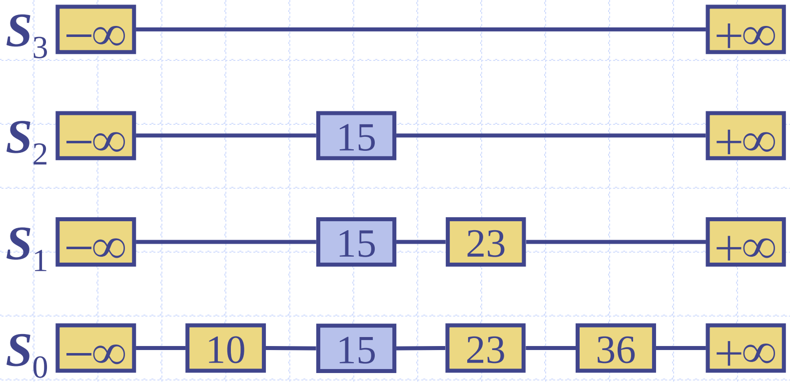
Ordered Maps and Skip Lists

Sections 9.3 – 9.4



Ordered Maps

- Ordered maps function as normal maps but also provides access to the order relationship on keys.
- They allow one to look up elements in the map based on the ordering. For example, one can find the element with the smallest key greater than some given key.
- The ordering relationship is often defined by a comparator for the keys, which can be provided when the map is created.
- For dealing with queries for nonexistent elements, there is a special sentinel entry called end.

The Ordered Map ADT

- Has all members that the Map ADT has, plus:
  - firstEntry(): Return an iterator to the entry with the smallest key value; if the map is empty, it returns end.
  - lastEntry(): Return an iterator to the entry with the largest key value; if the map is empty, it returns end.
  - ceilingEntry(k): Return an iterator to the entry with the least key value greater than or equal to k; if there is no such entry, it returns end.
  - floorEntry(k): Return an iterator to the entry with the greatest key value less than or equal to k; if there is no such entry, it returns end.
  - higherEntry(k): Return an iterator the entry with the least key value greater than k; if there is no such entry, it returns end.
  - lowerEntry(k): Return an iterator to the entry with the greatest key value less than k; if there is no such entry, it returns end.

Implementing Ordered Map

- Consider storing the Ordered Map's entries as an Array List where the entries are in sorted order from smallest to largest. This is called an ordered search table.

	find(k)	**put(k,v)**	**erase(k)**	**erase(p)**
hash table	O(1) expected*, O(n) worst	O(1) exp.* O(n)	O(1) exp.* O(n)	O(1)
unordered linked list	O(n)	O(1)	O(n)	O(1)
ordered search table	O(log n)	O(n)	O(n)	O(n)

*depends on load factor

Binary Search

- find(k) in an ordered search table is implemented by binary search.
- Example: search for 30 in the search table below.

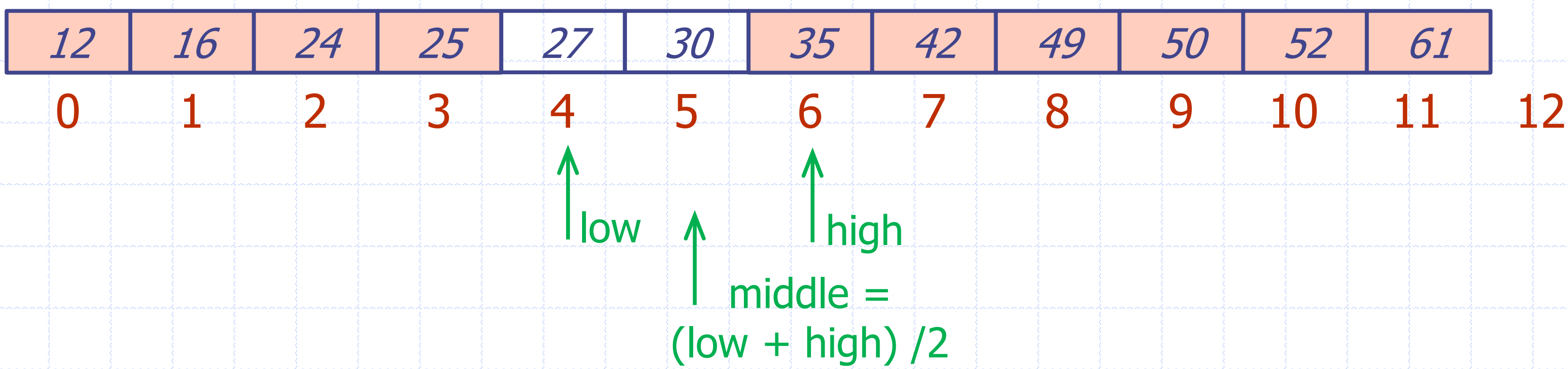
12	16	24	25	27	30	35	42	49	50	52	61	
0	1	2	3	4	5	6	7	8	9	10	11	12
low						middle = (low + high) /2						high

