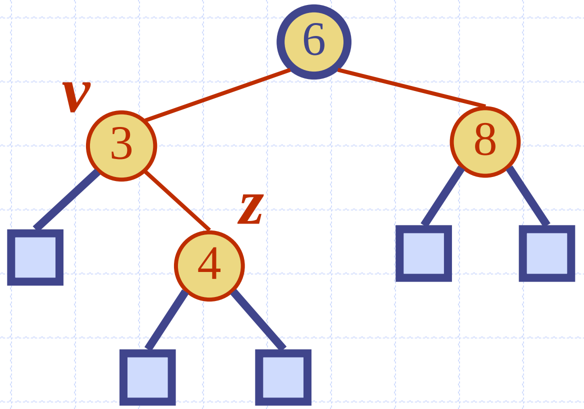
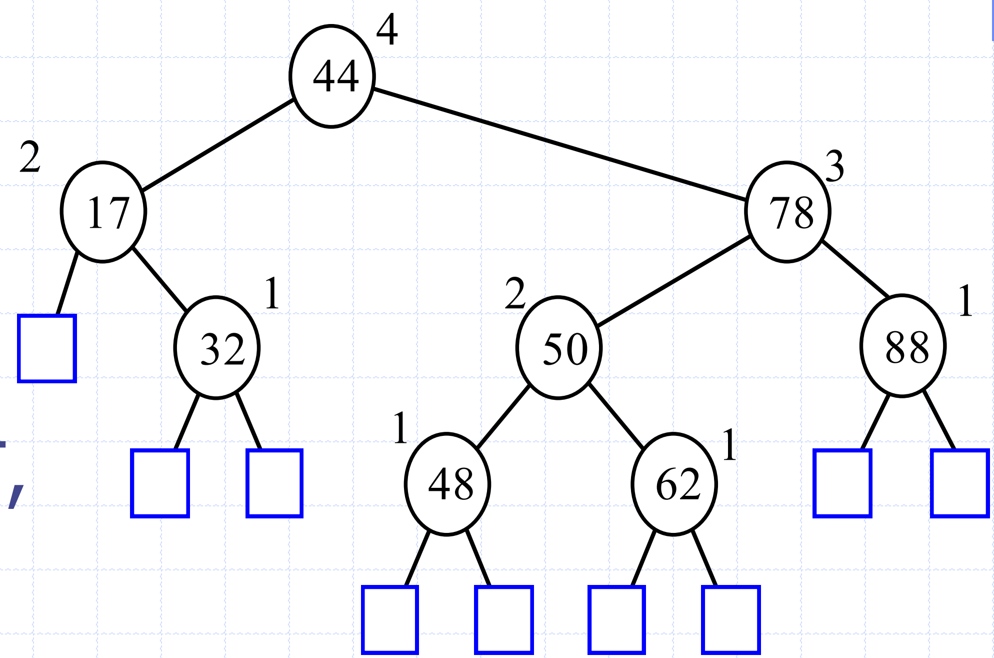
AVL Trees and (2,4) Trees

Sections 10.2 and 10.4



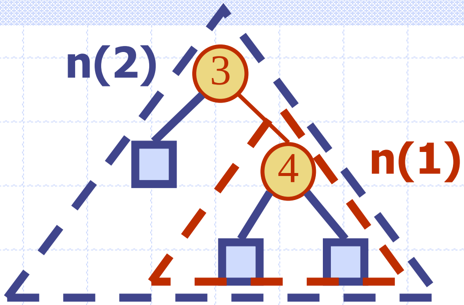
AVL Tree Definition

- AVL trees are balanced
- An AVL Tree is a binary search tree such that for every internal node v of T, the heights of the children of v can differ by at most 1



An example of an AVL tree where the heights are shown next to the nodes

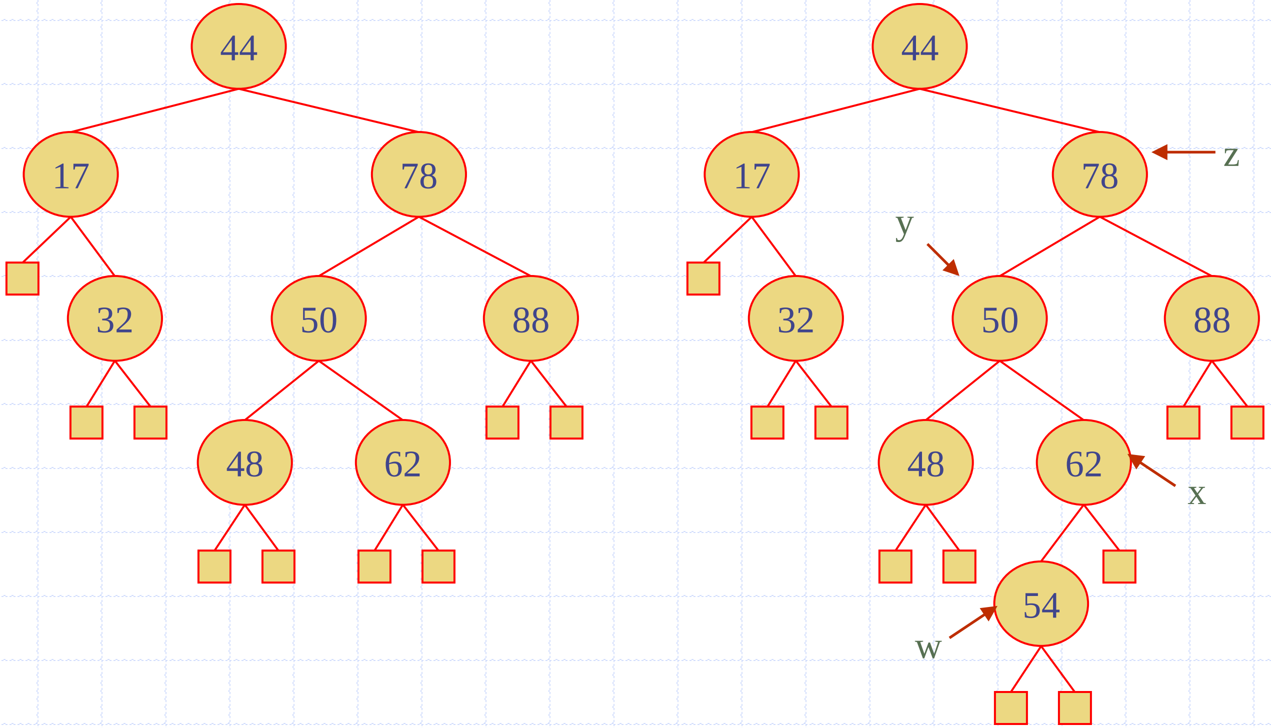
Height of an AVL Tree



- Fact: The height of an AVL tree storing n keys is O(log n).
- Proof: Let us bound g(h): the minimum number of internal nodes of an AVL tree of height h.
- We easily see that g(1) = 1 and g(2) = 2
- For h > 2, a minimal AVL tree of height h contains the root, one AVL subtree of height h-1 and another of height h-2.
- That is, g(h) = 1 + g(h-1) + g(h-2)
- Knowing g(h-1) > g(h-2), we get g(h) > 2g(h-2). So
  g(h) > 2g(h-2), g(h) > 4g(h-4), g(h) > 8g(h-6), … (by induction),
  $g(h) > 2^i g(h-2i)$
- Solving the base case we get: $g(h) > 2^{h/2-1}$
- Taking logarithms: log g(h) > h/2 – 1, or h < 2log g(h) +2
- Thus the height of an AVL tree is O(log g(h)) = O(log n)

