A simplified set of equations of motion for the Earth and an orbiting satellite is given by

\[
\ddot{r} = r \dot{\theta}^2 - \frac{k}{r^2} + u_1, \\
\dot{\theta} = -\frac{2r \dot{r} \dot{\theta}}{r} + \frac{1}{r} u_2,
\]

where \( r \) represents the Earth-satellite distance measured from their centres, and \( \theta \) represents the phase of the orbit. \( k \) is a positive constant.

a) Derive a state space model of this system in the form of a first-order ordinary differential equation.

Let \( x_1 = r, x_2 = \dot{r}, x_3 = \theta, x_4 = \dot{\theta} \). Then we have

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{2x_2 x_4}{x_1} + \frac{u_2}{x_1}
\end{align*}
\]

b) What are the equilibrium points of the state space model, under zero control input, \( u_1 = u_2 = 0 \)? Give a physical interpretation of the result.

Setting \( u_1 = u_2 = 0 \), we have

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 x_4^2 - \frac{k}{x_1^2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{2x_2 x_4}{x_1}
\end{align*}
\]

Equilibrium points are given by \( \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0 \). From \( \dot{x}_1 = \dot{x}_3 = 0 \), we get \( x_2 = x_4 = 0 \).

From \( \dot{x}_2 = 0 \) and \( x_4 = 0 \), the second component of the dynamics becomes

\[
\dot{x}_2 = \frac{k}{x_1^2},
\]

which is never zero for any positive \( k \).

Therefore, under zero control input, the system does not have any equilibrium points.

This is because the \( \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0 \) means that the satellite is staying a fixed distance from earth and is not rotating. However, to keep such a configuration, some thrust input is needed to balance out gravity.
c) What are the equilibrium points of the state space model, under zero control input, \( u_1 = \frac{k}{r^2}, u_2 = 0 \)? Give a physical interpretation of this control set point and of the equilibrium points.

Setting \( u_1 = \frac{k}{x_1^2}, u_2 = 0 \), we have

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 x_4^2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{2x_2 x_4}{x_1}
\end{align*}
\]

As before, equilibrium points are given by \( \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0 \). From \( \dot{x}_1 = \dot{x}_3 = 0 \), we get \( x_2 = x_4 = 0 \). Next, observe that \( x_2 = x_4 = 0 \) implies \( \dot{x}_2 = \dot{x}_4 = 0 \) for any \( x_1 \) and \( x_3 \).

The control set point represents thrust input that balances out gravity. As long as gravity can be counteracted, the system would be able to remain at any distance \( x_1 \) from the earth, at any phase angle \( x_3 \).

d) Linearize the model with respect to a reference orbit given by \( r(t) \equiv \rho, \theta(t) = \omega t, u_1 = u_2 = 0 \)

Based on the system dynamics

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_1 x_4^2 - \frac{k}{x_1^3} + u_1, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -\frac{2x_2 x_4}{x_1} + \frac{u_2}{x_1}
\end{align*}
\]

we can compute the Jacobians:

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\
\frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & 0 \\
x_2^2 + \frac{2k}{x_1^3} & 0 & 0 & 2x_1 x_4 \\
0 & 0 & 0 & 1 \\
-\frac{2x_2 x_4}{x_1^2} & \frac{u_2}{x_1^2} & \frac{2x_4}{x_1} & -\frac{2x_2}{x_1}
\end{bmatrix}
\]
$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{x_1} \end{bmatrix}$$

The set point implies $x_1 = \rho, x_2 = 0, x_3 = \omega t, x_4 = \omega, u_1 = u_2 = 0$. Evaluating at the set point, we have

$$\frac{\partial f}{\partial x} \bigg|_{x_1=\rho, x_2=0, x_3=\omega t, x_4=\omega, u_1=u_2=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 + \frac{2k}{\rho^3} & 0 & 0 & 2\rho \omega \\ 0 & 0 & 0 & 1 \\ 0 & \frac{2\omega}{\rho} & 0 & 0 \end{bmatrix}$$