Compare 30 to A[middle]: 30 < 35, so move high to middle

12	16	24	25	27	30	35	42	49	50	52	61	
0	1	2	3	4	5	6	7	8	9	10	11	12
low			middle = (low + high) /2			high						

Binary Search

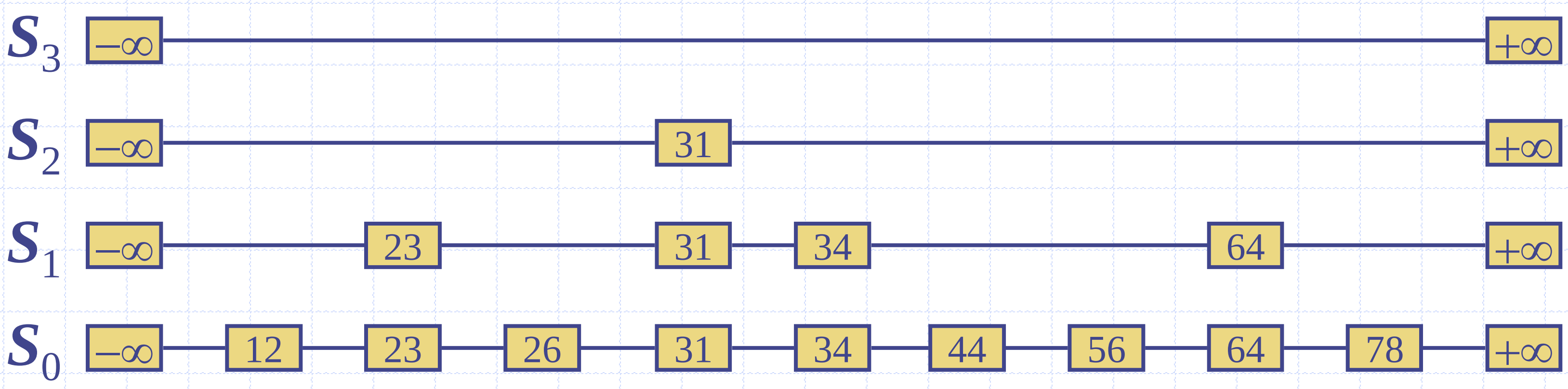
Compare 30 to A[middle]: 30 ≥ 25, so move low to middle + 1



Compare 30 to A[middle]: 30 = 30, so target is found.

Skip Lists

- A skip list for a set S of distinct (key, element) items is a series of lists $S_0, S_1, \ldots, S_h$ such that
  - Each list S_i contains the special keys $+\infty$ and $-\infty$
  - List S_0 contains the keys of S in nondecreasing order
  - Each list is a subsequence of the previous one, i.e.,
    $$S_0 \supseteq S_1 \supseteq \ldots \supseteq S_h$$
  - List S_h contains only the two special keys
- We can use a skip list to implement the Ordered Map ADT



