t(g)}$$

- Returning to C:

$$C < \sum_{g=1}^{\log^* n-1} \frac{n}{t(g)} \cdot (t(g) - t(g-1))$$

$$\le \sum_{g=1}^{\log^* n-1} \frac{n}{t(g)} \cdot t(g)$$

$$= \sum_{g=1}^{\log^* n-1} n$$

$$\le n \log^* n$$