Review: Linear Algebra

CMPT 882

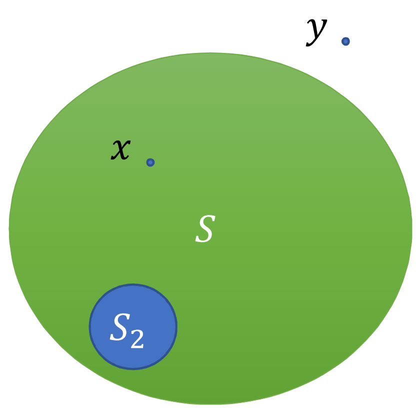
Jan. 7, 2018

Outline

- Notation
- Linear maps
- Norms
- Diagonalization and Jordan form
- Functions of matrices

Notation

- Sets of numbers: $\mathbb{Z}, \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}, \mathbb{R}_+, \mathbb{C}^n$
- Membership and Quantifiers: $\in, \notin, \forall, \exists, \exists!$
  - $x \in S, y \notin S$



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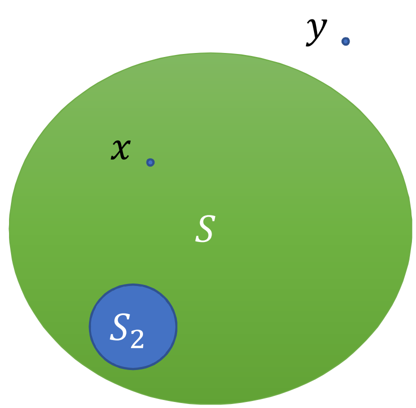
  - $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 0$



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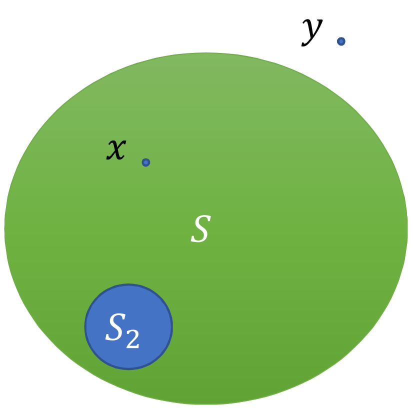

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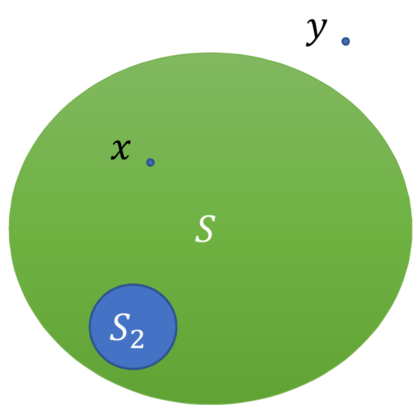

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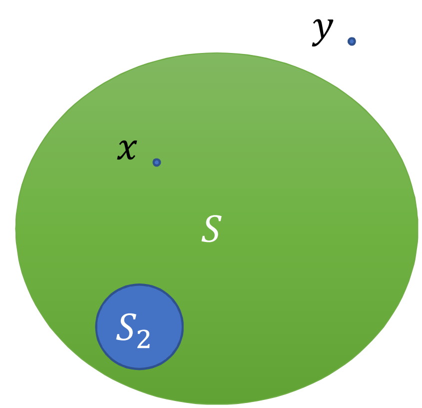
This is false



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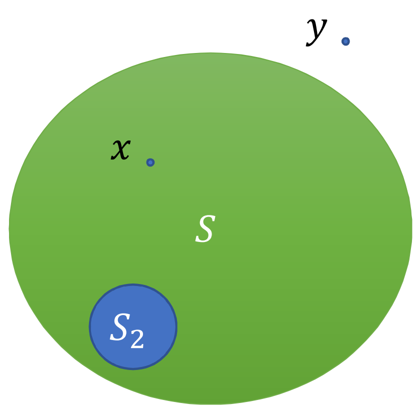
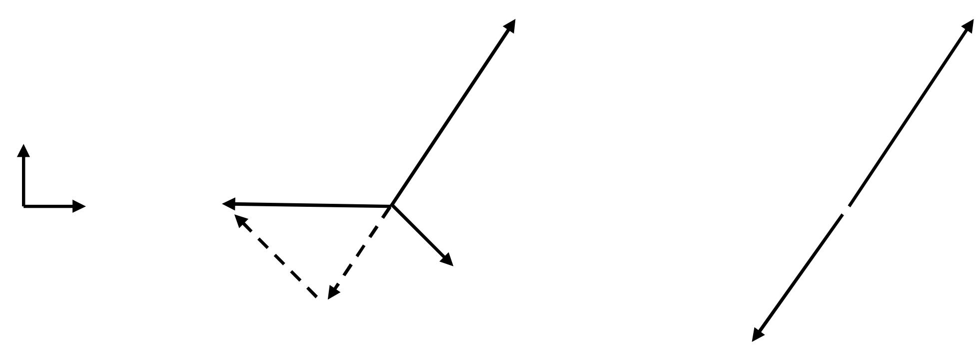