Insertion

- Insertion is as in a binary search tree
- Always done by expanding an external node w.



before insertion

after insertion

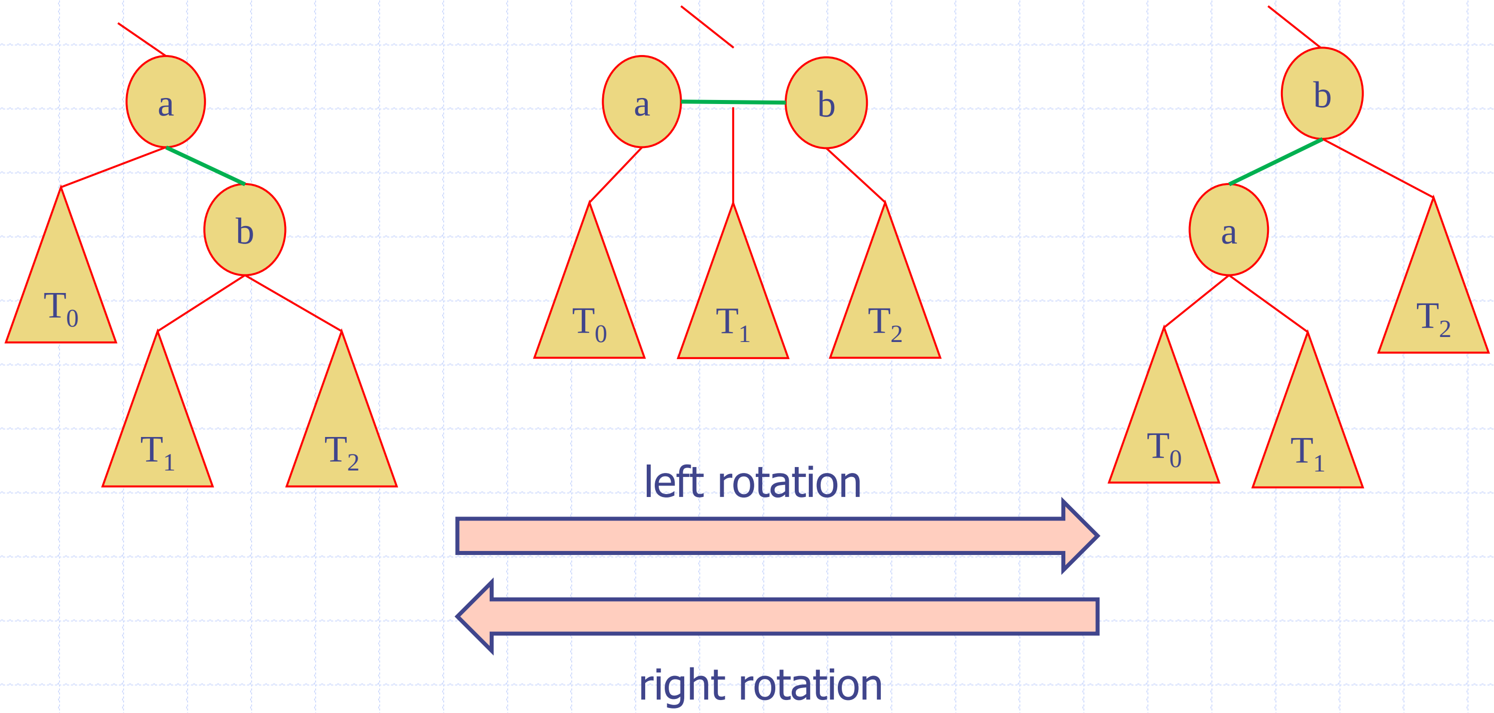
z is first unbalanced node encountered walking up tree from w.

y is the child of z with greater height.

x is the child of y with greater height. x might be w.

Tree Rotation

- Rotation is a fundamental restructuring operation for binary search trees.

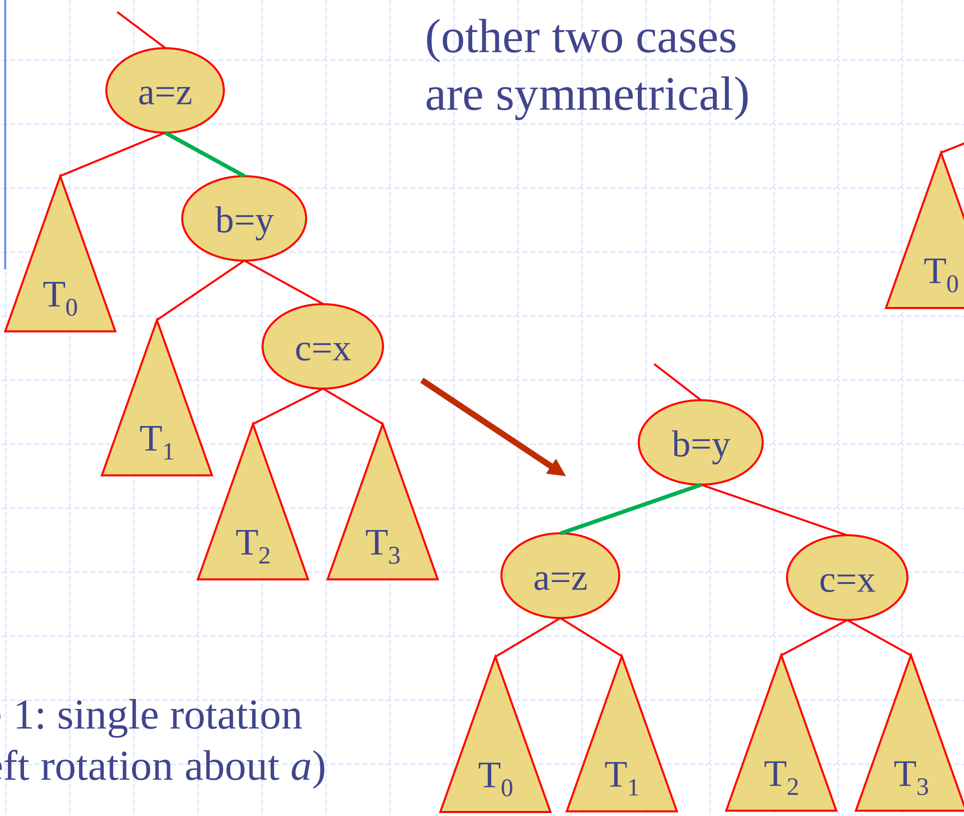


- Inorder listing is the same before and after rotation.

