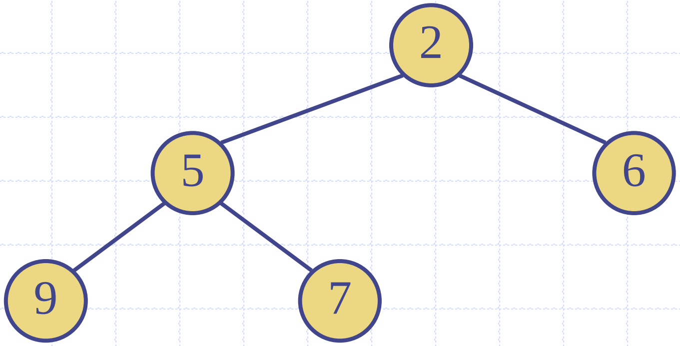
Heaps

Section 8.3



Recall Priority Queue ADT

- A priority queue stores a collection of entries
- Typically, an entry is a pair (key, value), where the key indicates the priority
- Methods of the Priority Queue ADT
  - insert(e) inserts an entry e
  - removeMin() removes the entry with smallest key (the one that would be returned by min)
  - min() returns, but does not remove, an entry with smallest key
  - size(), empty()

Recall PQ Sorting

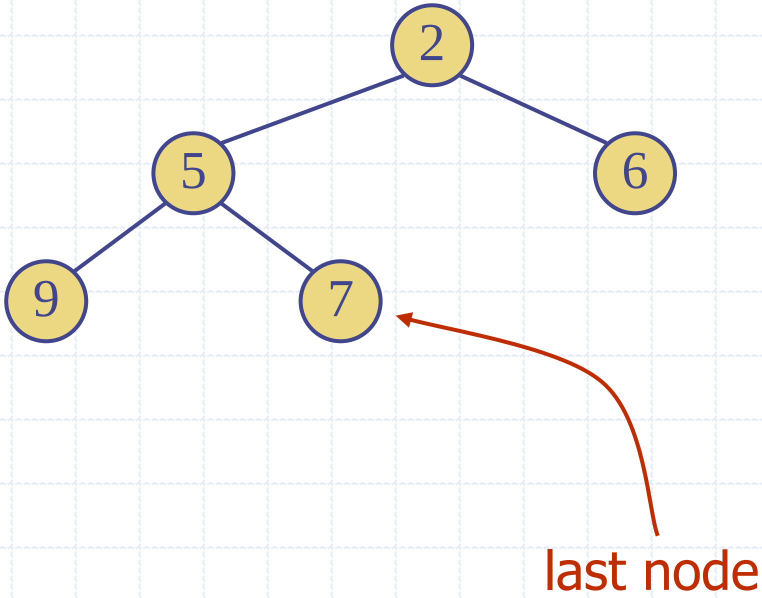
- We use a priority queue
  - Insert the elements with a series of insert operations
  - Remove the elements in sorted order with a series of removeMin operations
- The running time depends on the priority queue implementation:
  - Unsorted sequence gives selection-sort: $O(n^2)$ time
  - Sorted sequence gives insertion-sort: $O(n^2)$ time
- **Can we do better?**

Algorithm *PQ-Sort*(*S*, *C*)
    Input sequence ***S***, comparator ***C*** for the elements of ***S***
    Output sequence ***S*** sorted in increasing order according to ***C***
    P ← priority queue with comparator ***C***
    while ¬*S.empty* ()
        e ← *S.front*();
        S.eraseFront()
        P.insert (*e*)
    while ¬*P.empty*()
        e ← *P.min*(); *P.removeMin*()
        S.insertBack(*e*)

Heaps

- A heap is a binary tree storing keys at its nodes and satisfying the following properties:
- Heap-Order Property: for every internal node v other than the root, $\boldsymbol{key}(\boldsymbol{v}) \geq \boldsymbol{key}(\boldsymbol{parent}(\boldsymbol{v}))$ (this is a min-heap; there are also max-heaps.).
- Complete Binary Tree: let $\boldsymbol{h}$ be the height of the heap
  - for $\boldsymbol{i} = 0, \ldots, \boldsymbol{h} - 1$, there are 2^i nodes of depth $\boldsymbol{i}$
  - at depth $\boldsymbol{h}$, the leaves are as far to the left as possible

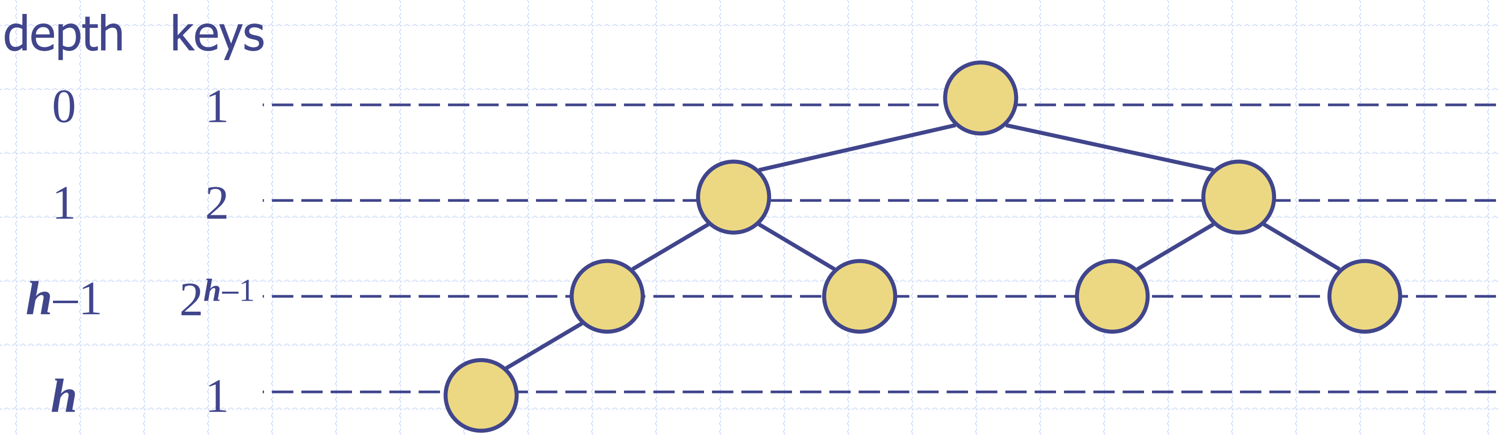
- The last node of a heap is the rightmost node of maximum depth



