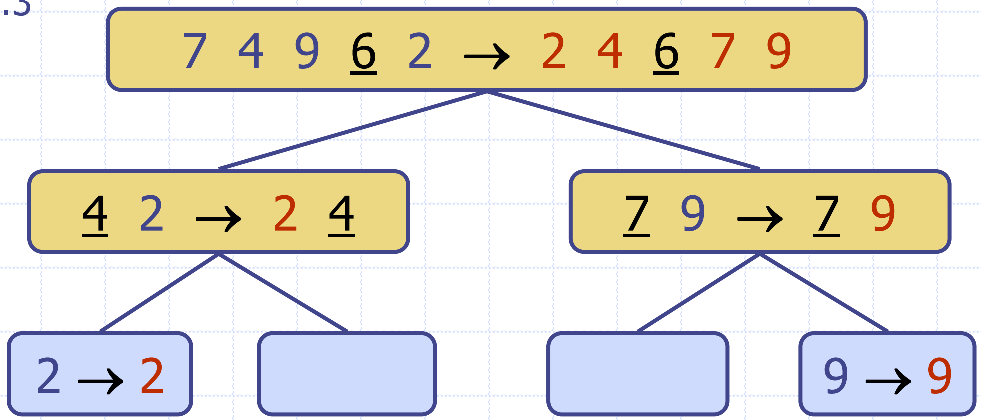
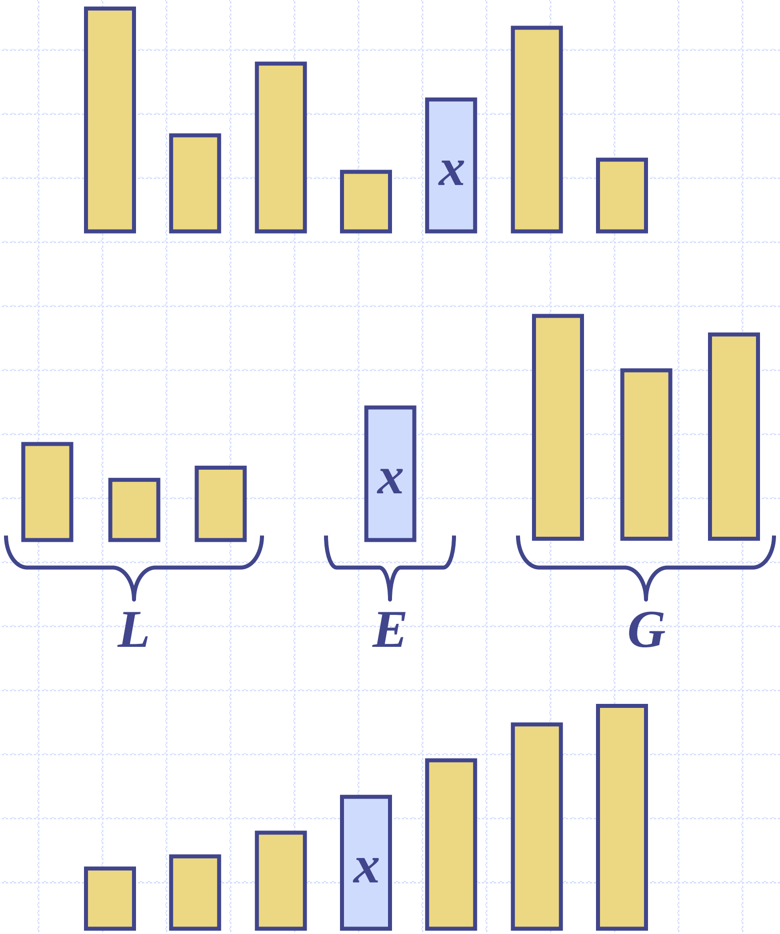
Quick-Sort, Bucket Sort, Radix Sort

Sections 11.2, 11.3.2, 11.3.3



Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



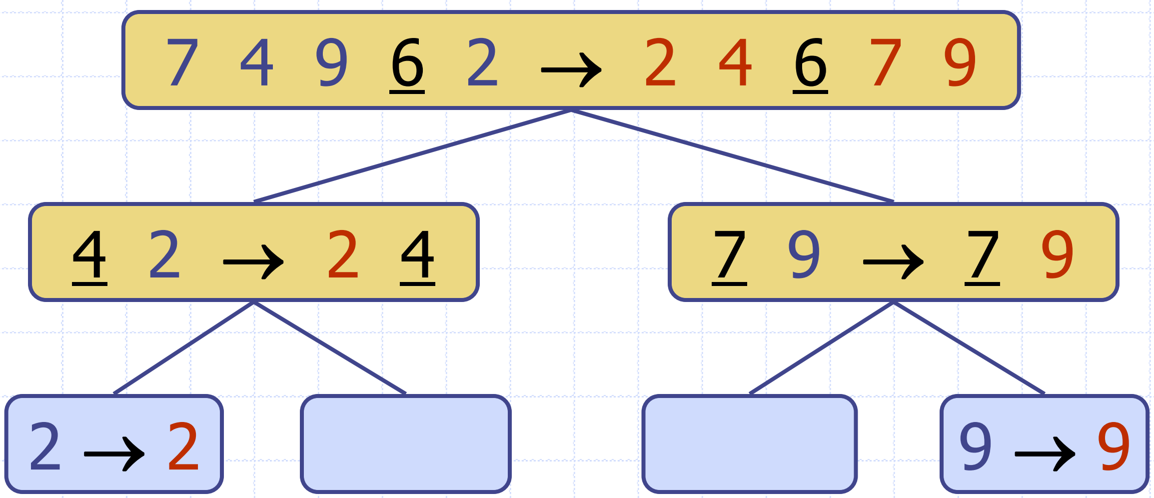
Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element *y* from *S* and
  - We insert *y* into *L*, *E* or *G*, depending on the result of the comparison with the pivot *x*
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the
        elements of S less than, equal to,
        or greater than the pivot, resp.