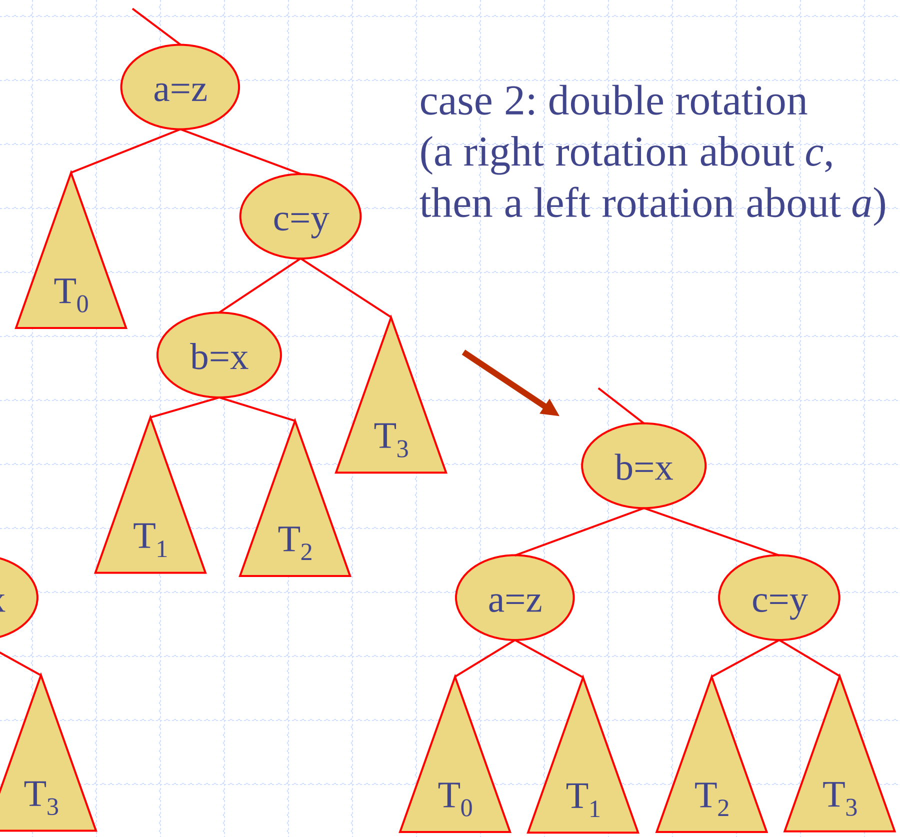
Trinode Restructuring

- let (*a*,*b*,*c*) be an inorder listing of *x*, *y*, *z*
- perform the rotations needed to make *b* the topmost node of the three

(other two cases are symmetrical)

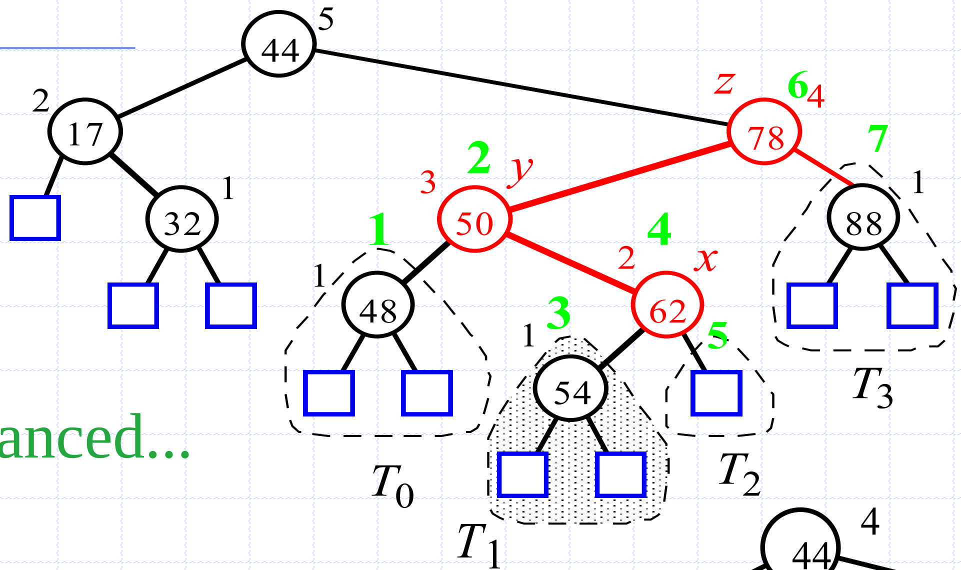


case 1: single rotation
(a left rotation about *a*)

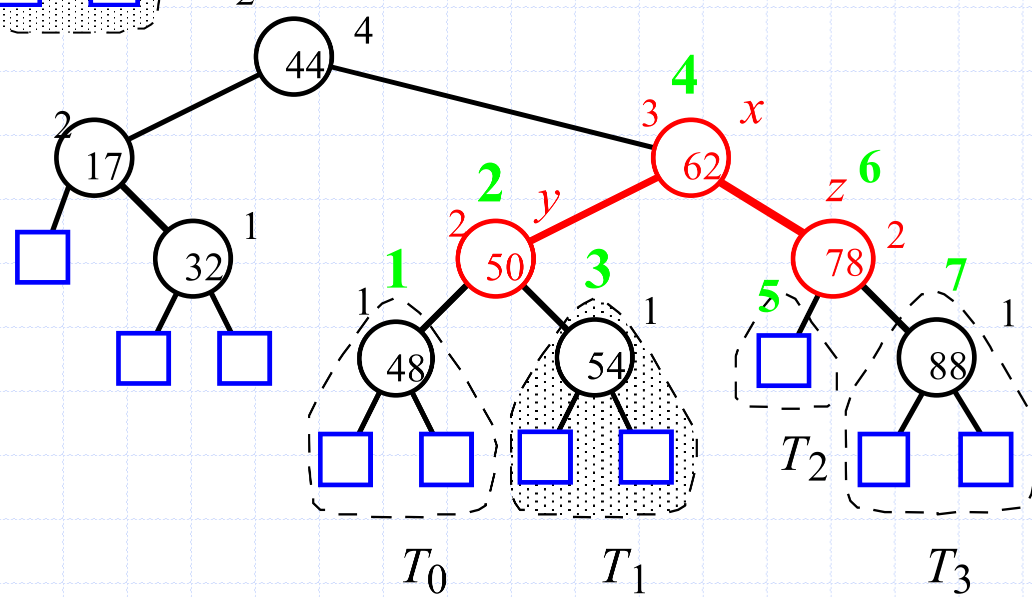


case 2: double rotation
(a right rotation about *c*, then a left rotation about *a*)

Insertion Example, continued



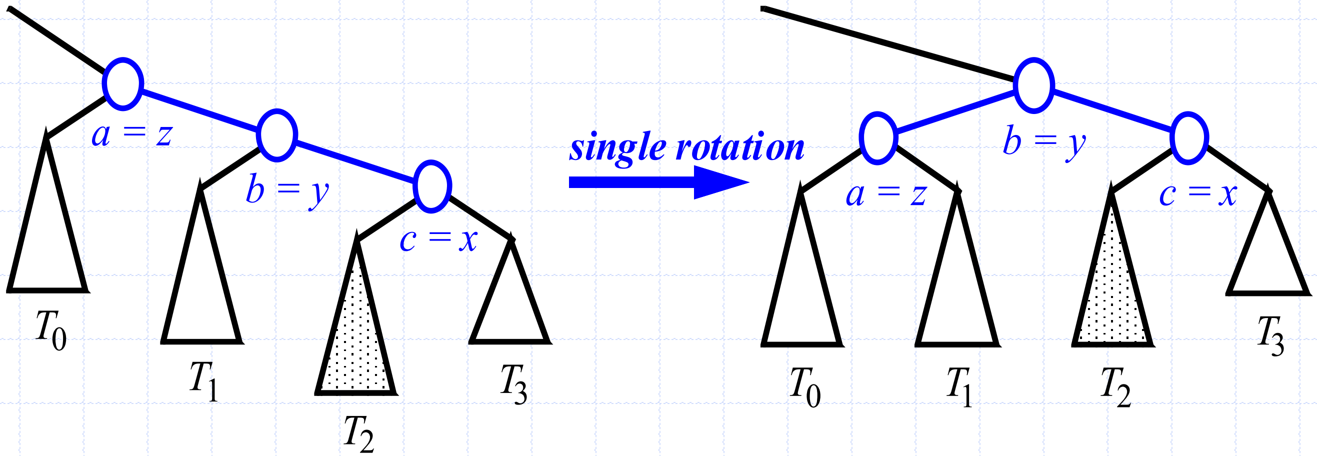
unbalanced...