Height of a Heap

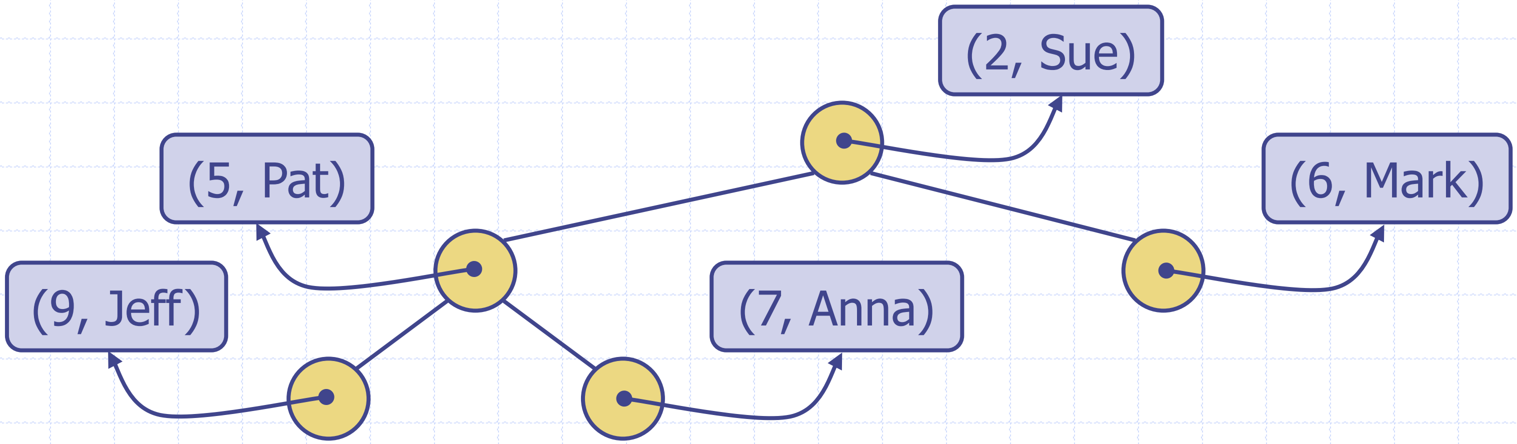
- **Theorem:** A heap storing n keys has height $O(\log n)$

  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h-1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
  - Thus, $n \geq 2^h$, i.e., $h \leq \log n$



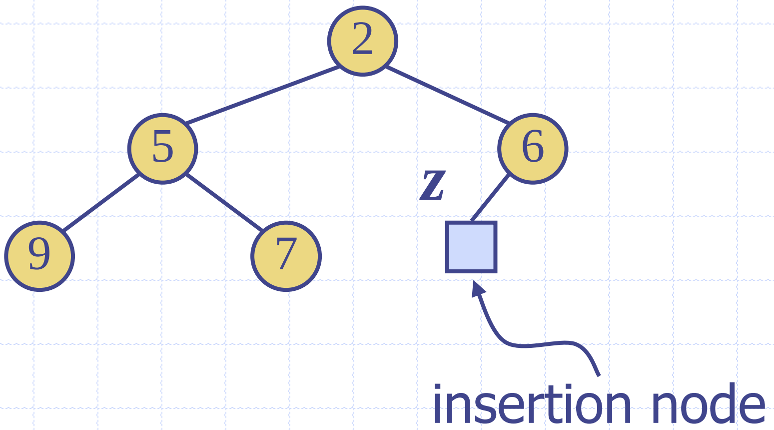
Heaps and Priority Queues

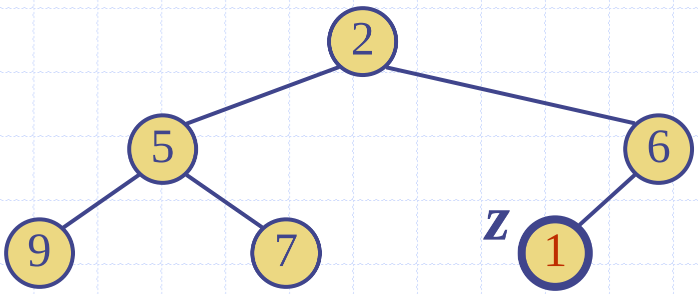
- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node