    L, E, G ← empty sequences
    x ← S.erase(p)
    while ¬S.empty()
        y ← S.eraseFront()
        if y < x
            L.insertBack(y)
        else if y = x
            E.insertBack(y)
        else { y > x }
            G.insertBack(y)
    return L, E, G
```

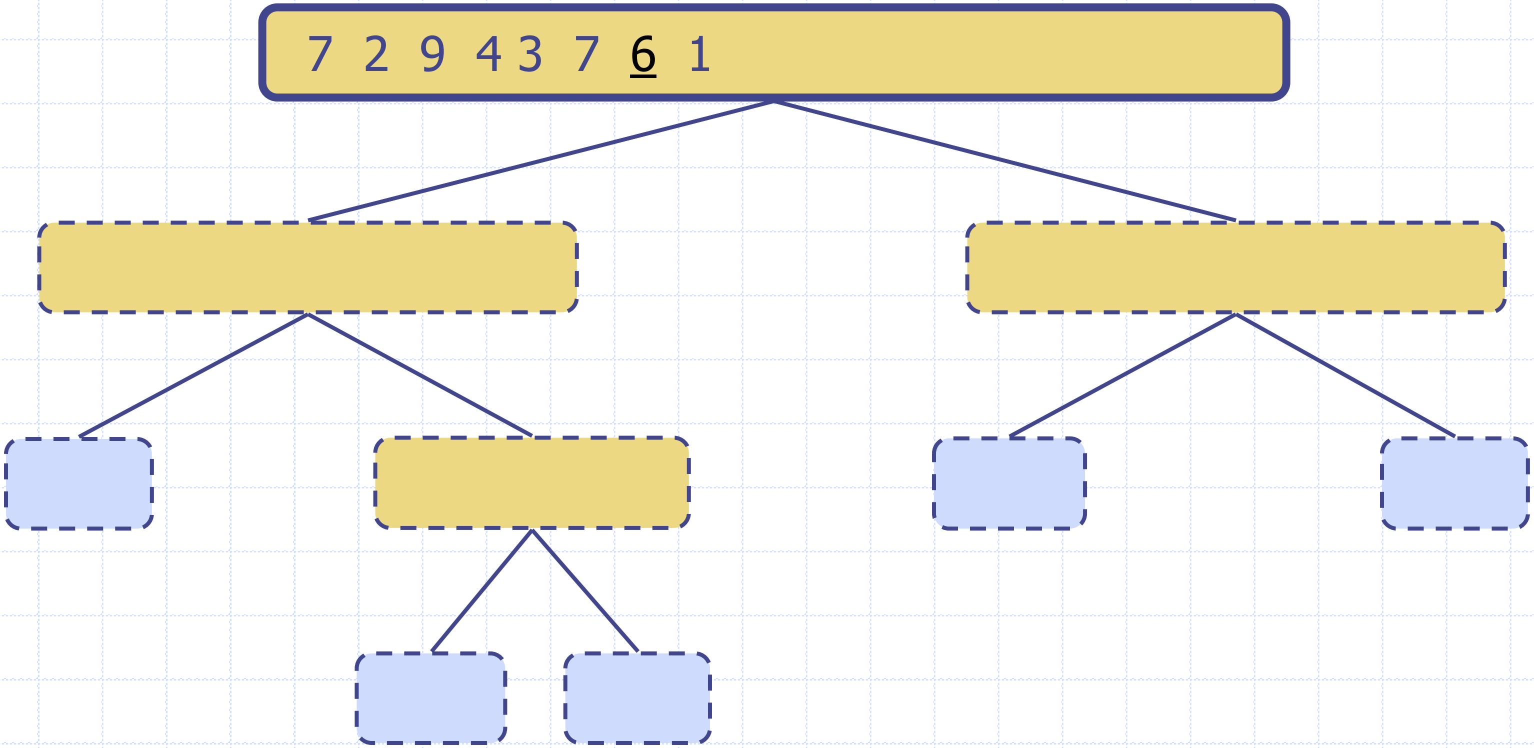
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



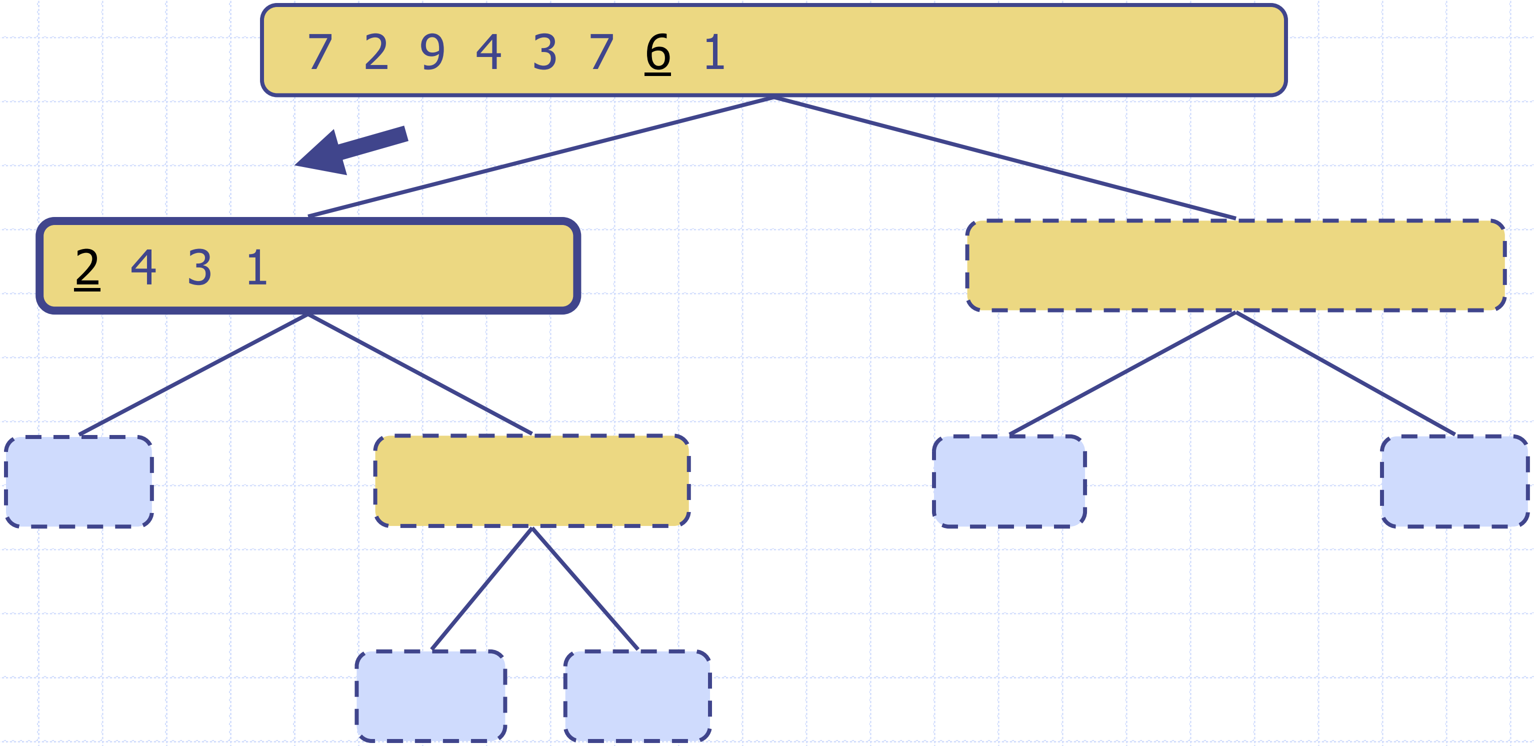
Execution Example

- Pivot selection



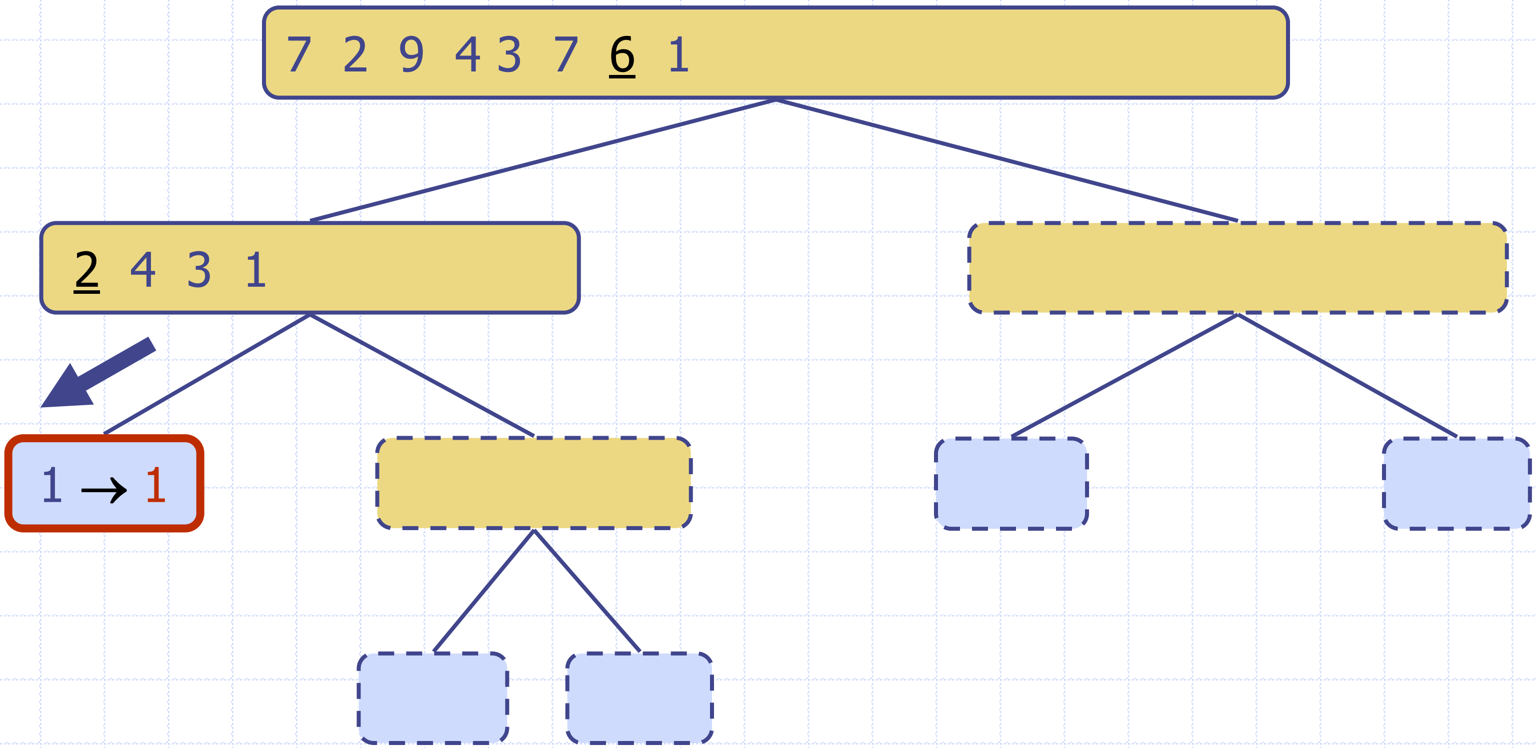
Execution Example (cont.)