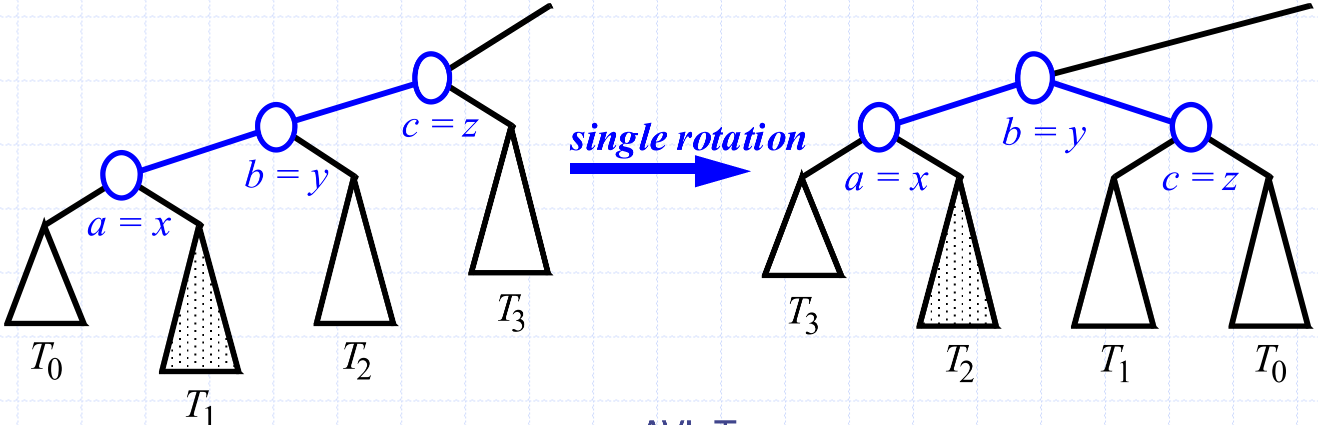


...balanced

Restructuring (as Single Rotations)

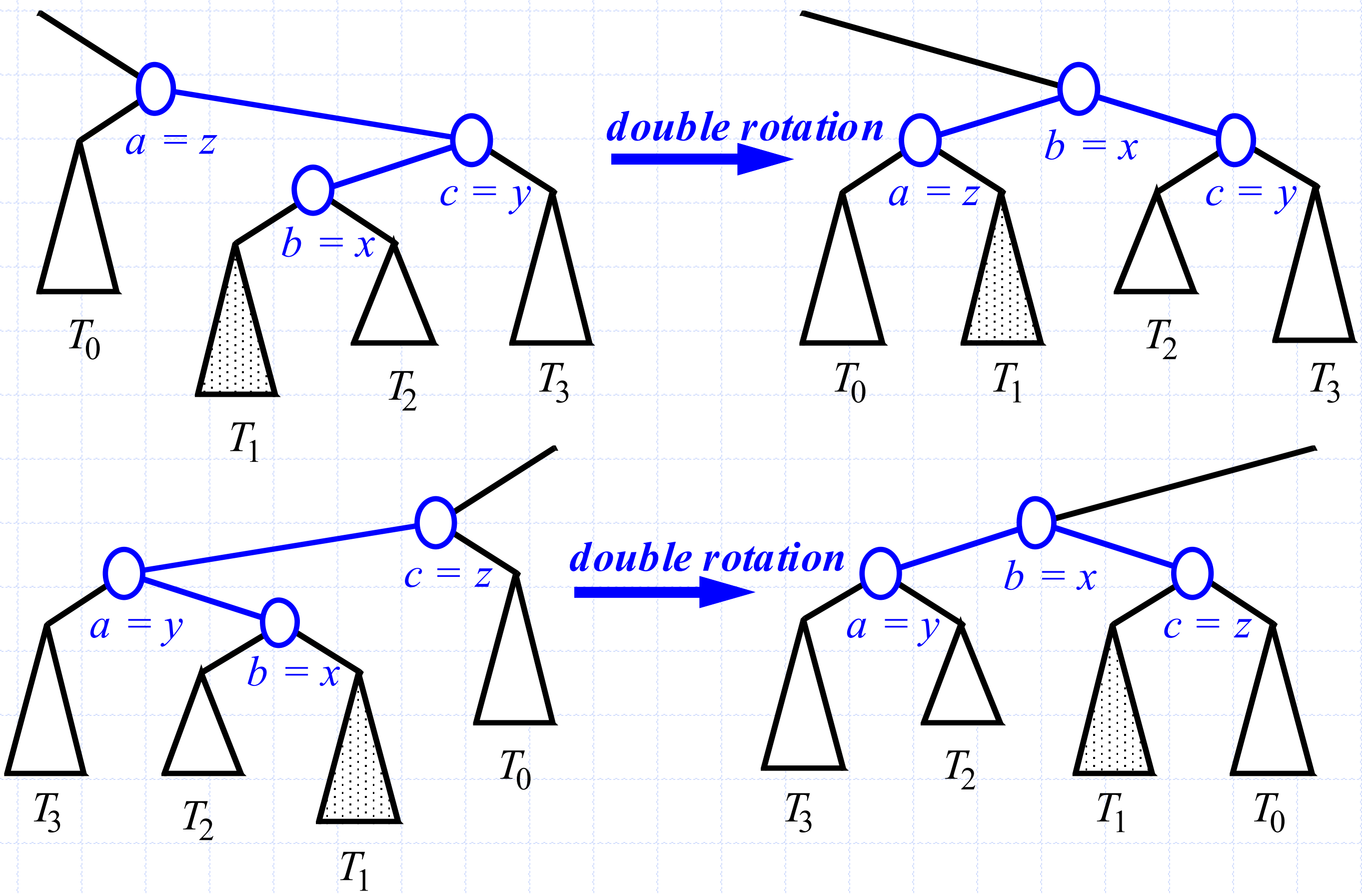
- Single Rotations:





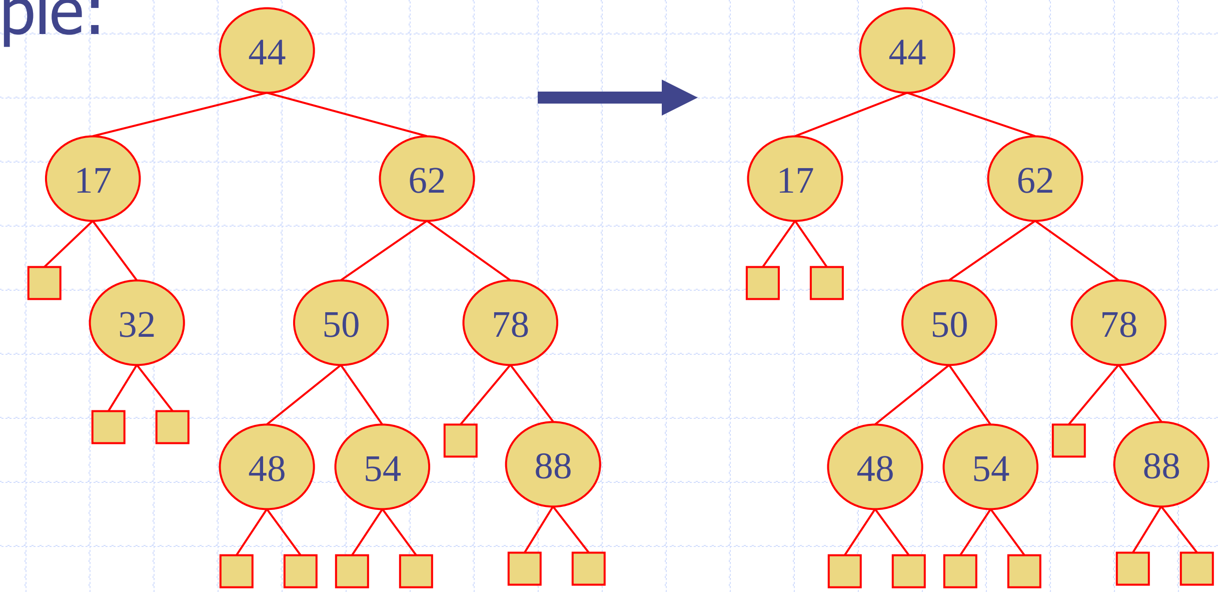
Restructuring (as Double Rotations)

- double rotations:



Removal

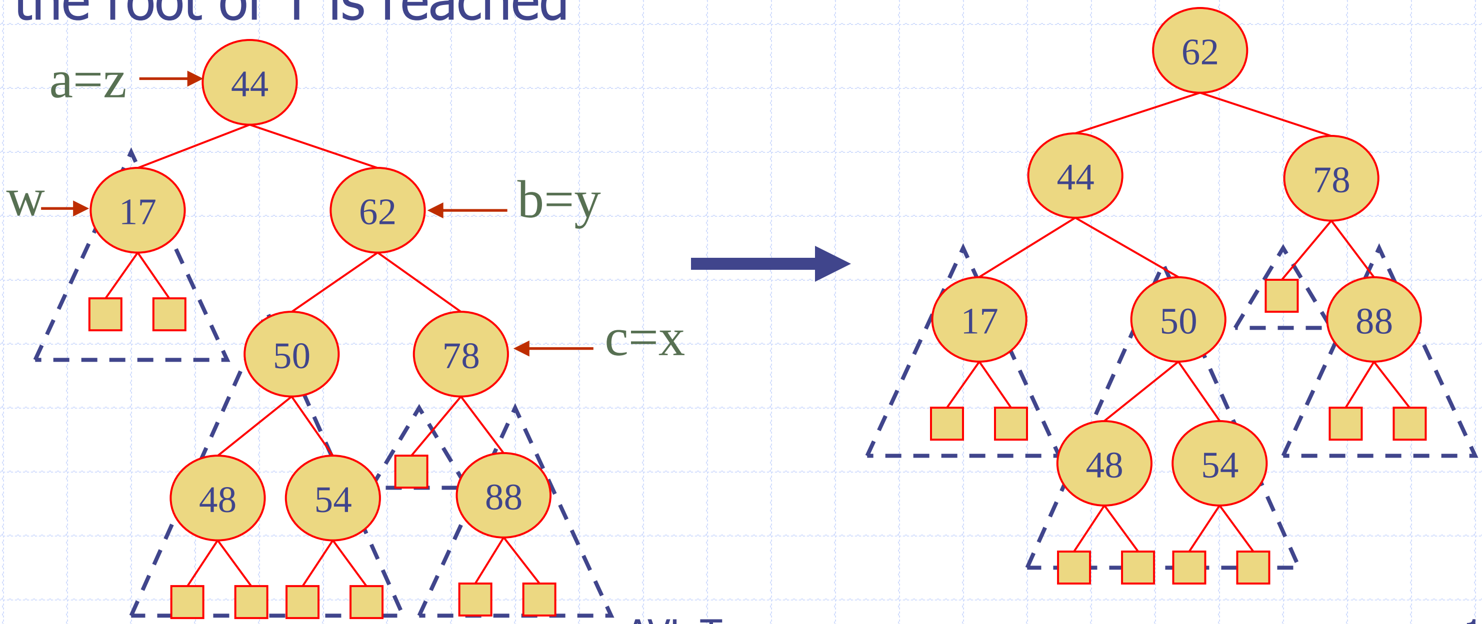
- Removal begins as in a binary search tree, which means the node removed will become an empty external node. Its parent, w, may cause an imbalance.
- Example:



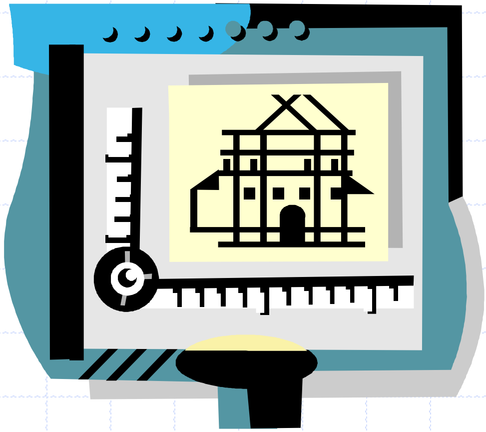
before deletion of 32 after deletion

Rebalancing after a Removal

- Let z be the first unbalanced node encountered while travelling up the tree from w. Also, let y be the child of z with the larger height, and let x be the child of y with the larger height
- We perform restructure(x) to restore balance at z
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached



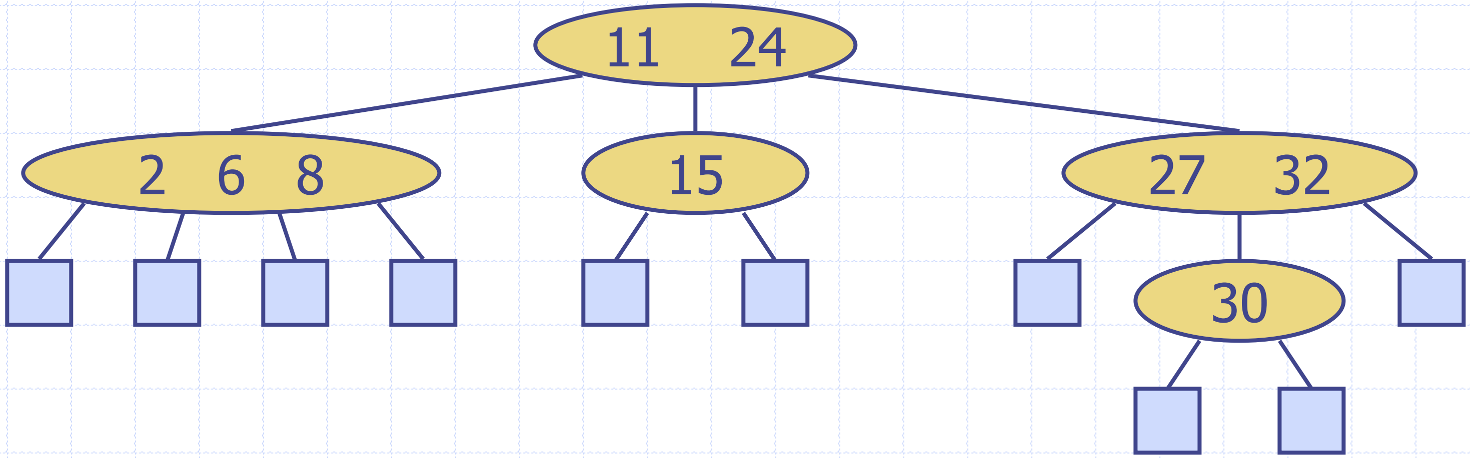
AVL Tree Performance



- a single restructure takes O(1) time
  - using a linked-structure binary tree
- find takes O(log n) time
  - height of tree is O(log n), no restructures needed
- put takes O(log n) time
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)
- erase takes O(log n) time
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)