Insertion into a Heap

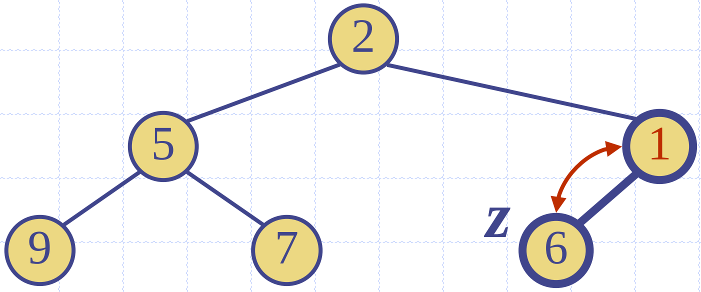
- Method insertItem of the priority queue ADT corresponds to the insertion of a key k to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z
  - Restore the heap-order property (discussed next)

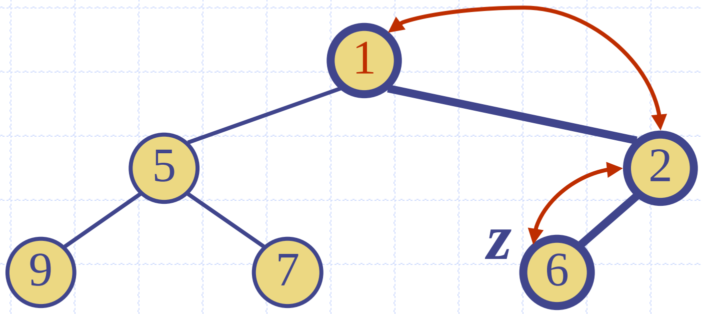




Upheap

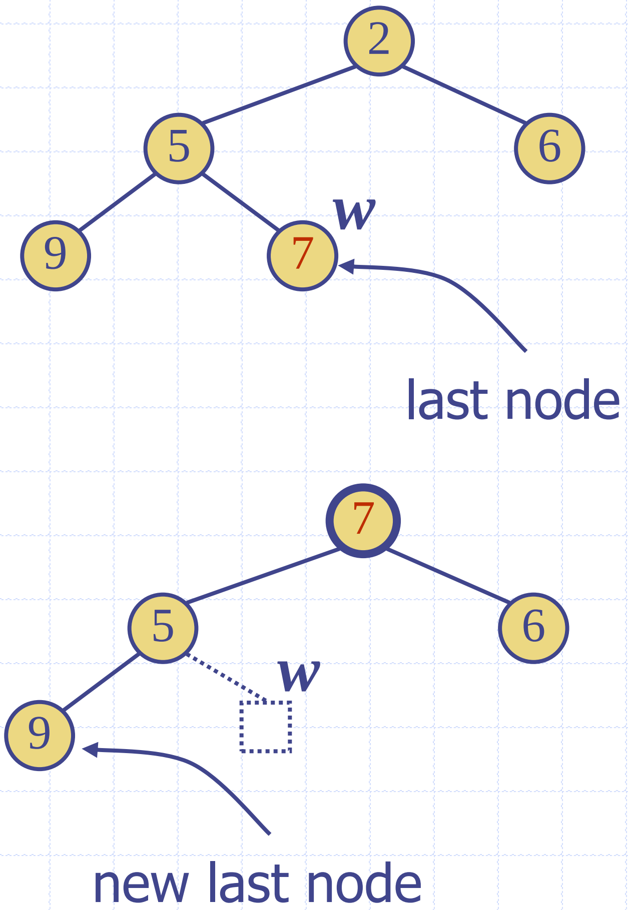
- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time





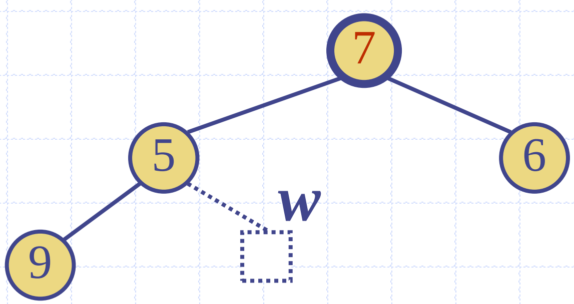
Removal from a Heap (§ 8.3.3)

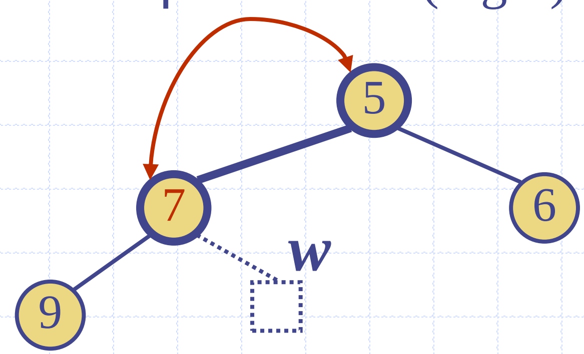
- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node *w*
  - Remove *w*
  - Restore the heap-order property (discussed next)



