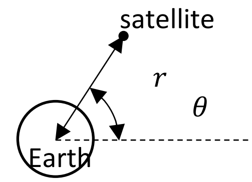
CMPT 882 Assignment 1

Due date: Monday, Feb. 4

1. A simplified set of equations of motion for the Earth and an orbiting satellite is given by

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1,$$
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{1}{r}u_2,$$



   where r represents the Earth-satellite distance measured from their centres, and θ represents the phase of the orbit. k is a positive constant.

   a) Derive a state space model of this system in the form of a first-order ordinary differential equation.

   b) What are the equilibrium points of the state space model, under zero control input, $u_1 = u_2 = 0$? Give a physical interpretation of the result.

   c) What are the equilibrium points of the state space model, under $u_1 = \frac{k}{x_1^2}, u_2 = 0$? Give a physical interpretation of this control set point and of the equilibrium points.

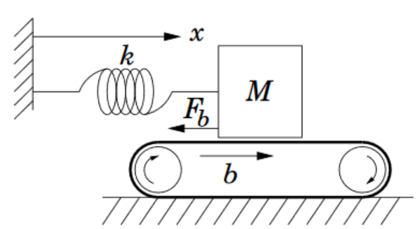
   d) Linearize the model with respect to a reference orbit given by $r(t) \equiv \rho, \theta(t) = \omega t, u_1 = u_2 = 0$

2. Given the system $\dot{x} = Ax + Bu$, with $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

   a) Is the system controllable?

   b) Construct a linear state feedback controller so that the closed loop system is stable.

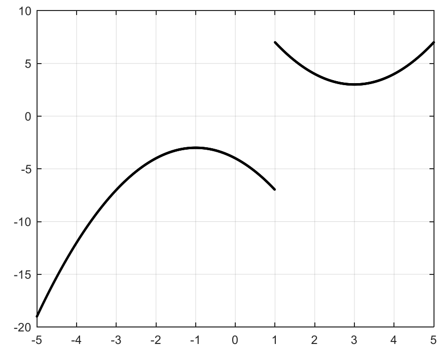
3. In the Rayleigh model of a violin string, the string is represented by a mass M, which vibrates with spring constant k. The bowing action is modeled using a conveyor belt that moves at a constant speed of b. Letting x be the position of the mass representing the string, the system dynamics are given by



$$M\ddot{x} + F_b(\dot{x}) + kx = 0,$$

where $F_b(\cdot)$ (shown on the right for $b = 1$) models the sticky friction during the bowing of the string. For this question, assume $M = 3, k = 3$, and

$$F_b(\dot{x}) = \begin{cases} -(\dot{x} - b + 2)^2 - 3, & \dot{x} < b, \\ (\dot{x} - b - 2)^2 + 3, & \dot{x} \geq b. \end{cases}$$



a) Derive a first order ODE model with states $(x, \dot{x})$.

b) Suppose $b = 1$, Calculate the equilibrium point in $(x, \dot{x})$ and determine its stability.

c) Numerically integrate the ODE using your own implementation of RK4 starting from a few different initial conditions $\big(x(0), \dot{x}(0)\big)$ and intuitively explain the behaviour. (Please attach your code.)

4. Bifurcations. Consider the planar system with dynamics

$$\dot{x}_1 = -x_2 + x_1(\mu - x_1^2 - x_2^2),$$
$$\dot{x}_2 = x_1 + x_2(\mu - x_1^2 - x_2^2),$$

where μ is a parameter.

a) Derive the polar coordinate (r, θ) representation of the system dynamics using the relationships

$$x_1 = r\cos\theta\,,$$
$$x_2 = r\sin\theta.$$

Hint: first compute $x_1\dot{x}_1 + x_2\dot{x}_2$ and $\dot{x}_2 x_1 - \dot{x}_1 x_2$.

b) Ignoring the θ dynamics, find the equilibrium points as a function of μ for the r component of the system

c) Find the branches of bifurcation, and describe and/or draw the behaviour of the system for different cases of the parameter μ. Take the stability of equilibrium points of the r subsystem into account.

5. Consider the linear system $\dot{x} = Ax$, with $A = \begin{bmatrix} 0 & 1 \\ -500 & -501 \end{bmatrix}$.
   a) Determine conditions on the time step size that must be satisfied for the forward Euler method to be stable.

   b) Repeat the above two steps for the backward Euler method.

6. Consider the following dynamical system:

$$\dot{x}_1 = x_1^2 + x_2$$
$$\dot{x}_2 = p(x_1, x_2) + q(x_1, x_2)u$$

   Find a state feedback control policy $u(x_1, x_2)$ such that the origin is asymptotically stable. Prove your result using a Lyapunov function. Hint: Use feedback stabilization.