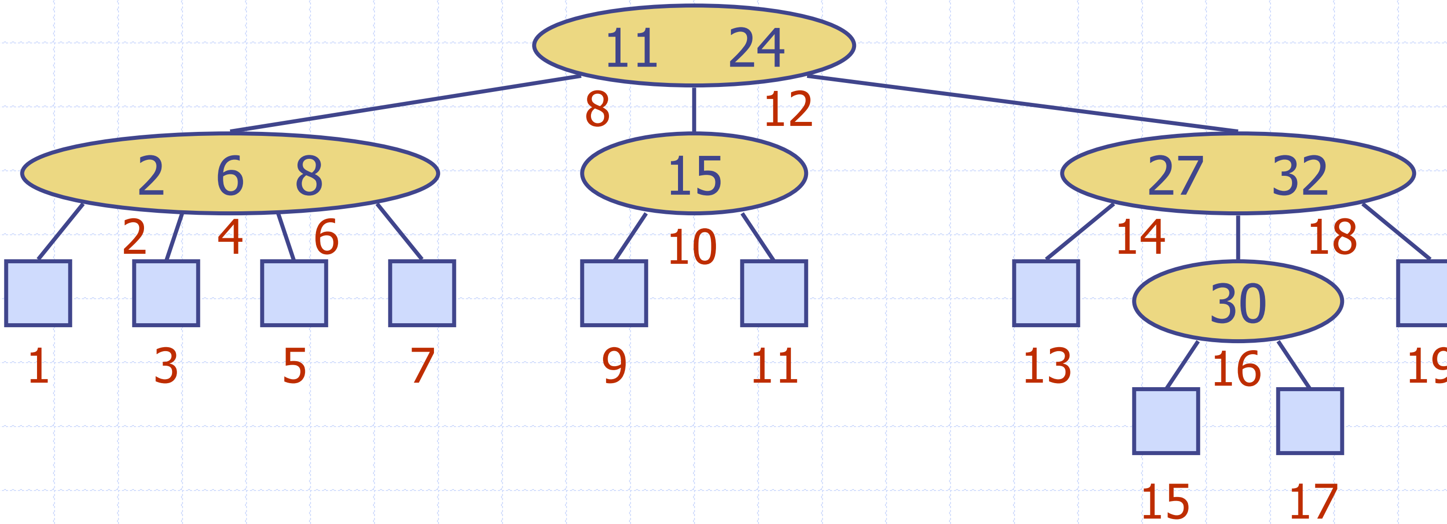
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\ v_2\ \ldots\ v_d$ storing keys $k_1\ k_2\ \ldots\ k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



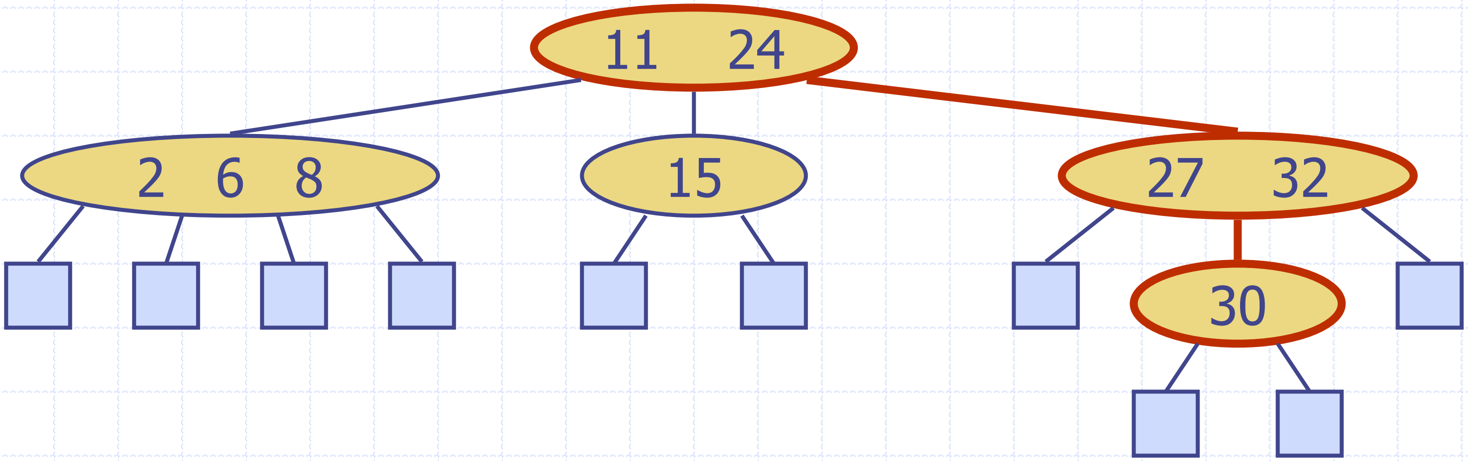
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



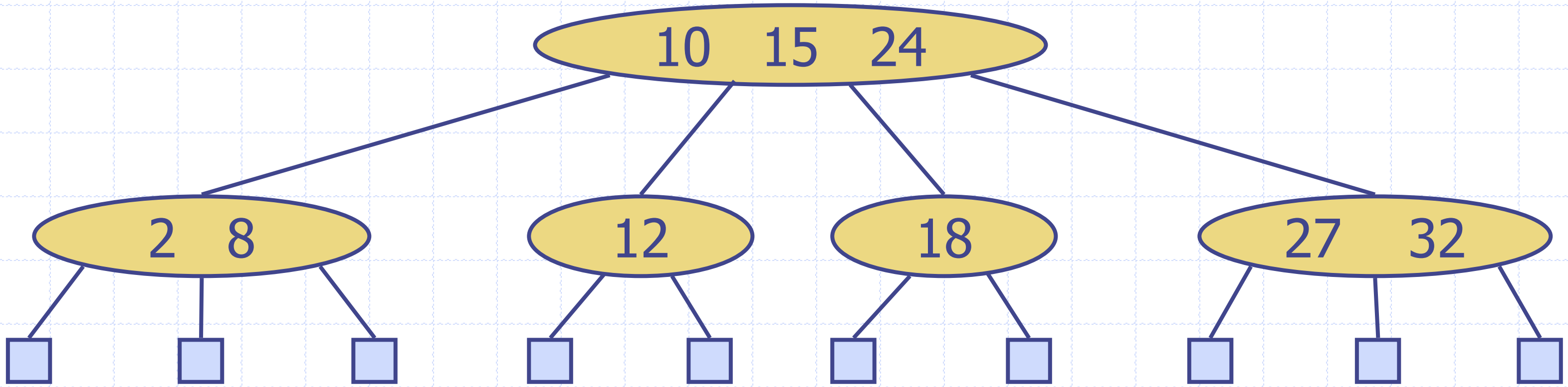
Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1\ v_2\ \ldots\ v_d$ and keys $k_1\ k_2\ \ldots\ k_{d-1}$
  - $k = k_i$ ($i = 1, \ldots, d-1$): the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ ($i = 2, \ldots, d-1$): we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Trees