- Implications and negation: $\Rightarrow, \Leftarrow, \Leftrightarrow, \neg$
  - $z \in S_2 \Rightarrow z \in S,\ \ y \notin S \Leftrightarrow \neg(y \in S)$
  - $p \Rightarrow q$ and $p \Leftarrow q$ means $p \Leftrightarrow q$

Basis

- Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$
  - They are **linearly independent** if and only if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$$



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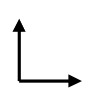
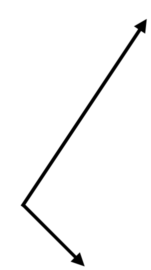
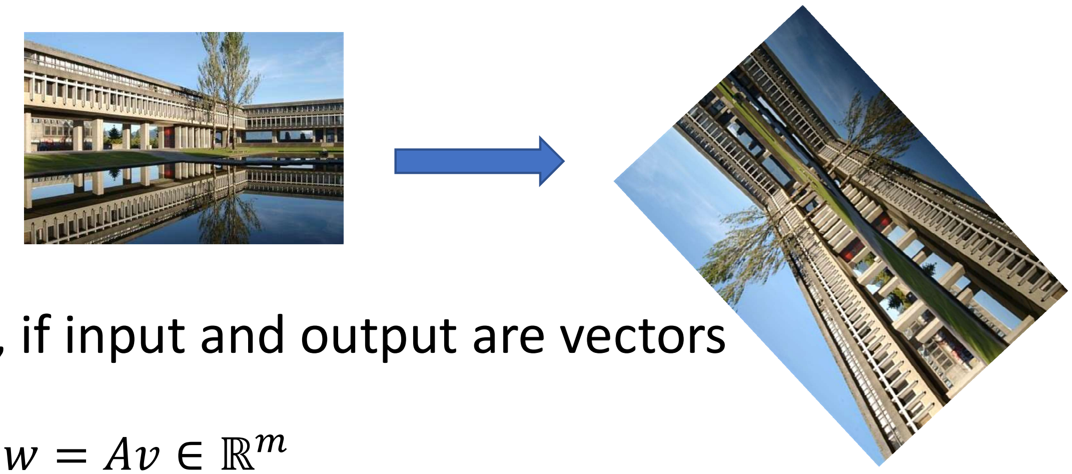
- A set of vectors $B = \{b_1, b_2, \ldots, b_n\}$ is a **basis** of $\mathbb{R}^n$ if and only if
  - $\forall v \in \mathbb{R}^n, \exists \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$
  - $\{b_1, b_2, \ldots, b_n\}$ is a linearly independent set of vectors

- Bases of $\mathbb{R}^2$:
  - $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$





Linear Maps



- Represented by a matrix $A \in \mathbb{R}^{m \times n}$, if input and output are vectors
  - $\mathcal{A}: v \to Av$
  - Operates on a vector $v \in \mathbb{R}^n$; outputs $w = Av \in \mathbb{R}^m$

- Linearity: $\mathcal{A}(a_1 v_1 + a_2 v_2) = a_1 \mathcal{A}(v_1) + a_2 \mathcal{A}(v_2)$
  - for all scalars $a_1, a_2 \in \mathbb{R}$, vectors $v_1, v_2 \in \mathbb{R}^n$

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- Range space: $R(\mathcal{A}) = \{w \mid w = \mathcal{A}(v), v \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$
  - Also known as the image of $\mathcal{A}$

- Null space: $N(\mathcal{A}) = \{v \mid \mathcal{A}v = 0\} \subseteq \mathbb{R}^n$
  - Also known as the kernel of $\mathcal{A}$

Linear Maps

- Example: $A = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & 2 \end{bmatrix}$
    - $R(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) =$ all vectors in the form $\begin{bmatrix} t \\ t \end{bmatrix}, t \in \mathbb{R}$
        - In general, the range of a matrix is given by all linear combinations of its columns

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- Matlab:
  - `A=sym([1 4 2; 1 4 2]);`
  - `colspace(A)`
  - `null(A)`

Linear Maps

- In general, an operator $\mathcal{A}$ may not have vectors as inputs and outputs
  - We use $\mathcal{A}$ to denote the map in this case
  - These maps may be linear!