Downheap

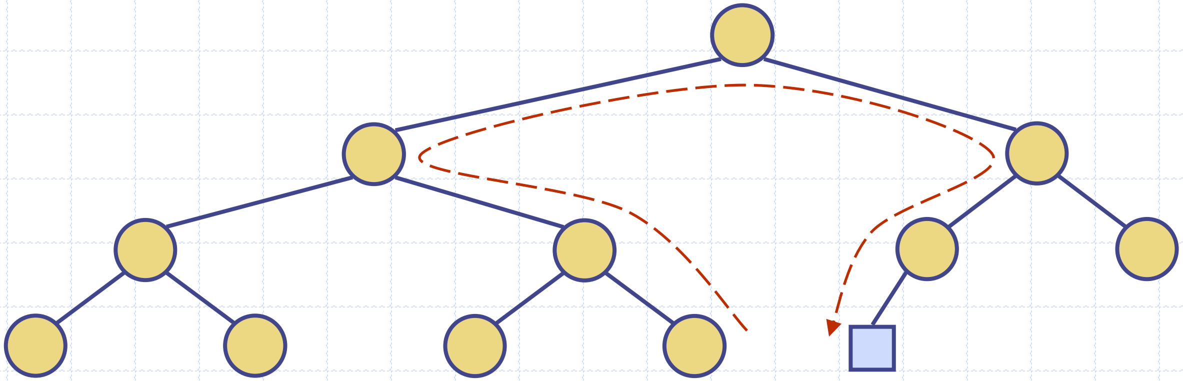
- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- If k is larger than any of its children, swap k with its smallest child.
- Downheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



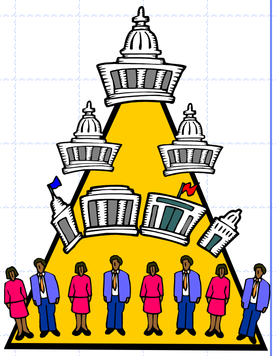


Updating the Last Node

- The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal



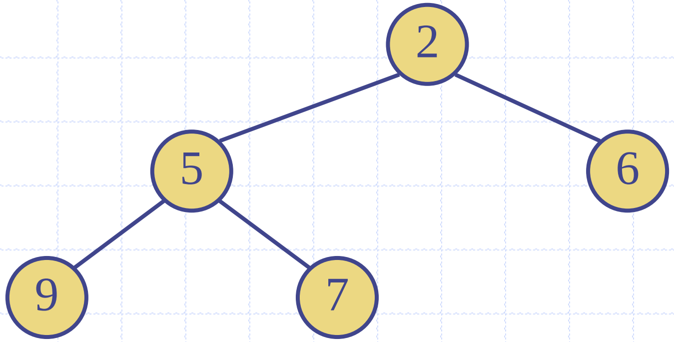
Heap-Sort

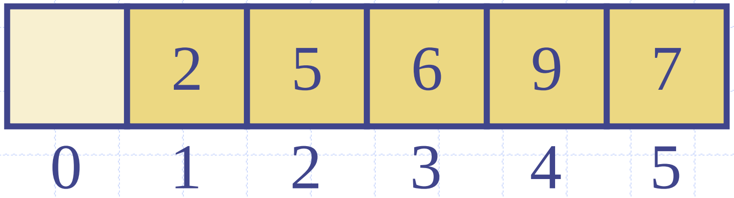


- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, empty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Vector-based Heap Implementation

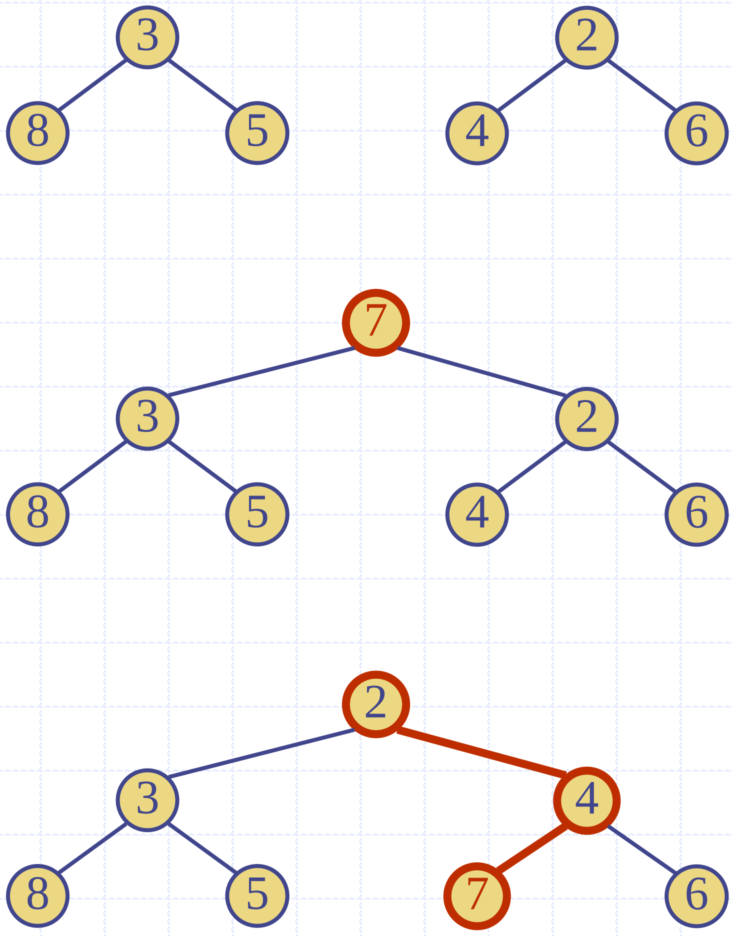
- We can represent a heap with n keys by means of a vector of length $n + 1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Links between nodes are not explicitly stored
- The cell of at rank 0 is not used
- Operation insert corresponds to inserting at rank $n + 1$
- Operation removeMin corresponds to removing at rank n
- Yields in-place heap-sort