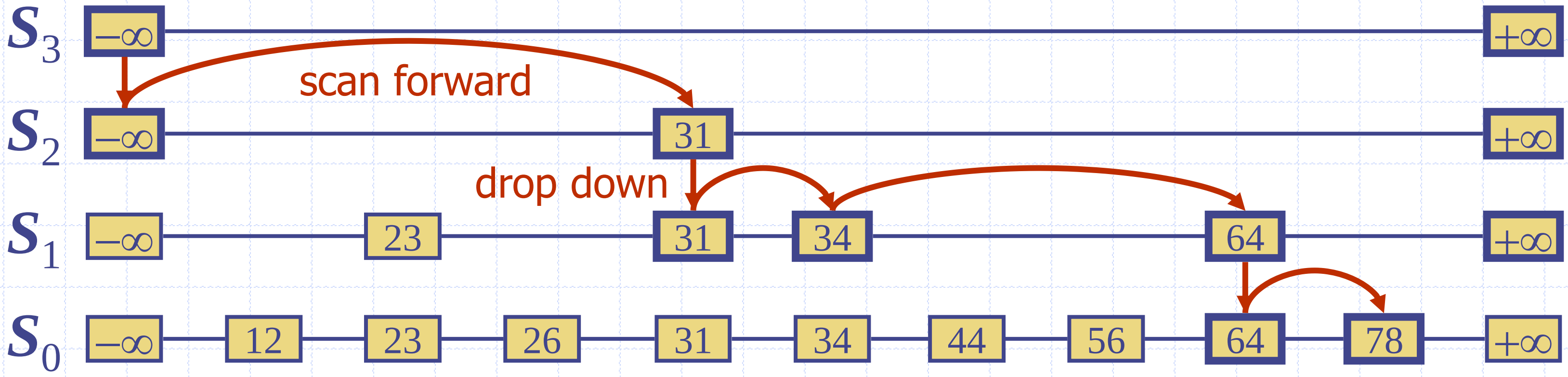
Search

- We search for a key x in a a skip list as follows:
  - We start at the first position of the top list
  - At the current position p, we compare x with $y \leftarrow key(next(p))$

    $x = y$: we return $element(next(p))$

    $x > y$: we "scan forward"

    $x < y$: we "drop down"
  - If we try to drop down past the bottom list, we return ***null***
- Example: search for 78



Randomized Algorithms

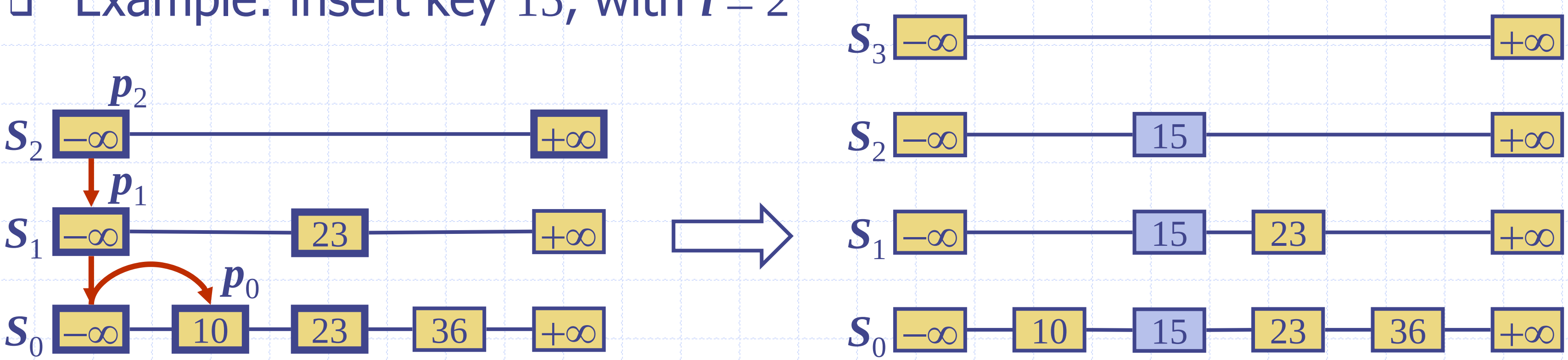
- A randomized algorithm performs coin tosses (i.e., uses random bits) to control its execution
- It contains statements of the type

  $b \leftarrow random()$
  if $b = 0$
  do A …
  else { $b = 1$}
  do B …
- Its running time depends on the outcomes of the coin tosses

- We analyze the expected running time of a randomized algorithm under the following assumptions
  - the coins are unbiased, and
  - the coin tosses are independent
- The worst-case running time of a randomized algorithm is often large but has very low probability (e.g., it occurs when all the coin tosses give "heads")
- We use a randomized algorithm to insert items into a skip list