- Partition, recursive call, pivot selection



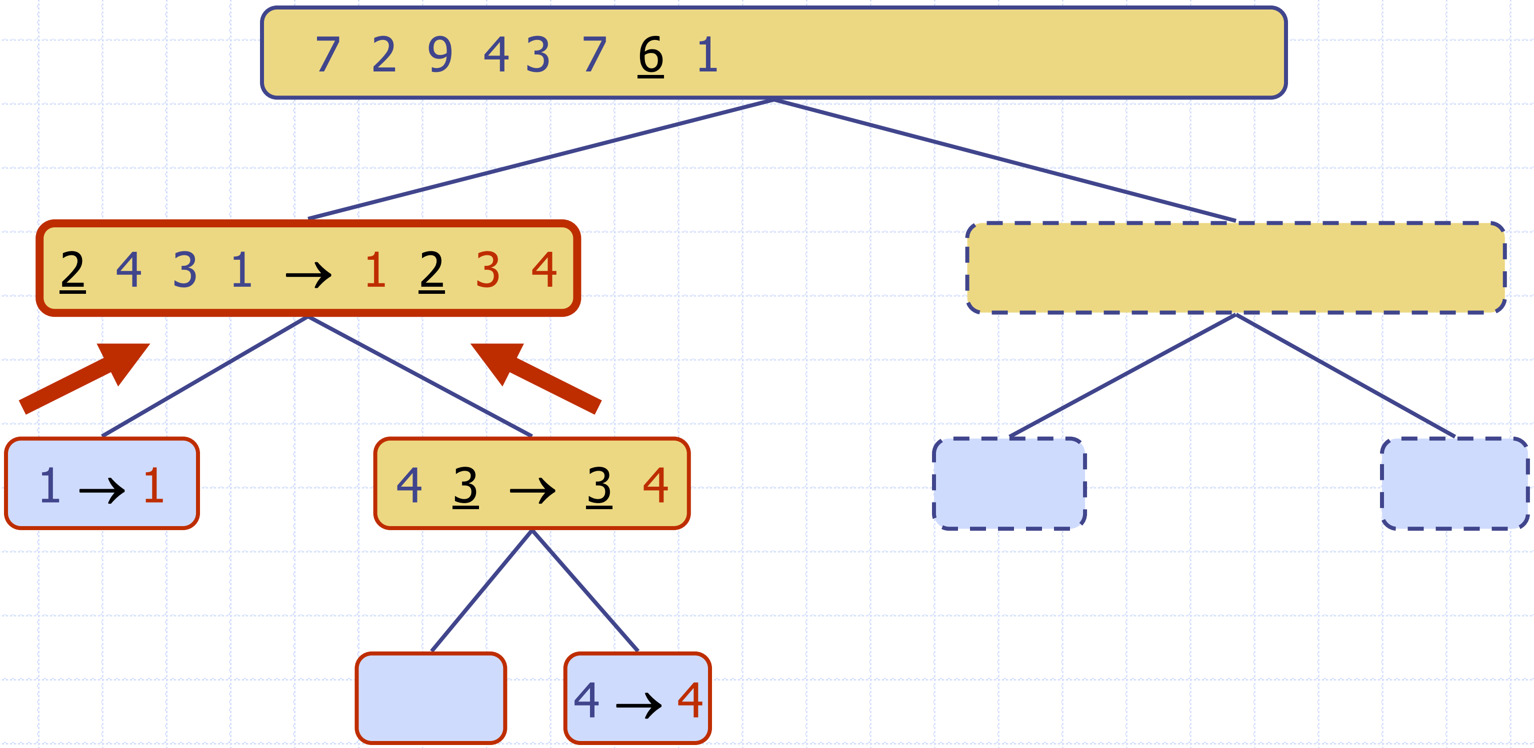
Execution Example (cont.)

- Partition, recursive call, base case



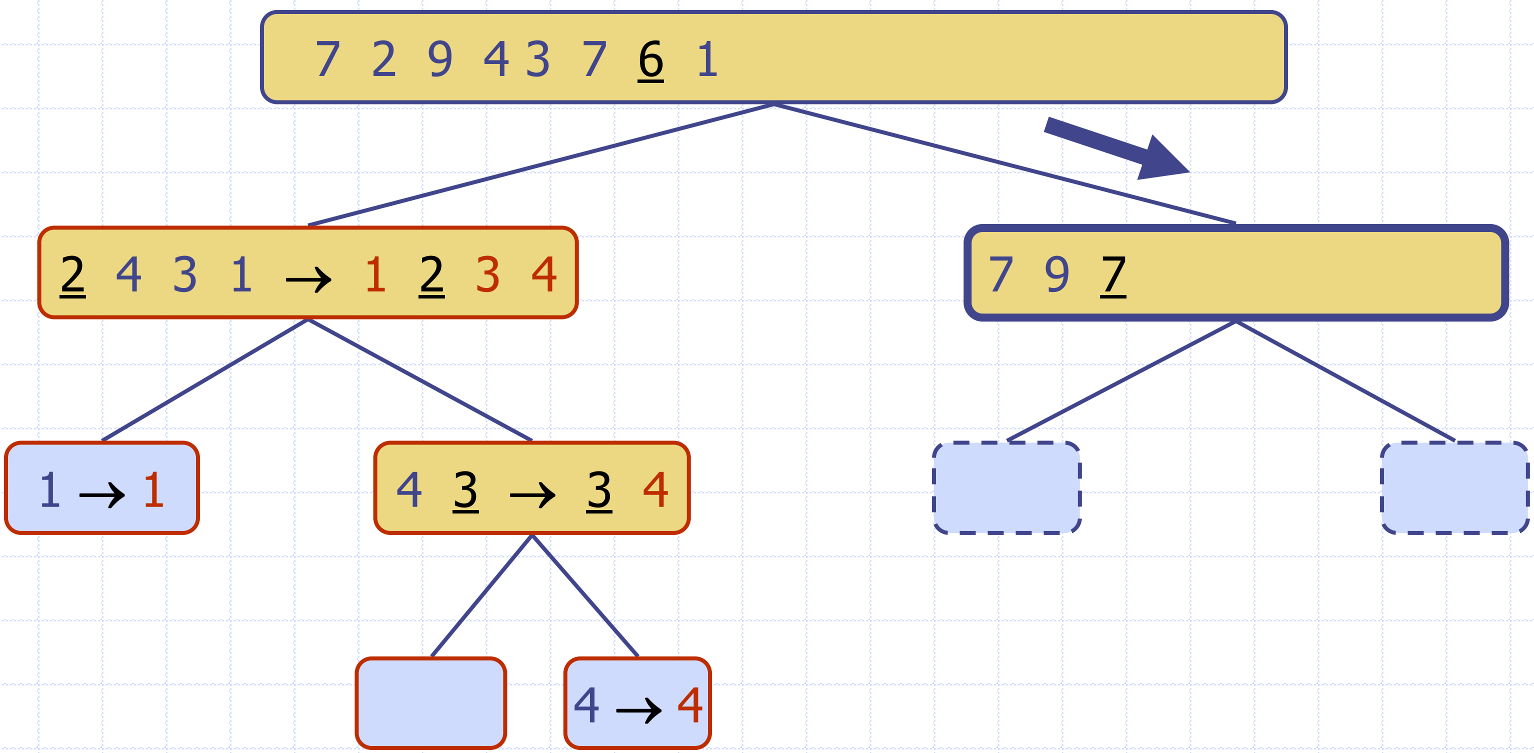
Execution Example (cont.)