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  - Maps quadratic functions to quadratic functions
  - Input and outputs are functions

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- Consider the map $\mathcal{A}: as^2 + bs + c \rightarrow cs^2 + bs + a$
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- Is the above map linear?
  - Let $v_1 = a_1 s^2 + b_1 s + c_1, v_2 = a_2 s^2 + b_2 s + c_2$
  - Check whether $\mathcal{A}(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \mathcal{A}(v_1) + \alpha_2 \mathcal{A}(v_2)$

Linear Maps: Example 1

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$$= \mathcal{A}\big((\alpha_1 a_1 + \alpha_2 a_2) s^2 + (\alpha_1 b_1 + \alpha_2 b_2) s + (\alpha_1 c_1 + \alpha_2 c_2)\big)$$

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&= \alpha_1 \mathcal{A}(v_1) + \alpha_2 \mathcal{A}(v_2)
\end{aligned}$$

- Therefore, the map is linear

Linear Map Properties

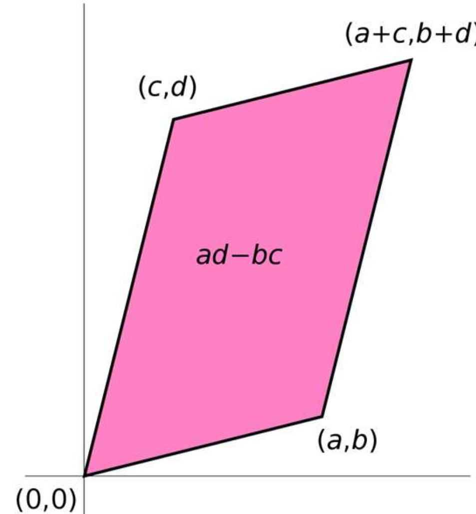
- Matrix inverse
  - System of equations, $Ax = b, A \in \mathbb{R}^{n \times n}$
  - Solution: $x = A^{-1}b$, if a solution exists
  - However, try not to do this in Matlab. Instead, use x=A\b
  - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- A is singular if it does not have an inverse
  - Columns of A are not linear independent $\Leftrightarrow$ A is singular

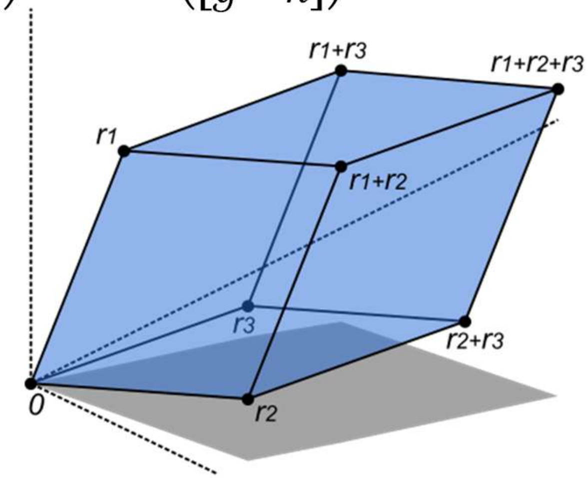
- Non-commutative in general
  - $\mathcal{A}\big(\mathcal{B}(x)\big) \neq \mathcal{B}\big(\mathcal{A}(x)\big)$

Linear Map Properties

- Matrix determinant
  - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det A = ad - bc$



  - If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then $\det A = a \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$
  - Can be defined recursively



- $\det A = 0 \Leftrightarrow A$ is singular

Linear Map Properties

- Let A be a linear map from $\mathcal{U} \to \mathcal{V}$, and $b \in \mathcal{V}$, then
  - $Au = b$ has at least one solution $\Leftrightarrow b \in R(A)$
  - If $b \in R(A)$, then
    - $Au = b$ has a unique solution $\Leftrightarrow N(A) = \{\theta_u\}$
    - Let x_0 be such that $Ax_0 = b$. Then, $Ax = b \Leftrightarrow x - x_0 \in N(A)$

Norms

- A norm is a map $\|\cdot\|: \mathcal{V} \to \mathbb{R}_+$ satisfying
  - $\forall v_1, v_2 \in \mathcal{V}, \|v_1 + v_2\| \le \|v_1\| + \|v_2\|$ (triangle inequality)
  - $\forall \alpha \in \mathbb{R}, v \in \mathcal{V}, \|\alpha v\| = |\alpha| \|v\|$
  - $\forall v \in \mathcal{V}, \|v\| = 0 \Leftrightarrow v = \theta_{\mathcal{V}}$