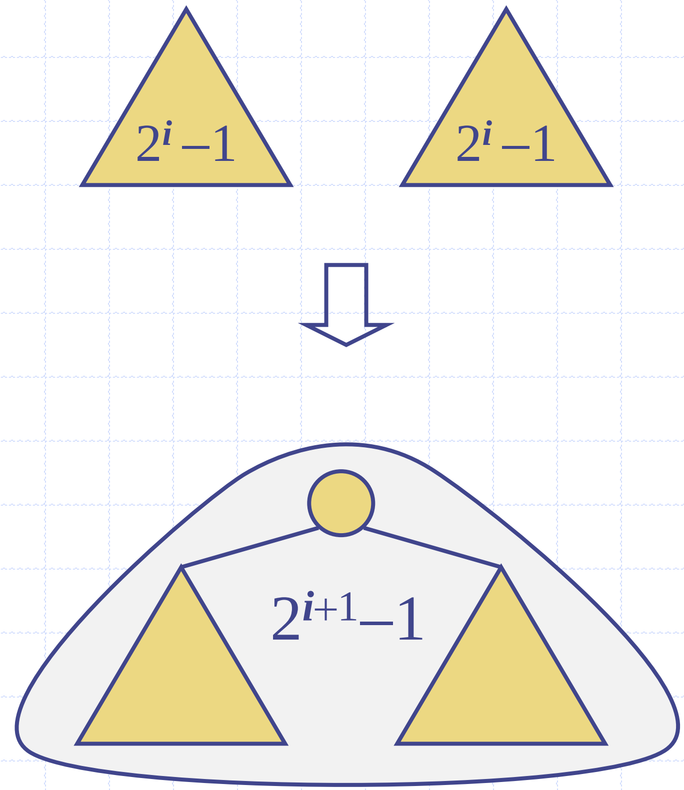
Merging Two Heaps

- We are given two heaps of the same size $2^{h+1} - 1$ and a key k
- We create a new heap with the root node storing k and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

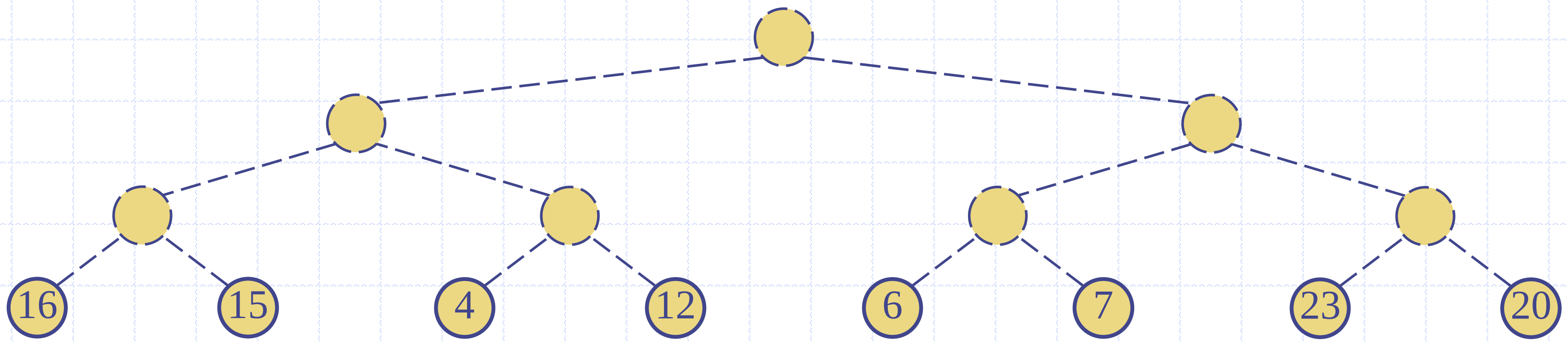


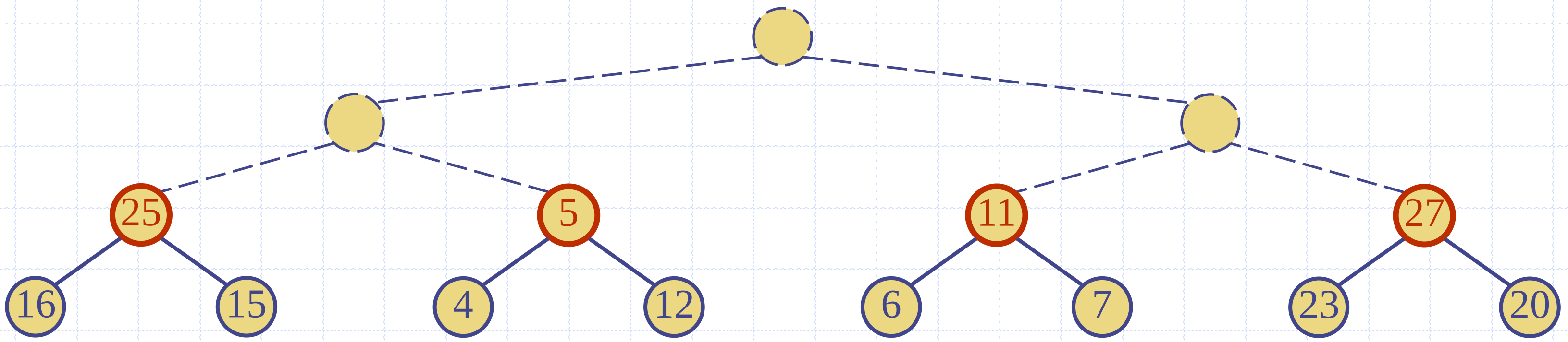
Bottom-up Heap Construction

- We can construct a heap storing n given keys in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys

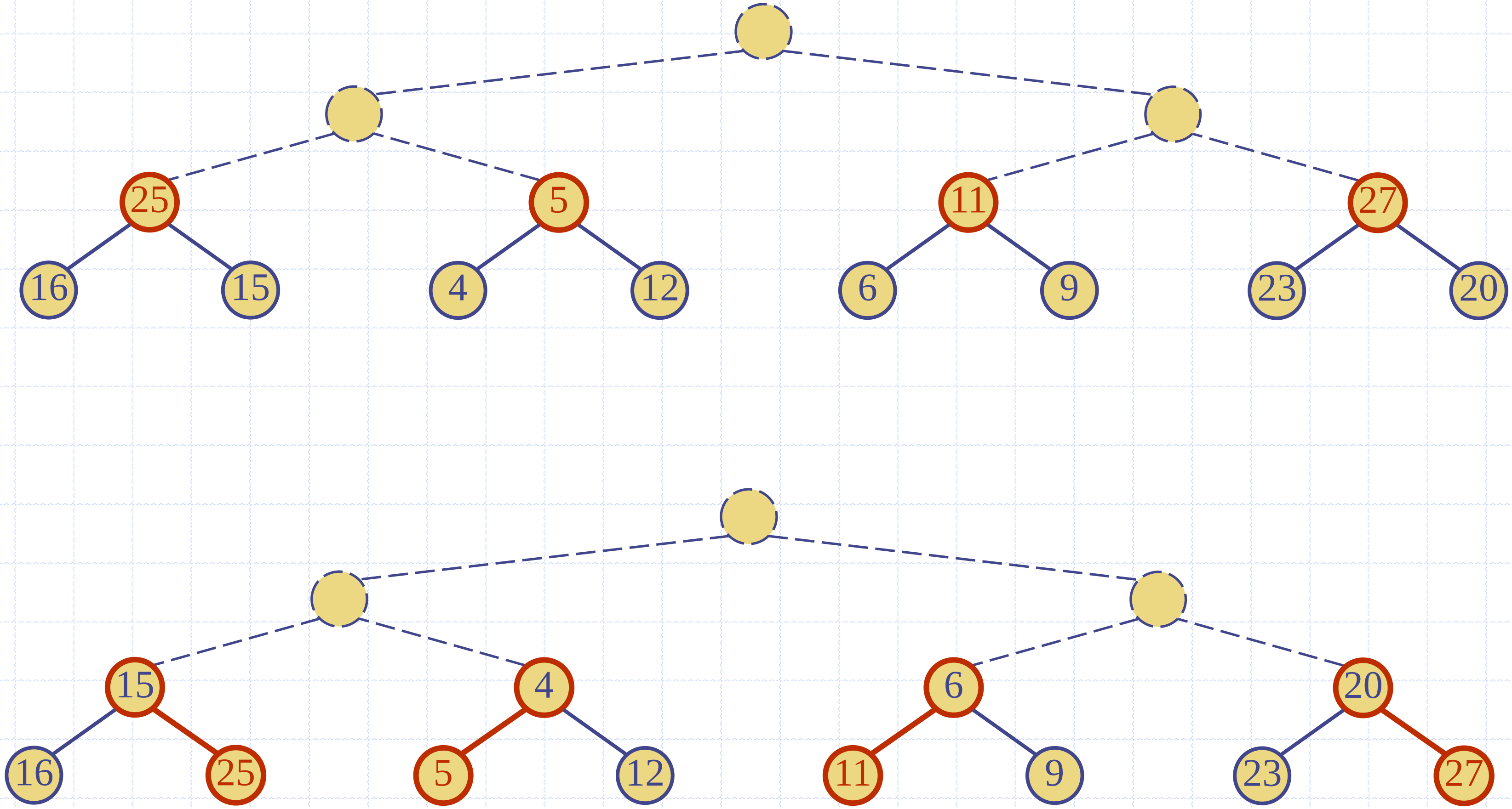


Example

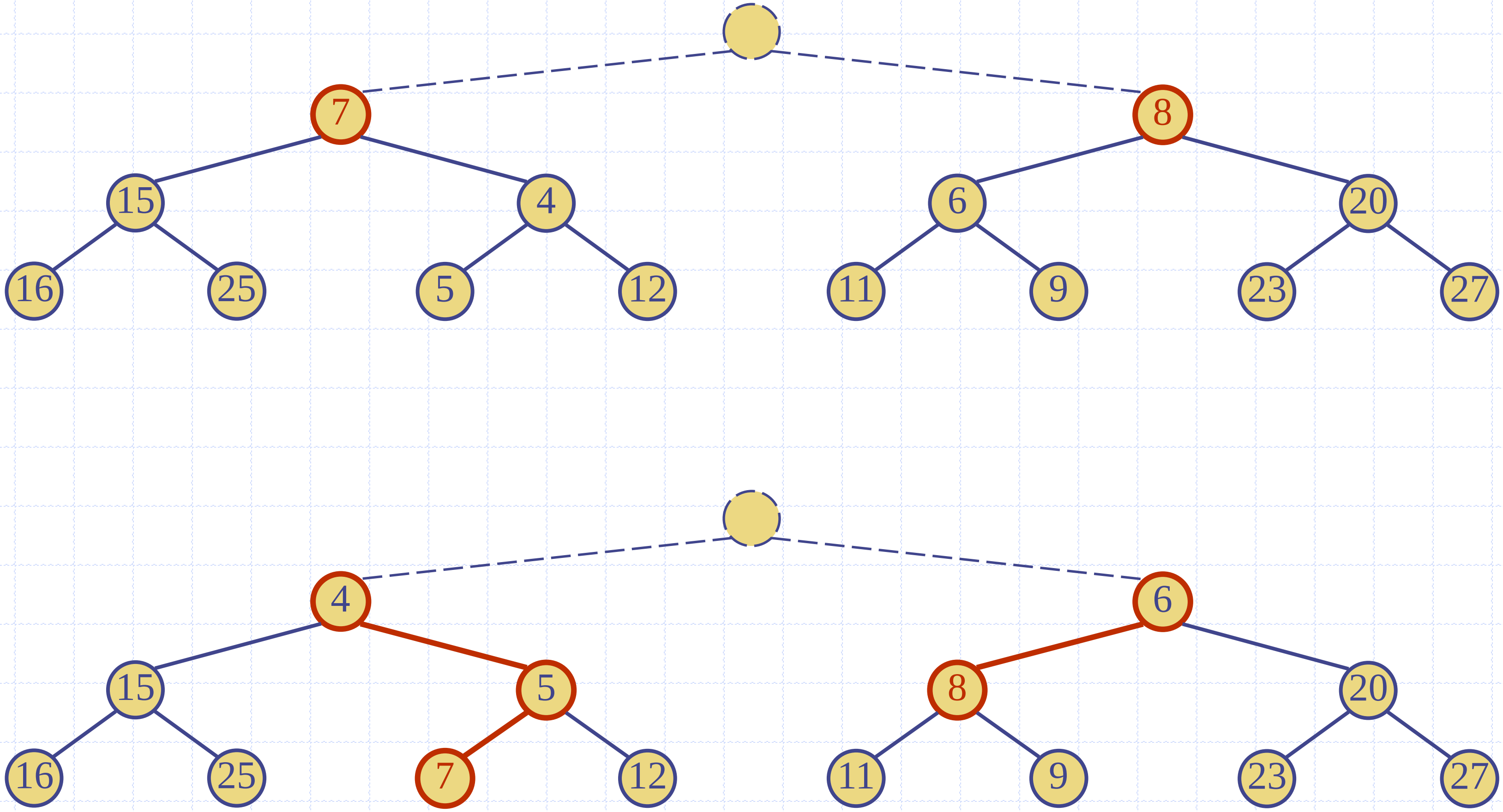