- Recursive call, …, base case, join



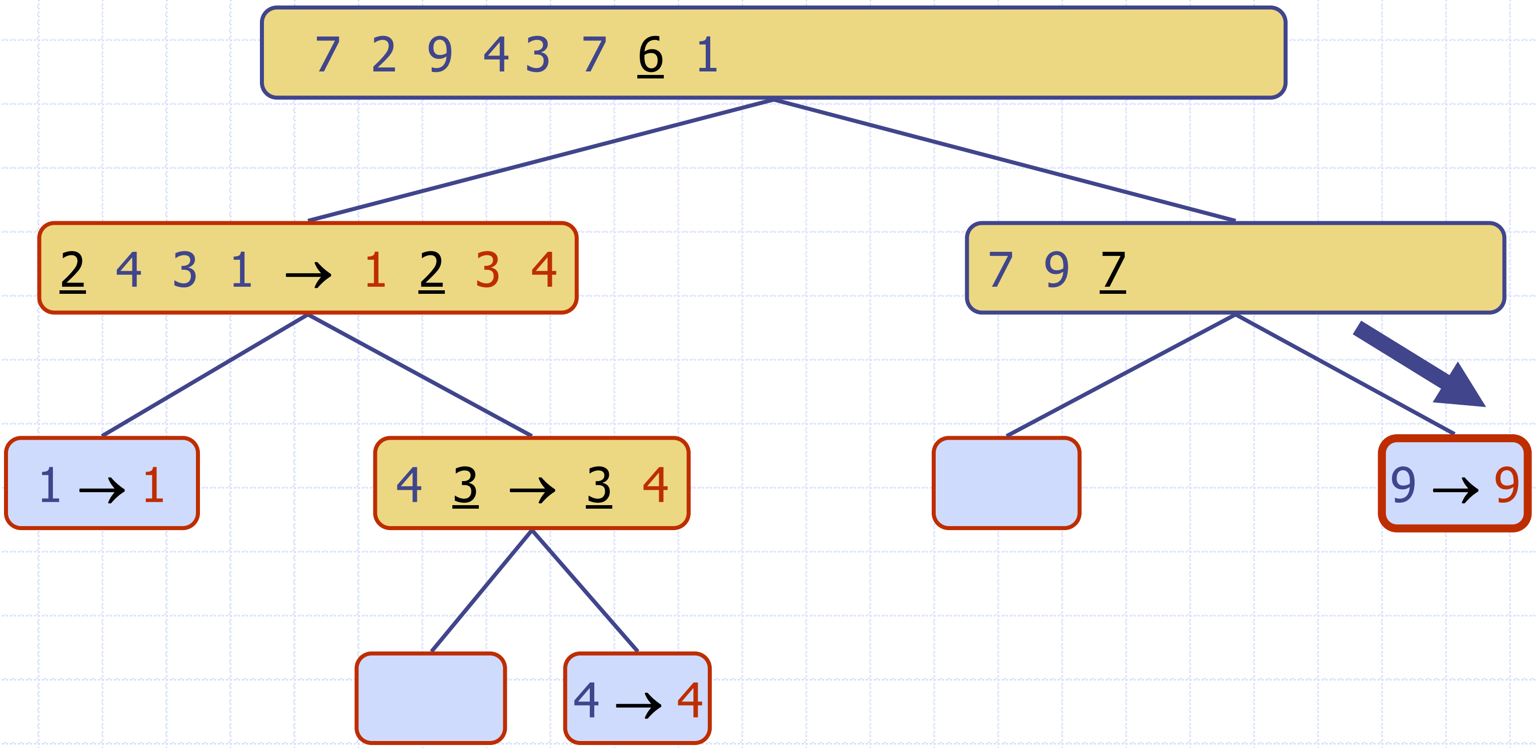
Execution Example (cont.)