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- Examples in $\mathbb{R}^n$
  - $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ ("1-norm", or "l_1 norm")
  - $\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ ("2-norm", or "l_2 norm")
  - $\|x\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$ ("p-norm", or "l_p norm")
  - $\|x\|_\infty = \max_i |x_i|$ ("∞-norm", or "l_∞ norm")

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- Examples in $\mathbb{R}^{m\times n}$
  - $\|A\|_a = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$
  - $\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}}$ ("Frobenius norm")
  - $\|A\|_b = \max_{i,j} |a_{ij}|$

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- Examples for continuous functions $f: [t_0, t_1] \to \mathbb{R}^n$
  - $\|f\|_1 = \int_{t_0}^{t_1} \|f(t)\| dt$, where $\|f(t)\|$ is any of the vector norms from before
  - $\|f\|_2 = \left(\int_{t_0}^{t_1} \|f(t)\|^2 dt\right)^{\frac{1}{2}}$ ("2-norm", or "l_2 norm")
  - $\|f\|_\infty = \max\{\|f(t)\|, t \in [t_0, t_1]\}$ ("∞-norm", or "l_∞ norm")

Induced Norms

- Let A be a linear operator. Then, the induced norm of $\mathcal{A}$ is defined as
$$\|A\|_{p,i} = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

- Properties
  - $\|A\|_{1,i} = \max_j \sum_{i=1}^{m} \left|a_{ij}\right|$ (maximum column sum)
  - $\|A\|_{2,i} = \max_j \mathrm{eig}(A^\top A)^{\frac{1}{2}}$ (maximum singular value)
  - $\|A\|_{\infty,i} = \max_i \sum_{j=1}^{n} \left|a_{ij}\right|$ (maximum row sum)

Eigenvalues and Eigenvectors

- Eigenvalues:
  - If there is some vector e and scalar λ such that $Ae = \lambda e$, then e is called the eigenvector corresponding to eigenvalue λ of the matrix A

- Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
  - $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
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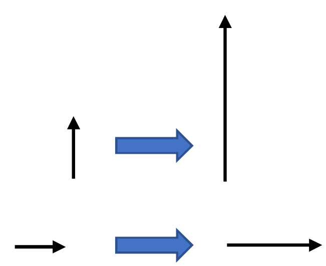

Eigenvalues and Eigenvectors

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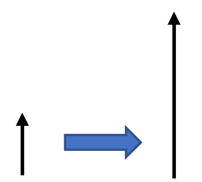

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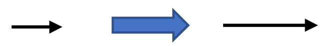

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- What if the matrix is applied to another vector?

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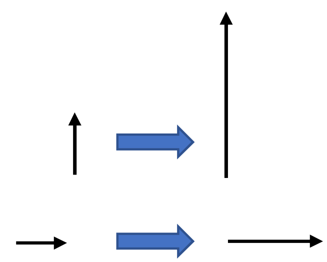

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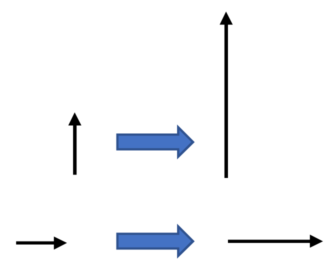

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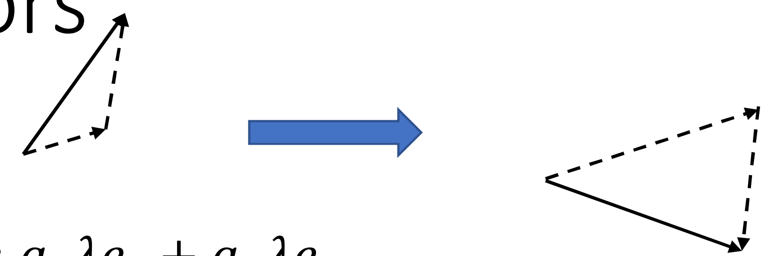
  - Looks silly, but we can apply the same idea for more complex matrices

Eigenvalues and Eigenvectors



- Geometric interpretation:
  - $Ax = A(a_1 e_1 + a_2 e_2) = a_1 A e_1 + a_2 A e_2 = a_1 \lambda e_1 + a_2 \lambda e_2$
  - Negative eigenvalues → reflection
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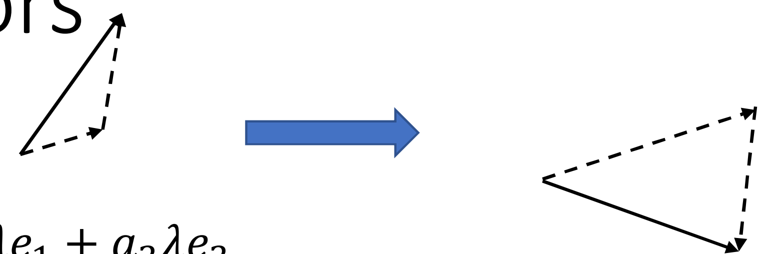
Eigenvalues and Eigenvectors