We can make a further simplification by observing that since $x_2 \equiv 0$, we also have $\dot{x}_2 = 0$, so

$$x_1 x_4^2 - \frac{k}{x_1^2} + u_1 = 0$$

$$x_3^2 x_4^2 = k$$

$$\omega^2 = \frac{k}{\rho^3}$$

Now, substituting $2\omega^2$ for $\frac{2k}{\rho^3}$ in $\frac{\partial f_2}{\partial x_1}$, we get

$$\frac{\partial f}{\partial x} \bigg|_{x_1=\rho, x_2=0, x_3=\omega t, x_4=\omega, u_1=u_2=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\rho \omega \\ 0 & 0 & 0 & 1 \\ 0 & \frac{2\omega}{\rho} & 0 & 0 \end{bmatrix}$$

Lastly,

$$\frac{\partial f}{\partial u} \bigg|_{x_1=\rho, x_2=0, x_3=\omega t, x_4=\omega, u_1=u_2=0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}$$

So the linearized system is

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\rho \omega \\ 0 & 0 & 0 & 1 \\ 0 & \frac{2\omega}{\rho} & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \bar{u},$$
where \( \ddot{x} = \begin{bmatrix} \dot{x}_1 - \rho \\ \dot{x}_2 \\ \dot{x}_3 - \omega t \\ \dot{x}_4 - \omega \end{bmatrix} \), and \( \ddot{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \).

2. Given the system \( \dot{x} = Ax + Bu \), with \( A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \), \( B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).
   
   a) Is the system controllable?

   The controllability matrix is given by \( \mathcal{C} = [B \ AB] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \), which is full rank. Therefore, the system is controllable.

   b) Construct a linear state feedback controller so that the closed loop system is stable.

   Preliminaries

   First, we compute the eigenvalues of \( A \) so that we can obtain the characteristic equation.

   \[
   \det(sI - A) = \det \begin{bmatrix} s - 1 & 0 \\ -1 & s + 2 \end{bmatrix}
   \]

   \[
   0 = (s - 1)(s + 2)
   \]

   Characteristic polynomial: \( 0 = s^2 + s - 2 \)

   Current eigenvalues assuming zero input \( (u(t) \equiv 0) \): \( s = 1, -1 \)

   This means that the zero-input system is not stable.

   Method 1: Controllable canonical form

   We look to transform the system into controllable canonical form, with

   \[
   \bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix},
   \]

   where based on the characteristic polynomial, \( \alpha_0 = -2, \alpha_1 = 1 \). Therefore,

   \[
   T^{-1} = \begin{bmatrix} \alpha_1 & 1 \\ 1 & 0 \end{bmatrix}
   \]

   \[
   = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}
   \]

   \[
   = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}
   \]

   Thus, \( T = \frac{1}{2(1-(-1))} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \).

   With the transformation \( z = Tx (x = T^{-1}z) \), the system dynamics becomes

   \[
   \dot{z} = T\ddot{x} = TAx + TBu
   \]

   \[
   \dot{z} = TAT^{-1}z + TBu
   \]

   \[
   \dot{z} = \bar{A}z + Bu
   \]

   \[
   \bar{B} = TB
   \]
\[
\begin{bmatrix}
1 & -1 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]
as expected.

Using the feedback controller \( u = -Kz \), where \( K = [k_0 \ k_1] \), we have \( \dot{z} = \dot{A}z - \dot{B}Kz = (\bar{A} - \bar{B}\bar{K})z \), or
\[
\dot{z} = \begin{bmatrix}
0 & 1 \\
2 - \bar{k}_0 & -1 - \bar{k}_1
\end{bmatrix} z.
\]
This means that the characteristic equation is
\[
s^2 + (1 + \bar{k}_1)s + (-2 + \bar{k}_0) = 0
\]
To stabilize the system, we require the closed-loop eigenvalues to have negative real parts. One valid choice of eigenvalues is \(-1, -2\), which gives us the following desired characteristic equation:
\[
(s + 1)(s + 2) = 0
\]
\[
s^2 + 3s + 2 = 0
\]
Matching coefficients, we have
\[
1 + \bar{k}_1 = 3 \Rightarrow \bar{k}_1 = 2
\]
\[
-2 + \bar{k}_0 = 2 \Rightarrow \bar{k}_0 = 4
\]
The gain matrix \( \bar{K} \) is applied to the transformed state \( z = T\bar{x} \). In the original states, the feedback controller is \( u = -Kx = -\bar{K}z = -\bar{K}T\bar{x} \), so
\[
K = \bar{K}T
\]
\[
= \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}
\]
\[
= \begin{bmatrix} 2 & 0 \end{bmatrix}
\]