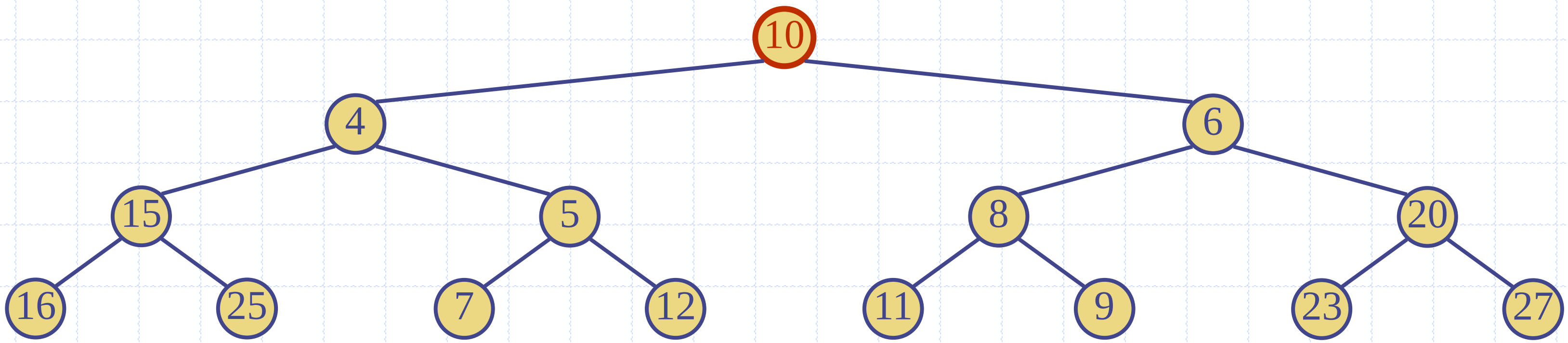
Example (contd.)