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- Property: $\det A =$ the product of eigenvalues
  - Recall $\det A = 0 \Leftrightarrow A$ is singular
  - If $\det A = 0$, then there must be an eigenvalue that's 0
  - $Ae = 0$, so $e \in N(A)$
  - Any vector in the direction of e gets scaled by a factor of 0
  - Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Eigenvalues and Eigenvectors

- Define $T^{-1} = [e_1 \quad e_2 \quad \cdots \quad e_n]$
  - Then, $AT^{-1} = T^{-1}\Lambda$, where $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
  - So, $A = T^{-1}\Lambda T$. This is a **similarity transform**

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  - So, $A = T^{-1}\Lambda T$. This is a **similarity transform**.
  - Define $z = Tx$, and we have $Ax = T^{-1}\Lambda Tx = T^{-1}\Lambda z$
    - In the coordinate system obtained from applying transformation T, the map A is diagonal
    - To obtain the result of applying A in the original coordinate system, transform back with T^{-1}

Obtaining Eigenvalues and Eigenvectors

- Hand calculation: $A = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$
  - Eigenvalues $Ae = \lambda e$

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Solve for λ in $\det(A - \lambda I) = 0$

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- Matlab: `eig(A)`

Generalized Eigenvalues and Eigenvectors

- Not all matrices are diagonalizable

$$Ae_1 = \lambda e_1$$
$$Ae_2 = \lambda e_2$$
$$Ae_3 = \lambda e_3$$

Generalized Eigenvalues and Eigenvectors

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$$
\begin{aligned}
Ae_1 &= \lambda e_1 & \qquad Av_1 &= \lambda v_1 + e_1 \\
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Ae_3 &= \lambda e_3 & &
\end{aligned}
$$

Generalized Eigenvalues and Eigenvectors

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$$
\begin{array}{lll}
Ae_1 = \lambda e_1 & Av_1 = \lambda v_1 + e_1 & Aw_1 = \lambda w_1 + v_1 \\
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\end{aligned}
$$

$$
J = TAT^{-1} = \begin{bmatrix} \lambda & 1 & 0 & & & \\ 0 & \lambda & 1 & & & \\ 0 & 0 & \lambda & & & \\ & & & \lambda & 1 & \\ & & & 0 & \lambda & \\ & & & & & \lambda \end{bmatrix}
$$

$$
T^{-1} = [e_1 \quad v_1 \quad w_1 \quad e_2 \quad v_2 \quad e_3]
$$

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$$\begin{aligned} Ae_1 &= \lambda e_1 \qquad & Av_1 &= \lambda v_1 + e_1 \qquad & Aw_1 &= \lambda w_1 + v_1 \\ Ae_2 &= \lambda e_2 & Av_2 &= \lambda v_2 + e_2 & & \\ Ae_3 &= \lambda e_3 & & & & \end{aligned}$$

$$T^{-1} = [e_1 \quad v_1 \quad w_1 \quad e_2 \quad v_2 \quad e_3]$$

  - This is the matrix structure for one eigenvalue
  - There may be more than one such blocks in general

- All matrices can be put into **Jordan form**

Functions of Matrices

- Consider a polynomial of a matrix, $f(A) = A^k$
  - $A^k = (T^{-1}JT)^k = (T^{-1}JT)\,(T^{-1}JT)\,(T^{-1}JT)\ldots\,(T^{-1}JT) = T^{-1}J^kT$
  - Adjacent T matrices and inverse cancel!

- This motivates general functions of matrices, like $f(A) = \sin A$, defined through Taylor series
  - $\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$

Functions of Matrices

- Suppose $J = \begin{array}{cc} \overset{\longleftarrow\; n \;\longrightarrow}{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}} & \updownarrow n \end{array}$
- Then, $f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ & f(\lambda) & \ddots & \vdots \\ & & \ddots & f'(\lambda) \\ & & & f(\lambda) \end{bmatrix}$
- And $f(A) = T^{-1} f(J) T$, where $A = T^{-1} J T$
  - Therefore, we also have that the eigenvalues of $f(A)$ are $\{f(\lambda)\}$, where $\{\lambda\}$ are eigenvalues of A

Linear Maps

- How about $\mathcal{A}: as^2 + bs + c \to \int_0^s (bt + a)dt$?