Method 2: Directly

With the controller \( u = -K\bar{x} \), the closed-loop system is \( \dot{x} = (A - BK)x \). To stabilize the system, we need to choose \( K \) so that \( A - BK \) has eigenvalues in the open left half plane. We will pick \(-1, -2\) for the eigenvalues as before.
\[
A - BK = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} k_0 & k_1 \end{bmatrix}
\]
\[
= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} k_0 & k_1 \\ k_0 & k_1 \end{bmatrix}
\]
\[
= \begin{bmatrix} 1 - k_0 & -k_1 \\ 1 - k_0 & -2 - k_1 \end{bmatrix}
\]
The eigenvalues are given by
\[
\det\left(\begin{bmatrix} 1 - k_0 & -k_1 \\ 1 - k_0 & -2 - k_1 \end{bmatrix} - sl\right) = 0
\]
\[
\det\left(\begin{bmatrix}
1 - k_0 - s & -k_1 \\
1 - k_0 & -2 - k_1 - s
\end{bmatrix}\right) = 0
\]

\[s^2 + (k_0 - 1 + 2 + k_1)s + (1 - k_0)(-2 - k_1) + k_1(1 - k_0) = 0\]

\[s^2 + (k_0 + k_1 + 1)s + k_0k_1 + 2k_0 - k_1 - 2 + k_1 - k_0k_1 = 0\]

\[s^2 + (k_0 + k_1 + 1)s + 2k_0 - 2 = 0\]

The desired characteristic equation is \(s^2 + 3s + 2 = 0\). Matching coefficients, we have

\[2k_0 - 2 = 2 \Rightarrow k_0 = 2\]

\[k_0 + k_1 + 1 = 3 \Rightarrow k_1 = 0\]

This is the same as before (of course).

Note that had the \(k_0k_1\) terms not cancel out, the equation may have been much harder to solve in general. In this particular case, it may have been easier to compute \(K\) directly, without transforming to controllable canonical form.

3. In the Rayleigh model of a violin string, the string is represented by a mass \(M\), which vibrates with spring constant \(k\). The bowing action is modeled using a conveyor belt that moves at a constant speed of \(b\). Letting \(x\) be the position of the mass representing the string, the system dynamics are given by

\[M\ddot{x} + F_b(x) + kx = 0,\]

where \(F_b(\cdot)\) models the sticky friction during the bowing of the string. For this question, assume \(M = 3, k = 3\), and

\[F_b(\dot{x}) = \begin{cases} 
-(\dot{x} - b + 2)^2 - 3, & \dot{x} < b, \\
(\dot{x} - b - 2)^2 + 3, & \dot{x} \geq b.
\end{cases}\]

a) Derive a first order ODE model with states \((x, \dot{x})\).

Let \(x_1 = x, x_2 = \dot{x}\), then we have

\[\dot{x}_1 = x_2,\]

\[\dot{x}_2 = -\frac{1}{3}F_b(x_2) - x_1,\]

b) Suppose \(b = 1\), Calculate the equilibrium point in \((x, \dot{x})\) and determine its stability.

Setting \(\dot{x}_1 = 0\), we get \(x_2 = 0\).

First, note that \(x_2 = 0 < b = 1\) means that we are in the region \(x_2 < b\) for the function \(F_b(x_2)\). Setting \(\dot{x}_2 = 0\) and \(x_2 = 0\), we have

\[0 = -\frac{1}{3}F_b(0) - x_1\]

\[x_1 = -\frac{1}{3}F_b(0)\]
\[
= -\frac{1}{3}(-(-b + 2)^2 - 3)
\]

When \( b = 1 \), \( x_1 = \frac{4}{3} \). Summarizing, the equilibrium point is at \( \left( \frac{4}{3}, 0 \right) \).