- Recursive call, pivot selection



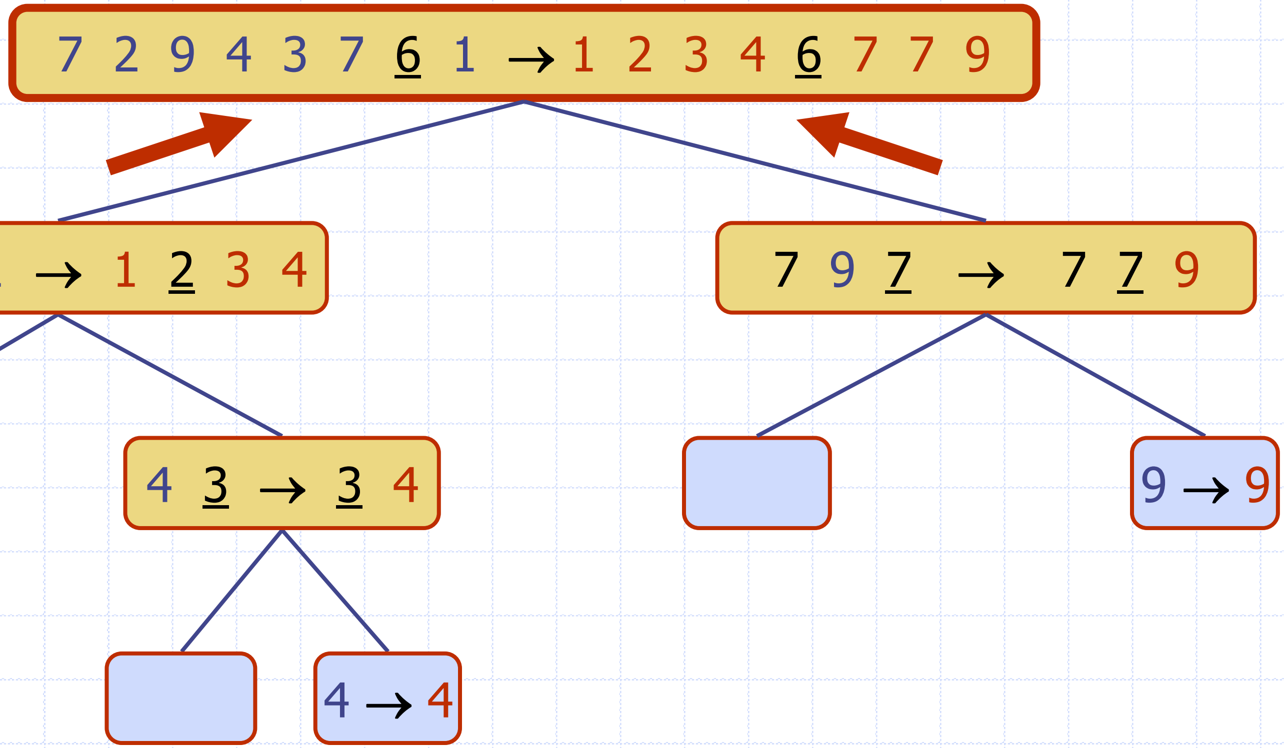
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



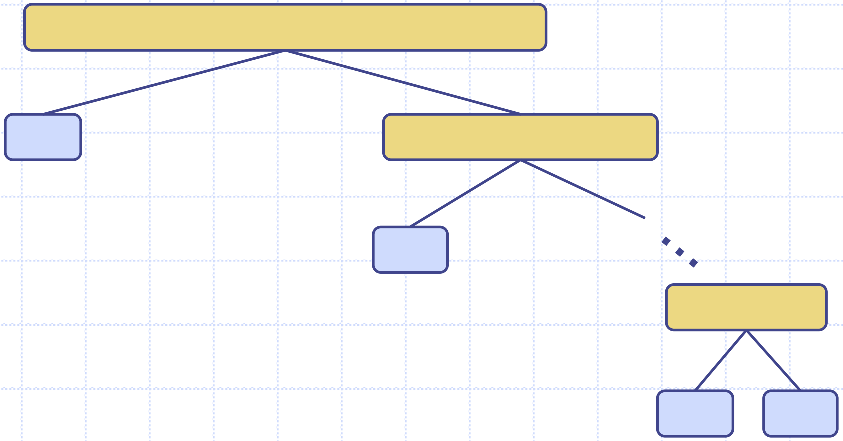
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

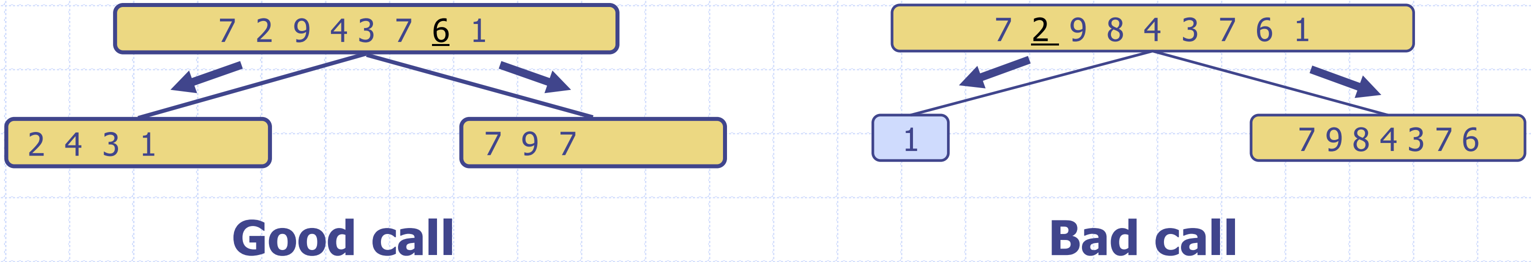
- Thus, the worst-case running time of quick-sort is $O(n^2)$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1

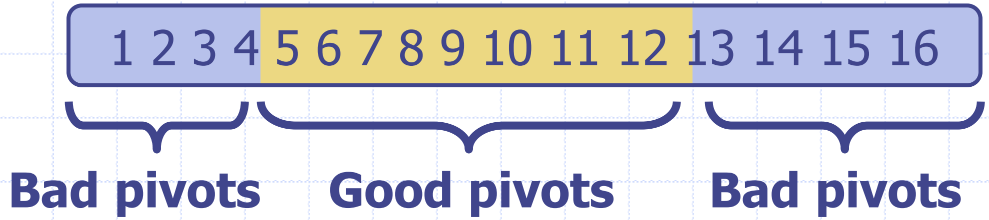


Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$

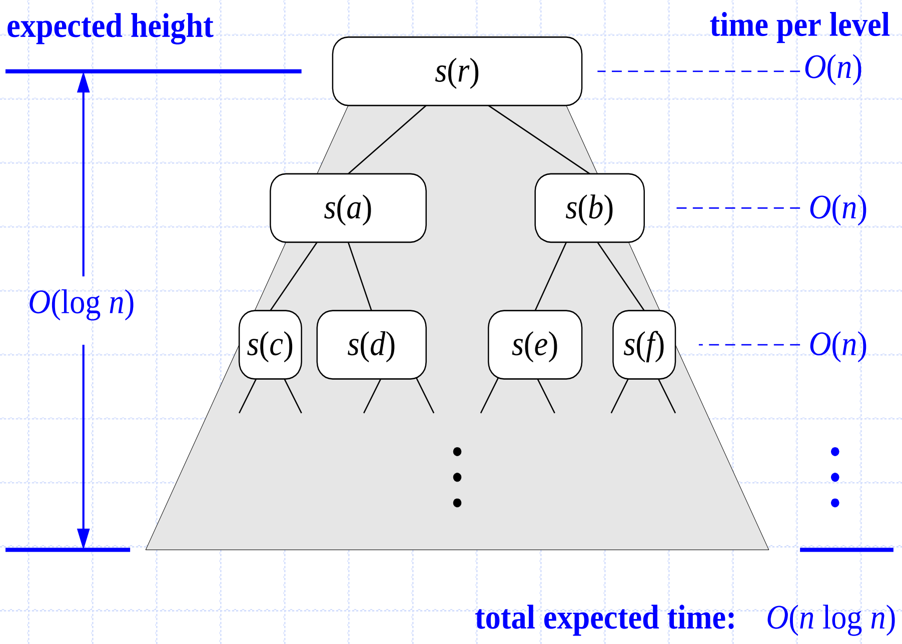


- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:

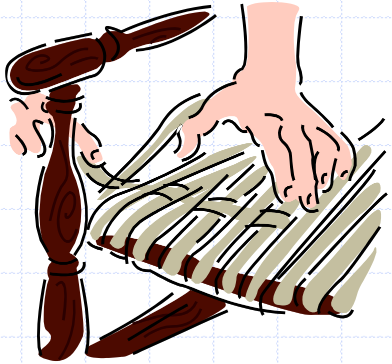


Expected Running Time, Part 2

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$
- For a node of depth i, we expect
  - $i/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



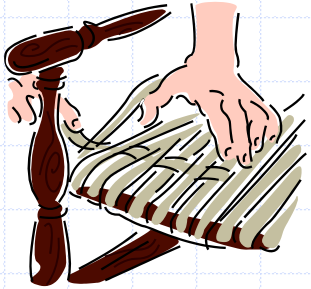
In-Place Quick-Sort



- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than ***h***
  - the elements equal to the pivot have rank between ***h*** and ***k***
  - the elements greater than the pivot have rank greater than ***k***
- The recursive calls consider
  - elements with rank less than ***h***
  - elements with rank greater than ***k***

Algorithm *inPlaceQuickSort*(*S, l, r*)
  Input sequence *S*, ranks *l* and *r*
  Output sequence *S* with the elements of rank between *l* and *r* rearranged in increasing order
  if $l \geq r$
    return
  $i \leftarrow$ a random integer between *l* and *r*
  $x \leftarrow$ *S.elemAtRank*(*i*)
  $(h, k) \leftarrow$ *inPlacePartition*(*x*)
  inPlaceQuickSort($S, l, h - 1$)
  inPlaceQuickSort($S, k + 1, r$)

In-Place Partitioning



- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

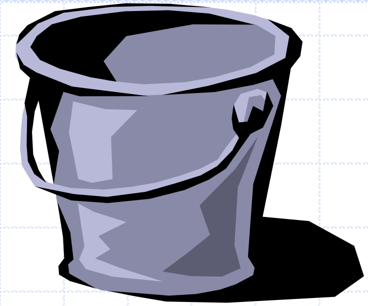
→ j k ←

3 2 5 1 0 [7] 3 5 9 [2] 7 9 8 9 7 **6** 9