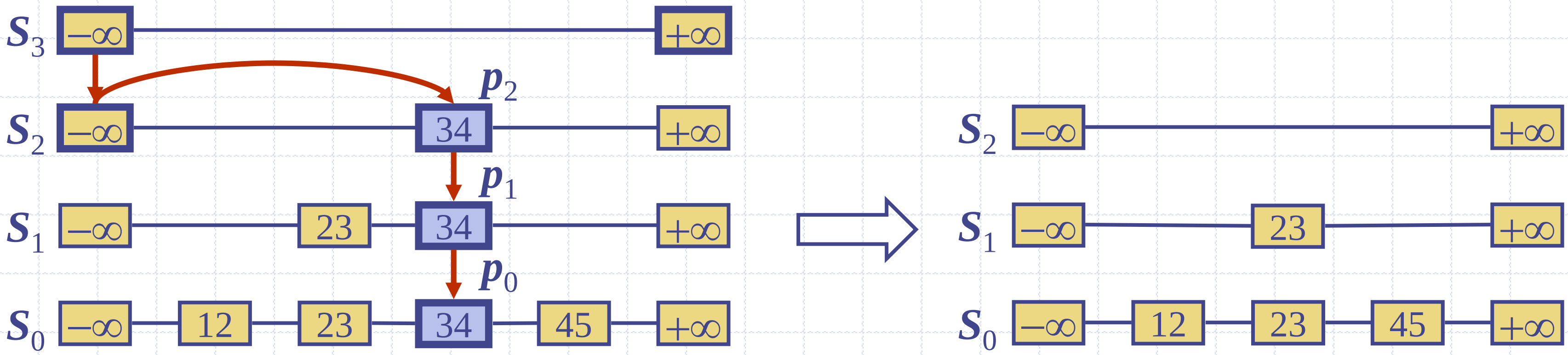
Insertion

- To insert an entry (x, o) into a skip list, we use a randomized algorithm:
  - We repeatedly toss a coin until we get tails, and we denote with i the number of times the coin came up heads
  - If $i \geq h$, we add to the skip list new lists $S_{h+1}, \ldots, S_{i+1}$, each containing only the two special keys
  - We search for x in the skip list and find the positions $p_0, p_1, \ldots, p_i$ of the items with largest key less than x in each list $S_0, S_1, \ldots, S_i$
  - For $j \leftarrow 0, \ldots, i$, we insert item (x, o) into list S_j after position p_j
- Example: insert key 15, with $i = 2$



Deletion

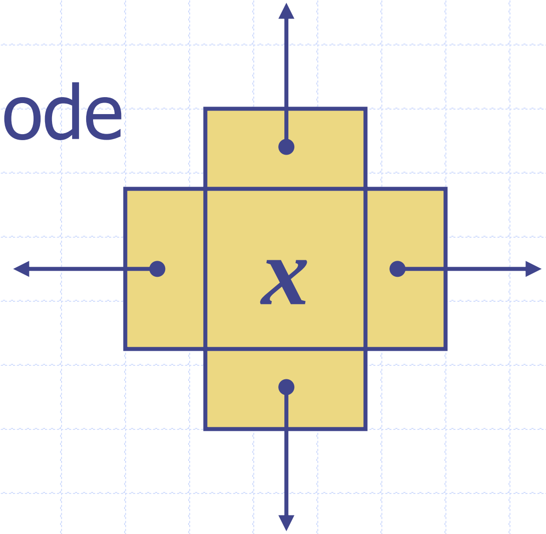
- To remove an entry with key x from a skip list, we proceed as follows:
  - We search for x in the skip list and find the positions $p_0, p_1, \ldots, p_i$ of the items with key x, where position p_j is in list S_j
  - We remove positions $p_0, p_1, \ldots, p_i$ from the lists $S_0, S_1, \ldots, S_i$
  - We remove all but one list containing only the two special keys
- Example: remove key 34



Implementation

- We can implement a skip list with quad-nodes
- A quad-node stores:
  - entry
  - link to the node prev
  - link to the node next
  - link to the node below
  - link to the node above
- Also, we define special keys PLUS_INF and MINUS_INF, and we modify the key comparator to handle them

quad-node



Space Usage

- The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
- We use the following two basic probabilistic facts:

  Fact 1: The probability of getting i consecutive heads when flipping a coin is $1/2^i$

  Fact 2: If each of n entries is present in a set with probability p, the expected size of the set is np

- Consider a skip list with n entries
  - By Fact 1, we insert an entry in list S_i with probability $1/2^i$
  - By Fact 2, the expected size of list S_i is $n/2^i$
- The expected number of nodes used by the skip list is