The dynamics in the region \( x_2 < b \) are \( \dot{x} = f(x) \), where \( f(x) = \left[ \frac{x_2}{3} \left( x_2 - b + 2 \right)^2 + 1 - x_1 \right] \). The Jacobian of \( f \) at \( \left( \frac{4}{3}, 0 \right) \) is

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}_{(x_1=\frac{4}{3},x_2=0)} = \begin{bmatrix}
0 & 1 \\
-1 & \frac{2}{3}(x_2 - b + 2)
\end{bmatrix}_{(x_1=\frac{4}{3},x_2=0)}
\]

For \( b = 1 \), we have

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
0 & 1 \\
-1 & \frac{2}{3}
\end{bmatrix}
\]

Note that this is in controllable canonical form, so we can immediately read off the characteristic equation. The eigenvalues are given by \( s^2 - \frac{2}{3}s + 1 = 0 \), or \( s = \frac{2 \pm \sqrt{4 - 36}}{6} \), a complex conjugate pair with positive real part. Therefore, the equilibrium is an unstable focus.

c) Numerically integrate the ODE using your own implementation of RK4 starting from a few different initial conditions \( (x(0), \dot{x}(0)) \) and intuitively explain the behaviour.

Matlab code:

```matlab
function RK4_demo()

% Number of initial conditions
N = 100;
colors = lines(N);
figure;

for i = 1:N
    % Random initial condition
    x0 = [-1+4.5*rand; -2+3.5*rand];
    % Use RK4 to obtain numerical solution
    [~,x] = RK4(@dxdt, [0 20], x0);
    % Plot trajectory
```
function [t, x] = RK4(dxdt, tspan, x0)
% RK4 implementation

% Inputs:
% dxdt - function handle representing dynamics
% tspan - 2-element vector specifying initial and final time
% x0 - initial condition

% Hard-coded time step size
dt = 0.01;
t = tspan(1):dt:tspan(2);

% Initial conditions
x = nan(2, length(t));
x(:,1) = x0;

% RK4 algorithm
for i = 1:length(t)-1
    k1 = dt*dxdt(t(i), x(:,i));
    k2 = dt*dxdt(t(i)+dt/2, x(:,i) + k1/2);
    k3 = dt*dxdt(t(i)+dt/2, x(:,i) + k2/2);
    k4 = dt*dxdt(t(i)+dt, x(:,i) + k3);
    x(:,i+1) = x(:,i) + 1/6*(k1 + 2*k2 + 2*k3 + k4);
end
end

function xdot = dxdt(~, x)
% Function containing dynamics of the system

xdot = zeros(2,1);
xdot(1) = x(2);
xdot(2) = -1/3*Fb(x(2)) - x(1);
end

function z = Fb(y)
% Sub-routine needed in the dynamics function, for b = 1
if y < 1
    z = -(y+1)^2 - 3;
else
end
$z = (y-3)^2 + 3$;

end

end

Plot:

Observations:

- We see that trajectories spiral away from the point $\left(\frac{4}{3}, 0\right)$; this validates our analysis that $\left(\frac{4}{3}, 0\right)$ is an unstable focus.
- There is a stable limit cycle. The limit cycle indicates that the position of the violin string is cyclical, which agrees with intuition.
- The limit cycle has a flat region at $\dot{x} = 1$. In this portion of the limit cycle, the speed of the string is equal to the speed of the bowing action. This represents the string “sticking” to the bow due to friction.
- When the position of the string gets too large, the elastic force of the string overcomes the friction, and the string “bounces” in a manner similar to a harmonic oscillator (simple mass spring system). This continues until the string once again reaches the same speed as the bowing action.
4. Bifurcations. Consider the planar system with dynamics

\[
\begin{align*}
\dot{x}_1 &= -x_2 + x_1(\mu - x_1^2 - x_2^2), \\
\dot{x}_2 &= x_1 + x_2(\mu - x_1^2 - x_2^2),
\end{align*}
\]

where \(\mu\) is a parameter

a) Derive the polar coordinate \((r, \theta)\) representation of the system dynamics using the relationships

\[
\begin{align*}
x_1 &= r \cos \theta, \\
x_2 &= r \sin \theta.
\end{align*}
\]

Hint: compute \(x_1\dot{x}_1 + x_2\dot{x}_2\) and \(\dot{x}_2 x_1 - \dot{x}_1 x_2\).