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
insertion-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	▪ in-place, randomized ▪ fastest (good for large inputs)
heap-sort	$O(n \log n)$	▪ in-place ▪ fast (good for large inputs)
merge-sort	$O(n \log n)$	▪ sequential data access ▪ fast (good for huge inputs)

Bucket-Sort



- Let be S be a sequence of n (key, element) entries with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)

  Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$

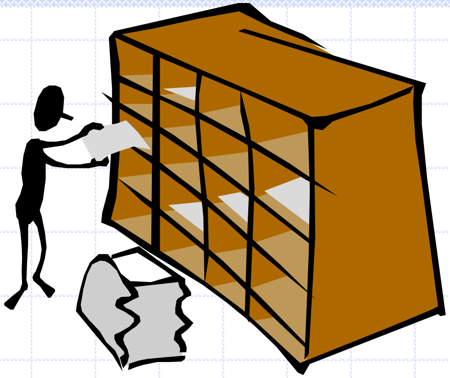
  Phase 2: For $i = 0, \ldots, N-1$, move the entries of bucket $B[i]$ to the end of sequence S

- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n+N)$ time

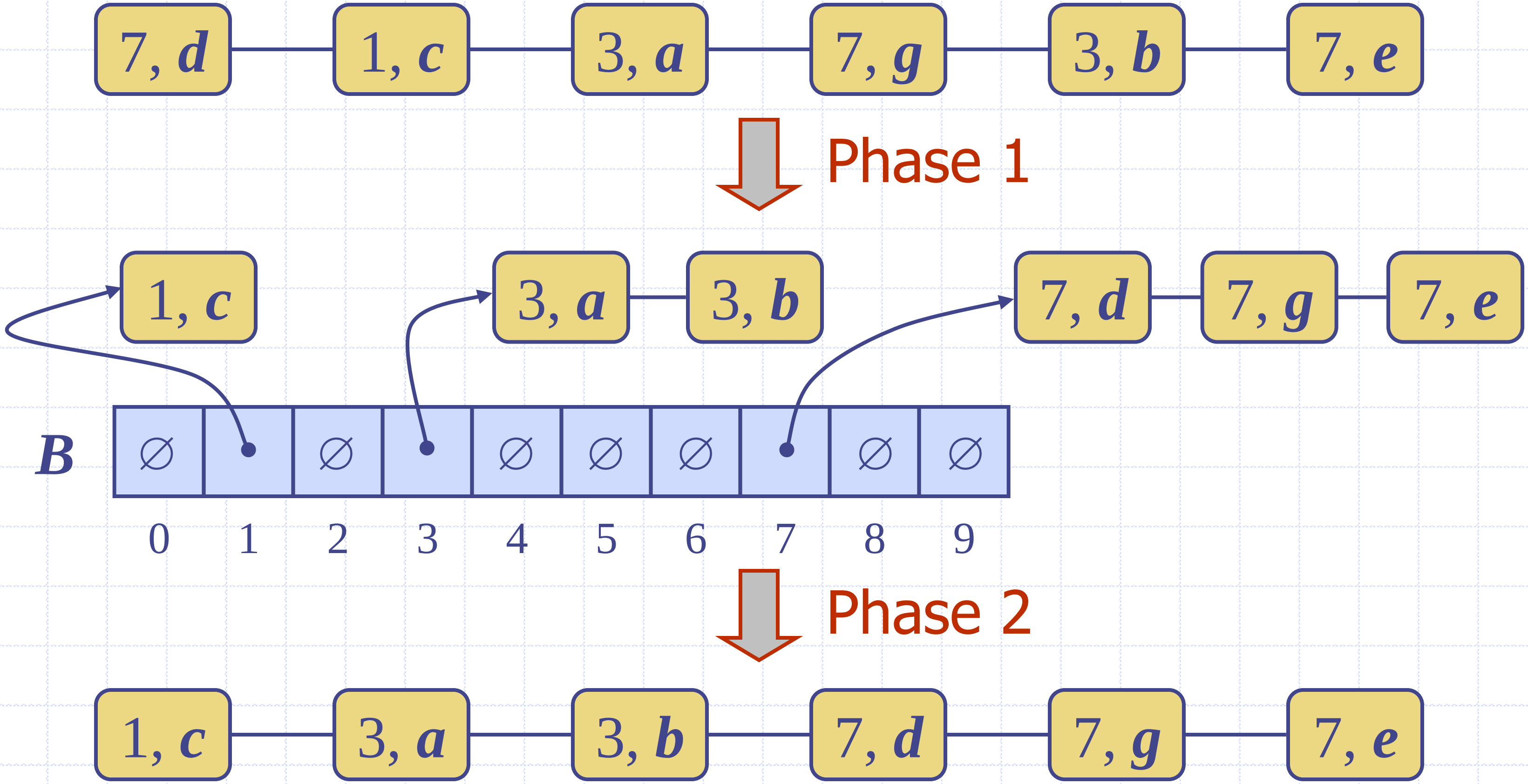
  Bucket-sort takes $O(n+N)$ time

```
Algorithm bucketSort(S, N)
    Input sequence S of (key, element)
        items with keys in the range
        [0, N – 1]
    Output sequence S sorted by
        increasing keys
    B ← array of N empty sequences
    while ¬S.empty()
        (k, o) ← S.front()
        S.eraseFront()
        B[k].insertBack((k, o))
    for i ← 0 to N – 1
        while ¬B[i].empty()
            (k, o) ← B[i].front()
            B[i].eraseFront()
            S.insertBack((k, o))
```

Example



- Key range [0, 9]



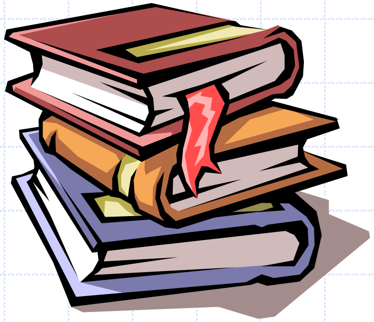
Properties and Extensions

- Key-type Property
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- Stable Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions

- Integer keys in the range $[a, b]$
  - Put entry (k, o) into bucket $B[k - a]$
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 13 provinces and territories)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put entry (k, o) into bucket $B[r(k)]$

Lexicographic Order



- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$

$$\Leftrightarrow$$

$$x_1 < y_1 \;\vee\; x_1 = y_1 \wedge (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let *stableSort*(S, C) be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm ***stableSort***, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of ***stableSort***

Algorithm *lexicographicSort*(S)
  Input sequence S of d-tuples
  Output sequence S sorted in lexicographic order

  for $i \leftarrow d$ **downto** 1
    stableSort(S, C_i)

Example:

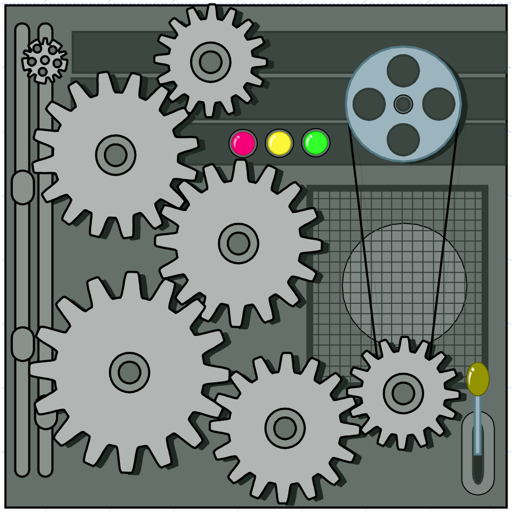
(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)

(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)

(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

Radix-Sort



- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d(n + N))$

Algorithm ***radixSort***(S, N)

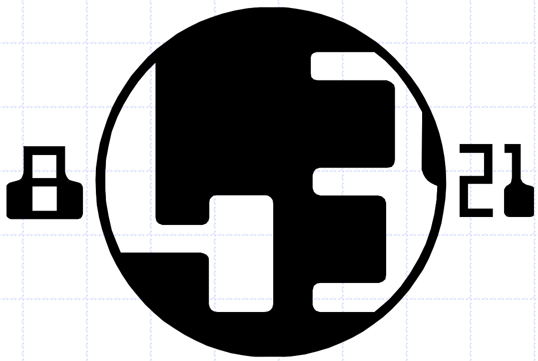
Input sequence S of d-tuples such that $(0, \ldots, 0) \le (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \le (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S

Output sequence S sorted in lexicographic order

for $i \leftarrow d$ **downto** 1

bucketSort(S, N)

Radix-Sort for Binary Numbers



- Consider a sequence of $\boldsymbol{n}$ $\boldsymbol{b}$-bit integers

  $$x = x_{b-1} \dots x_1 x_0$$

- We represent each element as a $\boldsymbol{b}$-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm ***binaryRadixSort***(*S*)
  Input sequence ***S*** of ***b***-bit integers
  Output sequence ***S*** sorted
  replace each element ***x*** of ***S*** with the item (0**,** ***x***)
  for ***i*** ← 0 **to** ***b*** – 1
    replace the key ***k*** of each item (***k*****,** ***x***) of ***S*** with bit x_i of ***x***
    bucketSort(***S*****,** 2)

Example

- Sorting a sequence of 4-bit integers