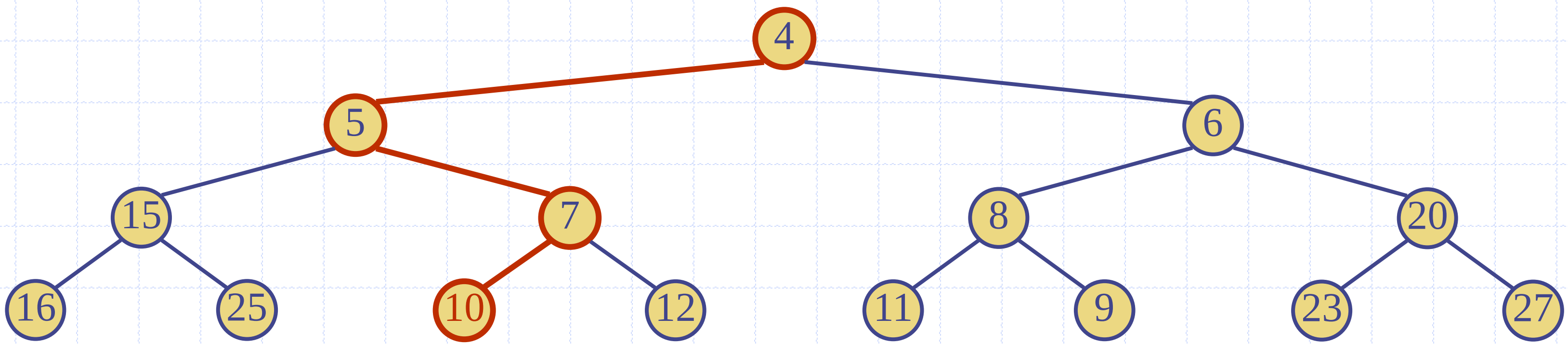


Example (contd.)

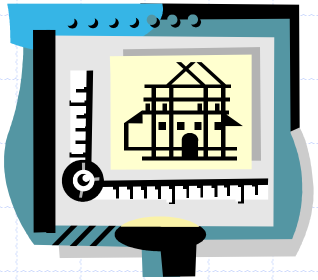


Example (end)

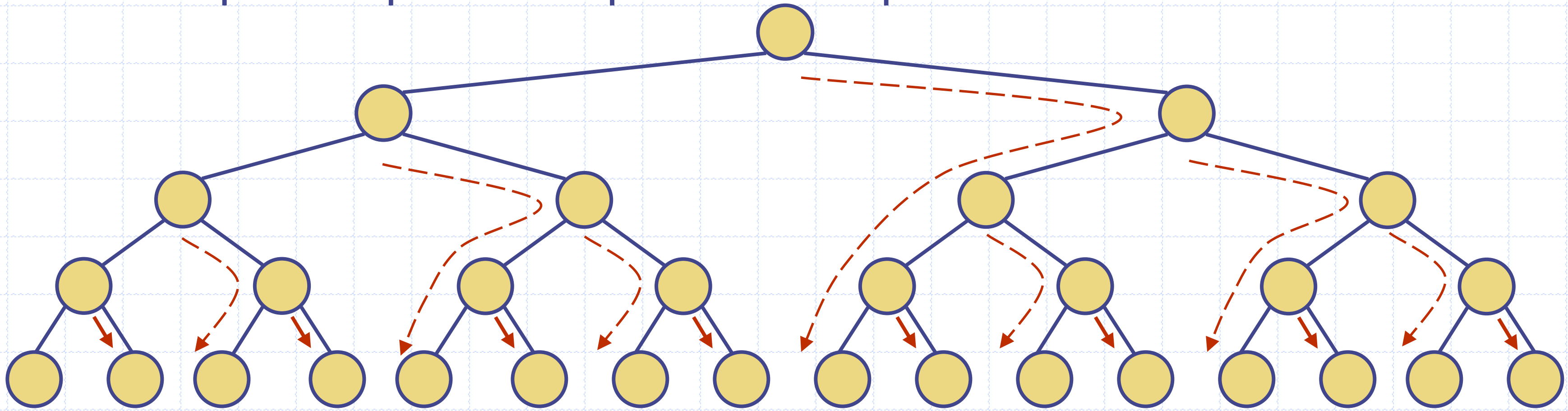




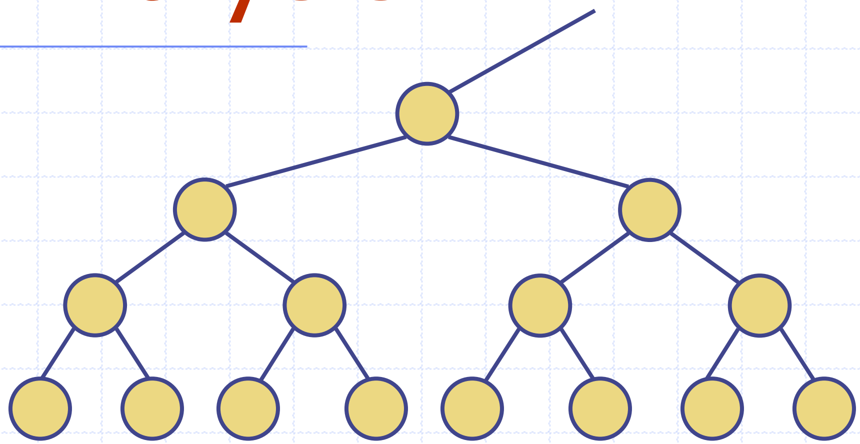
Analysis



- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort



Analysis



…

n/16 nodes * 4 units of work / node

n/8 nodes * 3 units of work / node

n/4 nodes * 2 units of work / node

n/2 nodes * 1 unit of work / node

- Total work $\leq \frac{n}{2}1 + \frac{n}{4}2 + \frac{n}{8}3 + \frac{n}{16}4 + \frac{n}{32}5 + \frac{n}{64}6 + \ \ldots$

$$\leq n(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \ \ldots\ )$$

$$\leq n(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \ \ldots\ )$$

$$= n\left(\frac{1}{2} + 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ \ldots\right)\right)$$

$$\leq n\left(\frac{1}{2} + 2(1)\right) = 2.5n$$