First, observe that \(r^2 = x_1^2 + x_2^2\), so

\[
\begin{align*}
\dot{x}_1 &= -x_2 + x_1(\mu - r^2), \\
\dot{x}_2 &= x_1 + x_2(\mu - r^2),
\end{align*}
\]

Using the hint, we have

\[
\begin{align*}
x_1\dot{x}_1 + x_2\dot{x}_2 &= -x_1 x_2 + x_1^2(\mu - r^2) + x_1 x_2 + x_2^2(\mu - r^2) \\
&= x_1^2(\mu - r^2) + x_2^2(\mu - r^2) \\
&= (x_1^2 + x_2^2)(\mu - r^2)
\end{align*}
\]

\[
\begin{align*}
\dot{x}_2 x_1 - \dot{x}_1 x_2 &= x_1^2 + x_1 x_2(\mu - r^2) + x_2^2 - x_1 x_2(\mu - r^2) \\
&= x_1^2 + x_2^2 \\
&= r^2
\end{align*}
\]

Now, using implicit differentiation on \(r^2 = x_1^2 + x_2^2\), we can obtain the dynamics in \(r\):

\[
\begin{align*}
2r\dot{r} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\
\dot{r} &= \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} \\
&= \frac{r^2(\mu - r^2)}{r} \\
&= r(\mu - r^2)
\end{align*}
\]

Next, observe that \(\frac{x_2}{x_1} = \tan \theta\); implicitly differentiating again, we get

\[
\begin{align*}
\frac{\dot{x}_2 x_1 - \dot{x}_1 x_2}{x_1^2} &= \frac{\dot{\theta}}{\cos^2 \theta} \\
\frac{\dot{r}^2}{r^2 \cos^2 \theta} &= \frac{\dot{\theta}}{\cos^2 \theta} \\
\dot{\theta} &= \cos \theta = 1
\end{align*}
\]

Therefore, the dynamics in \((r, \theta)\) are

\[
\dot{r} = r(\mu - r^2)
\]
\[ \dot{\theta} = 1 \]

b) Ignoring the \( \theta \) dynamics, find the equilibrium points as a function of \( \mu \) for the \( r \) component of the system

Equilibrium in the \( r \) component means \( \dot{r} = 0 \). Note that \( \dot{\theta} \) can never be zero; the system simply rotates with a rate of 1. In addition, the \( r \) and \( \theta \) dynamics are decoupled.

\[
\begin{align*}
0 &= \dot{r} \\
&= r(\mu - r^2) \\
&= r(\sqrt{\mu} + r)(\sqrt{\mu} - r) \\
r &= 0, \sqrt{\mu}, -\sqrt{\mu}
\end{align*}
\]

However, since \( r > 0 \), we discard the negative solution, so \( r = 0, \sqrt{\mu} \).

c) Find the branches of bifurcation, and describe and/or draw the behaviour of the system for different cases of the parameter \( \mu \). Take the stability of equilibrium points of the \( r \) subsystem into account.

If \( \mu < 0 \), then \( r = 0 \) is the only equilibrium point. In addition, since \( \mu - r^2 < 0 \), we also have \( \dot{r} = r(\mu - r^2) < 0 \). Therefore, the system spirals inwards towards the origin. One can also see this stability result clearly pictorially by looking at \( \dot{r} \) vs \( r \):

![Graph showing \( \dot{r} \) vs \( r \)]

If \( \mu > 0 \), then the equilibrium points of the \( r \) subsystem are at \( 0, \sqrt{\mu} \). To assess the stability, one can compute \( f'(r) \) and evaluate it at \( 0 \) and \( \sqrt{\mu} \), or plot \( f(r) \) vs. \( r \).

First, let’s compute \( f'(r) \).

\[
f'(r) = \mu - 3r^2
\]

At \( r = 0 \), \( f'(0) = \mu > 0 \), so the equilibrium \( r = 0 \) is unstable.

At \( r = \sqrt{\mu} \),

\[
f'(\sqrt{\mu}) = \mu - 3\mu
\]
\[ = -2\mu < 0, \]
so the equilibrium \( r = \sqrt{\mu} \) is stable. Pictorially on an \( \dot{r} \) vs \( r \) graph, this makes sense.

Therefore, any point on \( r = \sqrt{\mu} \) would orbit at a constant \( r \) with a rotation rate of 1, while starting at any other \( r \) except \( r = 0 \) would result in trajectories that spiral towards \( r = \sqrt{\mu} \).