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search tree with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node

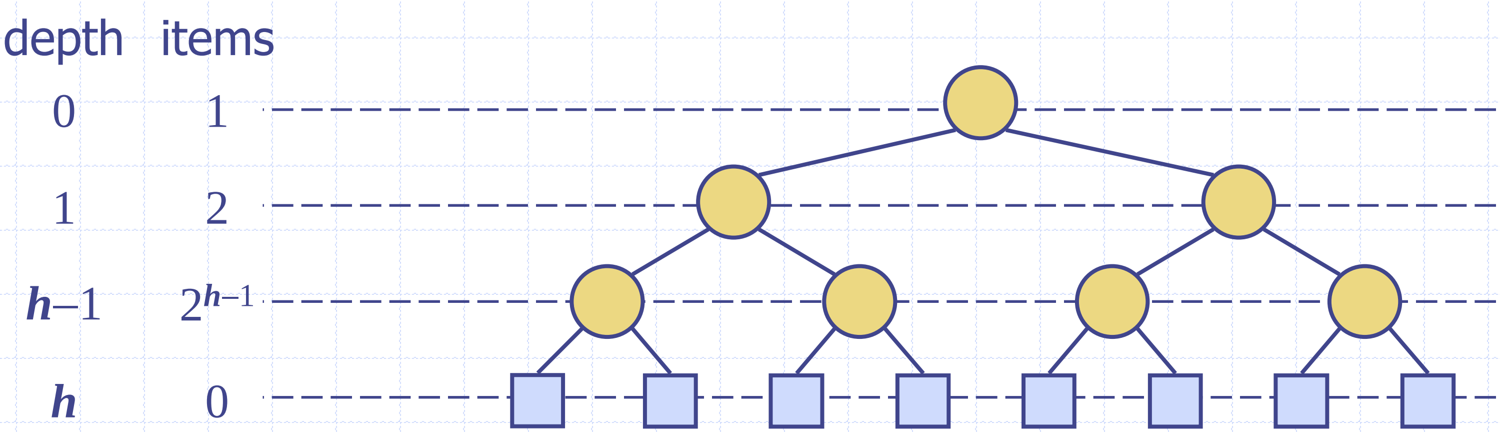


Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$

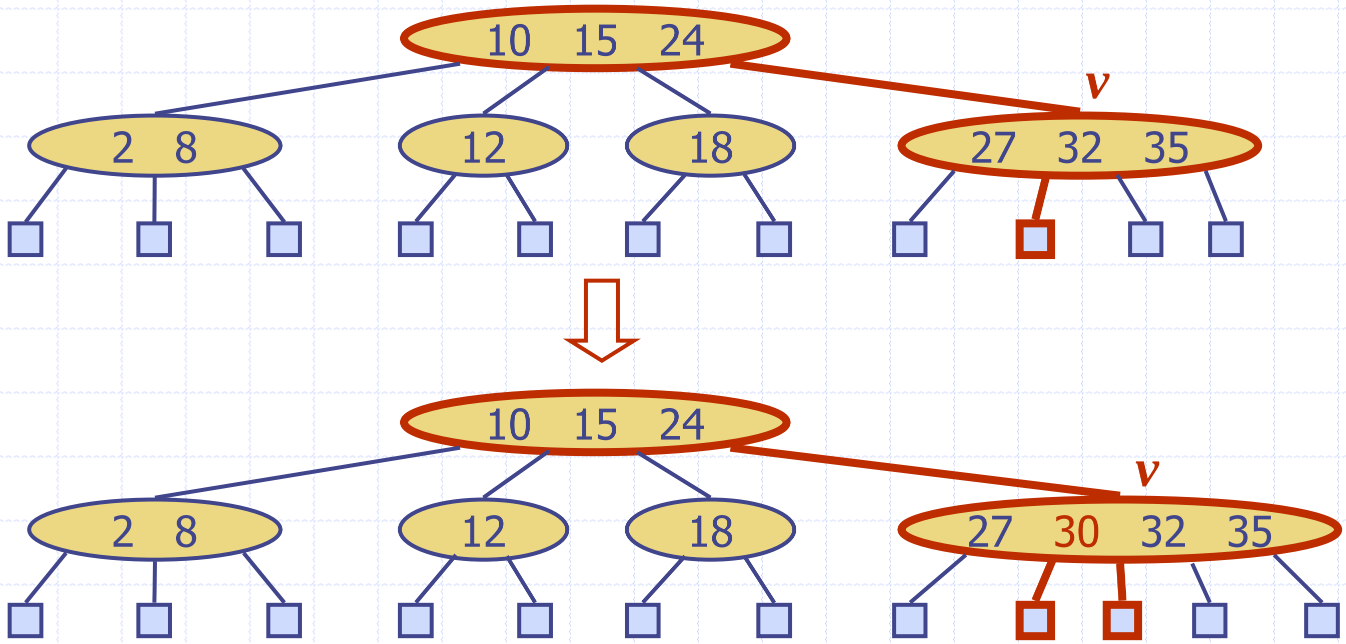
  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have

    $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n+1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



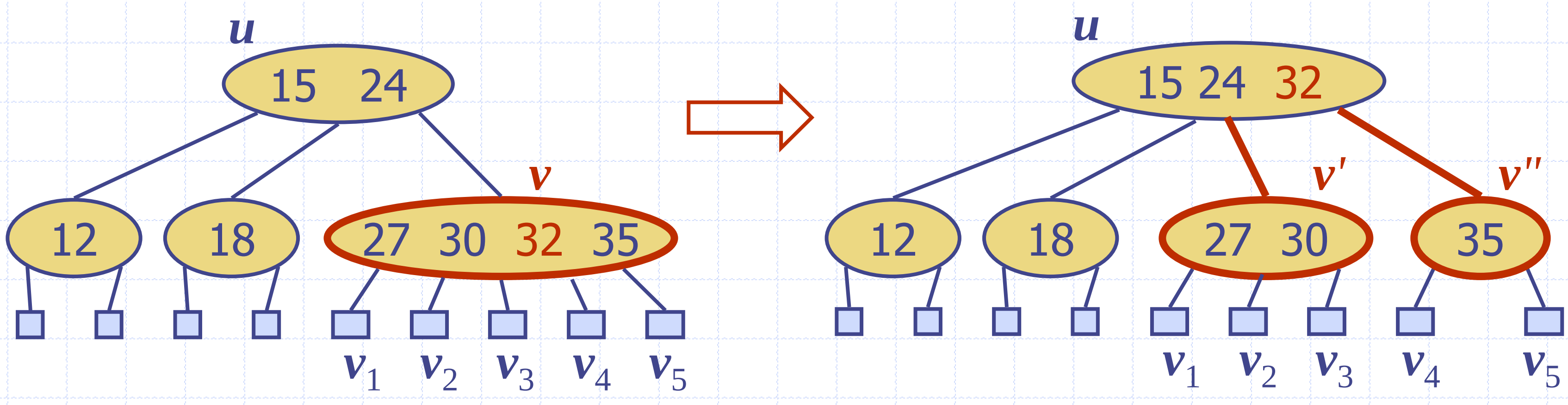
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow



Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



Analysis of Insertion

Algorithm *put*(k, o)

1. We search for key k to locate the insertion node v
2. We add the new entry (k, o) at node v
3. **while** *overflow*(v)

    if *isRoot*(v)