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i} < 2n$$

- Thus, the expected space usage of a skip list with n items is $O(n)$

Height

- The running time of the search and insertion algorithms is affected by the height h of the skip list
- We show that with high probability, a skip list with n items has height $O(\log n)$
- We use the following additional probabilistic fact:

  Fact 3: If each of n events has probability p, the probability that at least one event occurs is at most np

- Consider a skip list with n entires
  - By Fact 1, we insert an entry in list S_i with probability $1/2^i$
  - By Fact 3, the probability that list S_i has at least one item is at most $n/2^i$
- By picking $i = 3\log n$, we have that the probability that $S_{3\log n}$ has at least one entry is at most

  $$n/2^{3\log n} = n/n^3 = 1/n^2$$

- Thus a skip list with n entries has height at most $3\log n$ with probability at least $1 - 1/n^2$

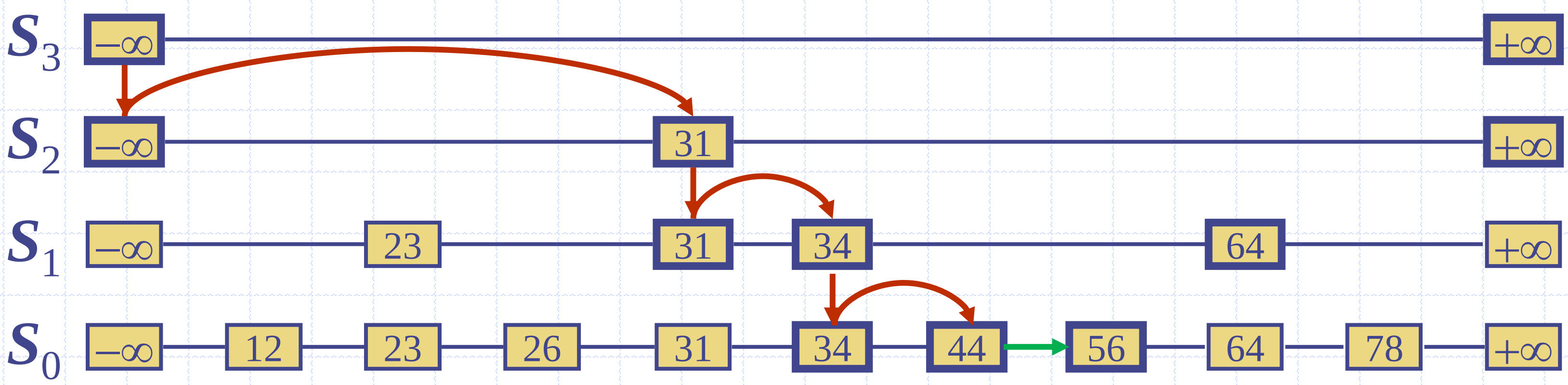
Search and Update Times

- The search time in a skip list is proportional to
  - the number of drop-down steps, plus
  - the number of scan-forward steps
- The drop-down steps are bounded by the height of the skip list and thus are $O(\log n)$ with high probability
- To analyze the scan-forward steps, we use yet another probabilistic fact:

  Fact 4: The expected number of coin tosses required in order to get heads is 2
- When we scan forward in a list, the destination key does not belong to a higher list
  - A scan-forward step is associated with a former coin toss that gave tails
- By Fact 4, in each list the expected number of scan-forward steps is 1
- Thus, the expected number of scan-forward steps is $O(\log n)$
- We conclude that a search in a skip list takes $O(\log n)$ expected time
- The analysis of insertion and deletion gives similar results

Implementing Ordered Map

- The (unordered) map functions are implemented as we've seen.
- firstEntry() and lastEntry() can be found in O(1) time from the sentinels of S_0.
- The other OrderedMap functions are easily implemented by starting with a search for the given key k.
- For example, ceilingEntry(50) on our sample skip list searches for 50. It ends the search at the 44 node on S_0 with a *not found* result; the ceiling entry (56) is the next node from the search's end.



Summary

- A skip list is a data structure for ordered maps that uses a randomized insertion algorithm
- In a skip list with n entries
  - The expected space used is $O(n)$
  - The expected search, insertion and deletion time is $O(\log n)$
- Using a more complex probabilistic analysis, one can show that these performance bounds also hold with high probability
- Skip lists are fast and simple to implement in practice