In the following plots, the black line indicates the origin, \((x_1, x_2) = (0,0)\), and trajectories are plotted in planes of constant \( \mu \) to show the system behaviour as \( \mu \) changes. To improve clarity, each colour represents a different \( \mu \).
5. Consider the linear system $\dot{x} = Ax$, with $A = \begin{bmatrix} 0 & 1 \\ -500 & -501 \end{bmatrix}$.

a) Determine conditions on the time step size that must be satisfied for the forward Euler method to be stable.

First, note that the eigenvalues of $A$ are $\lambda = -1, -500$.

Discretizing $\dot{x}$ with the forward Euler method, we have

$$x^{k+1} = x^k + \Delta t A x^k = (I + \Delta t A) x^k$$

Let $\lambda$ represent an eigenvalue of $A$, then the eigenvalue of $I + \Delta t A$ would be represented by $1 + \lambda \Delta t$. For stability, we need $|1 + \lambda \Delta t| < 1$ for all eigenvalues $\lambda$.

For $\lambda = -1$, we have $|1 - \Delta t| < 1$, so $\Delta t < 2$.

For $\lambda = -500$, we have $|1 - 500 \Delta t| < 1$, so $500 \Delta t < 2$, or $\Delta t < \frac{1}{250}$. Since this is a stricter requirement than $\Delta t < 2$, it takes precedence, and therefore for stability we need $\Delta t < \frac{1}{250}$.

b) Repeat the above two steps for the backward Euler method.
The backward Euler method is given by
\[
x^{k+1} = x^k + \Delta t A x^{k+1}
\]
\[
x^{k+1} - \Delta t A x^{k+1} = x^k
\]
\[
(I - \Delta t A) x^{k+1} = x^k
\]
\[
x^{k+1} = (I - \Delta t A)^{-1} x^k
\]

Now, the eigenvalues of \((I - \Delta t A)^{-1}\) is \((1 - \Delta t \lambda)^{-1}\), where \(\lambda = -1, -500\). Again, for stability we need \(|(1 - \Delta t \lambda)^{-1}| < 1\), or \(|1 - \Delta t \lambda| > 1\).

For \(\lambda = -1\), we have \(|1 + \Delta t| > 1\). This is satisfied by any \(\Delta t > 0\).

For \(\lambda = -500\), we have \(|1 + 500\Delta t| > 1\). This is also satisfied by any \(\Delta t > 0\).

Therefore, our choice of \(\Delta t\) only needs to account for integration error, and does not need to be as small as with forward Euler.

6. Consider the following dynamical system:
\[
\dot{x}_1 = x_1^2 + x_2
\]
\[
\dot{x}_2 = p(x_1, x_2) + q(x_1, x_2) u
\]

Find a state feedback control policy \(u(x_1, x_2)\) such that the origin is asymptotically stable. Prove your result using a Lyapunov function. Hint: Use feedback stabilization.

Treating \(x_2\) as a virtual control in \(\dot{x}_1\), we have
\[
\dot{x}_1 = x_1^2 + u
\]

By inspection, the control policy \(u = -x_1^2 - \bar{k} x_1\) for some \(\bar{k} > 0\) would stabilize the origin. A simple Lyapunov function that proves this is \(\bar{V}(x_1) = \frac{1}{2} x_1^2\).

We know that if \(\dot{x}_2 = u\), then choosing \(u = \dot{x}_1 - \frac{\partial \bar{V}}{\partial x_1} G(x_1) - k z\), where \(z = x_1 - \alpha(x_1)\) would stabilize the origin, with a Lyapunov function given by \(V(x_1, z) = \bar{V}(x_1) + \frac{1}{2} z^2\).

In this case, however, we have \(\dot{x}_2 = p(x_1, x_2) + q(x_1, x_2) u\). Thus, if we set the entire right-hand side to \(\dot{x}_1 - \frac{\partial \bar{V}}{\partial x_1} G(x_1) - k z\), we should have a stabilizing controller:
\[
p(x_1, x_2) + q(x_1, x_2) u = \dot{x}_1 - \frac{\partial \bar{V}}{\partial x_1} G(x_1) - k z
\]
\[
u = \frac{\dot{x}_1 - \frac{\partial \bar{V}}{\partial x_1} G(x_1) - k z - p(x_1, x_2)}{q(x_1, x_2)}
\]

The Lyapunov function would still be \(V(x_1, z) = \bar{V}(x_1) + \frac{1}{2} z^2\).