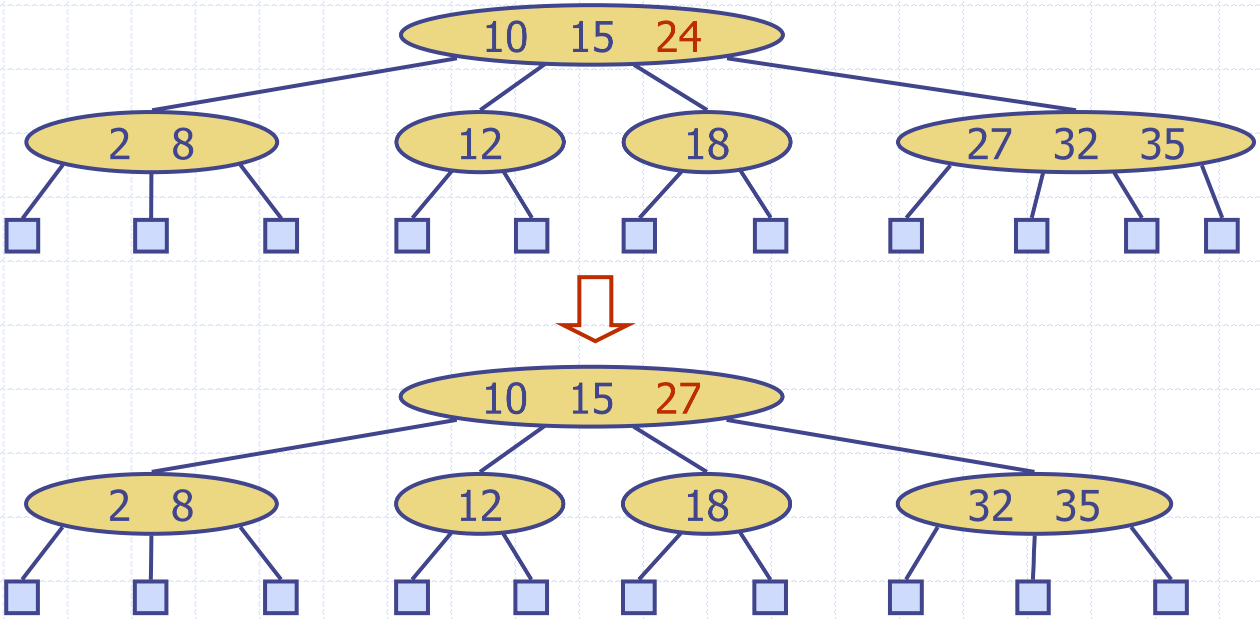
        create a new empty root above v

    $v \leftarrow$ *split*(v)

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

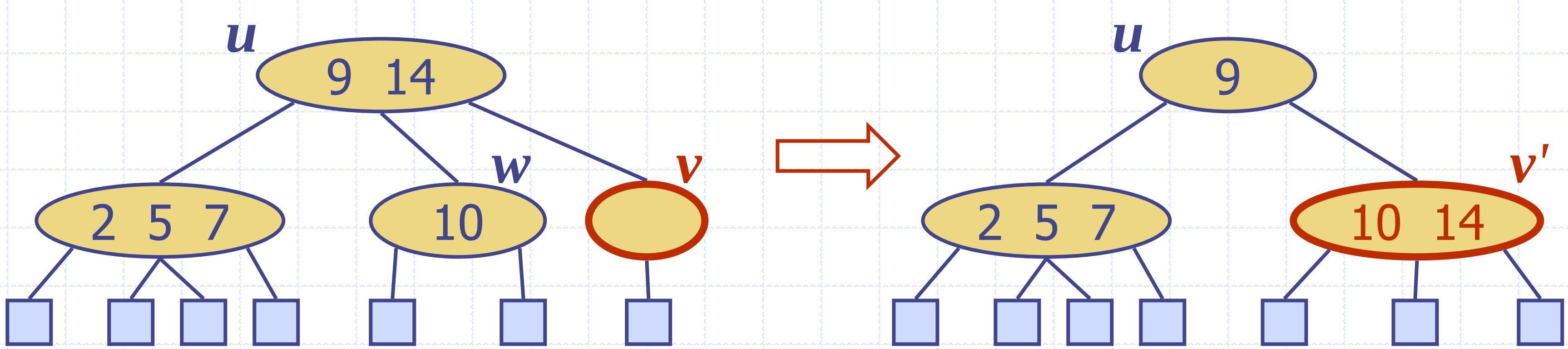
Deletion

- We reduce deletion of an entry to the case where the item is at the node with leaf children
- Otherwise, we replace the entry with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter entry
- Example: to delete key 24, we replace it with 27 (inorder successor)



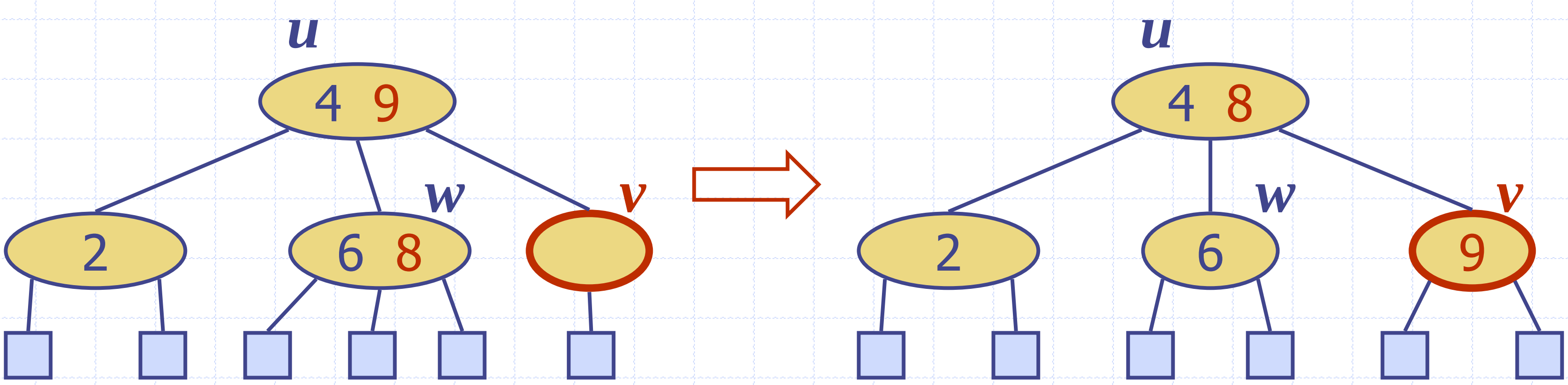
Underflow and Fusion

- Deleting an entry from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Underflow and Transfer

- To handle an underflow at node ***v*** with parent ***u***, we consider two cases
- Case 2: an adjacent sibling ***w*** of ***v*** is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of ***w*** to ***v***
    2. we move an item from ***u*** to ***v***
    3. we move an item from ***w*** to ***u***
  - After a transfer, no underflow occurs



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the entry
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

Comparison of Map Implementations

	Find	Put	Erase	Notes
Hash Table	1 expected	1 expected	1 expected	○ no ordered map methods ○ simple to implement
Skip List	log ***n*** high prob.	log ***n*** high prob.	log ***n*** high prob.	○ randomized insertion ○ simple to implement
AVL and (2,4) Tree	log ***n*** worst-case	log ***n*** worst-case	log ***n*** worst-case	○ complex to implement