Review: Linear Algebra (Cont'd)

CMPT 882

Jan. 9, 2018

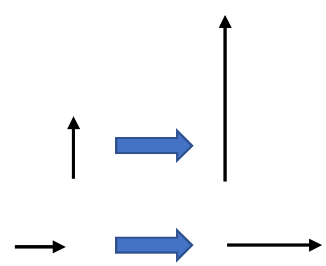
Outline

- Eigenvalues and Eigenvectors
- Jordan form of matrices
- Functions of matrices

Eigenvalues and Eigenvectors

- Eigenvalues:
  - If there is some vector e and scalar λ such that $Ae = \lambda e$, then e is called the eigenvector corresponding to eigenvalue λ of the matrix A

- Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
  - $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - When a matrix is applied to eigenvectors, the effect is simple!



Eigenvalues and Eigenvectors

- Define $T^{-1} = [e_1 \quad e_2 \quad \cdots \quad e_n]$
  - Then, $AT^{-1} = T^{-1}\Lambda$, where $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
  - So, $A = T^{-1}\Lambda T$. This is a **similarity transform**

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  - So, $A = T^{-1}\Lambda T$. This is a **similarity transform**.
  - Define $z = Tx$, and we have $Ax = T^{-1}\Lambda T x = T^{-1}\Lambda z$
    - In the coordinate system obtained from applying transformation T, the map A is diagonal
    - To obtain the result of applying A in the original coordinate system, transform back with T^{-1}

Obtaining Eigenvalues and Eigenvectors

- Hand calculation: $A = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$
  - Eigenvalues

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$$(A - \lambda I)e = 0$$

This means the matrix $A - \lambda I$ has an eigenvalue of 0

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Solve for λ in $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2-\lambda & 3 \\ -1 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)(2-\lambda) - 9 = 0$$

$$2 - \lambda = \pm 3$$
$$\lambda = 2 \pm 3 = -1 \text{ or } 5$$

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- Matlab: `eig(A)`

Generalized Eigenvalues and Eigenvectors

- Not all matrices are diagonalizable

$$
\begin{aligned}
Ae_1 &= \lambda e_1 \\
Ae_2 &= \lambda e_2 \\
Ae_3 &= \lambda e_3
\end{aligned}
$$

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&Ae_1 = \lambda e_1 \qquad && Av_1 = \lambda v_1 + e_1 \\
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&Ae_3 = \lambda e_3 &&
\end{aligned}
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\begin{array}{lll}
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$$

$$
J = TAT^{-1} = \begin{bmatrix} \lambda & 1 & 0 & & & \\ 0 & \lambda & 1 & & & \\ 0 & 0 & \lambda & & & \\ & & & \lambda & 1 & \\ & & & 0 & \lambda & \\ & & & & & \lambda \end{bmatrix}
$$

$$
T^{-1} = [e_1 \quad v_1 \quad w_1 \quad e_2 \quad v_2 \quad e_3]
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$$T^{-1} = [e_1 \quad v_1 \quad w_1 \quad e_2 \quad v_2 \quad e_3]$$

  - This is the matrix structure for one eigenvalue
  - There may be more than one such blocks in general

- All matrices can be put into **Jordan form**
  - Note that the eigenvalues of J are the same as those of A

Imagine $\det(J - sI) = 0$

Functions of Matrices

- Consider a polynomial of a matrix, $f(A) = A^k$
  - $A^k = (T^{-1}JT)^k = (T^{-1}JT)\,(T^{-1}JT)\,(T^{-1}JT)\dots\,(T^{-1}JT) = T^{-1}J^kT$
  - Adjacent T matrices and inverse cancel!

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  - Adjacent T matrices and inverse cancel!

- This motivates general functions of matrices, like $f(A) = \sin A$, defined through Taylor series
  - $\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$

Functions of Matrices

- Suppose $J = \begin{array}{cc} \xleftrightarrow{\qquad n \qquad} & \\ \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} & \updownarrow n \end{array}$

- Then, $f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ & f(\lambda) & \ddots & \vdots \\ & & \ddots & f'(\lambda) \\ & & & f(\lambda) \end{bmatrix}$

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- And $f(A) = T^{-1}f(J)T$, where $A = T^{-1}JT$
- **Spectral theorem**: the eigenvalues of $f(A)$ are $\{f(\lambda)\}$, where $\{\lambda\}$ are eigenvalues of A

Imagine $\det(f(J) - sI) = 0$

Linear Systems

CMPT 882

Jan. 9

Outline

- Differential equations

- Linear time-invariant differential equations

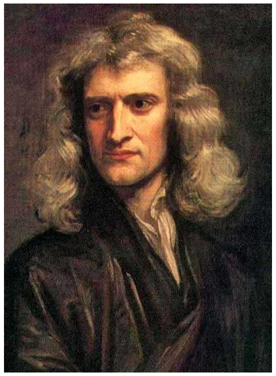
References for Linear Systems

- F. Callier & C. A. Desoer, Linear System Theory, Springer-Verlag, 1991.
- W. J. Rugh, Linear System Theory, Prentice-Hall, 1996.

Differential equations

- Continuous time model of robotic systems
  - In general, nonlinear systems
  - One may construct discrete time models from continuous time models

Differential equations



- Continuous time model of robotic systems
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  - One may construct discrete time models from continuous time models

- **Dynamics:** $\dot{x} = f(t, x, u, d), x \in \mathbb{R}^n, t \geq t_0$
  - Specifies how the robot state or configuration changes over time
  - In some ways, the most "natural" model, since $F = ma = m\ddot{x}$
  - Defining $x_1 = x, x_2 = \dot{x}$, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{F}{m} \end{bmatrix}$$

Differential equations

- **State:** $x(t) \in \mathbb{R}^n, x(t_0) = x_0$
  - Contains all information needed to specify the configuration of the robot
  - Most common: position, velocity, angular position, angular velocity

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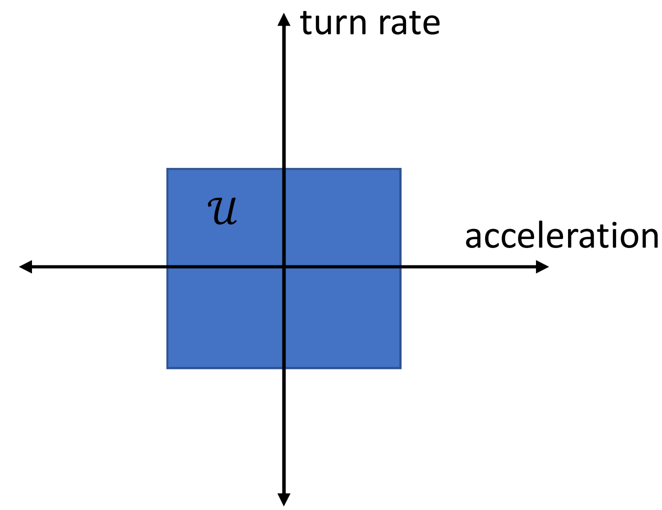
- **Control:** $u(t) \in \mathcal{U}$
  - Examples: steering, accelerating, decelerating
  - Usually constrained to be within some set



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- **Control:** $u(t) \in \mathcal{U}$
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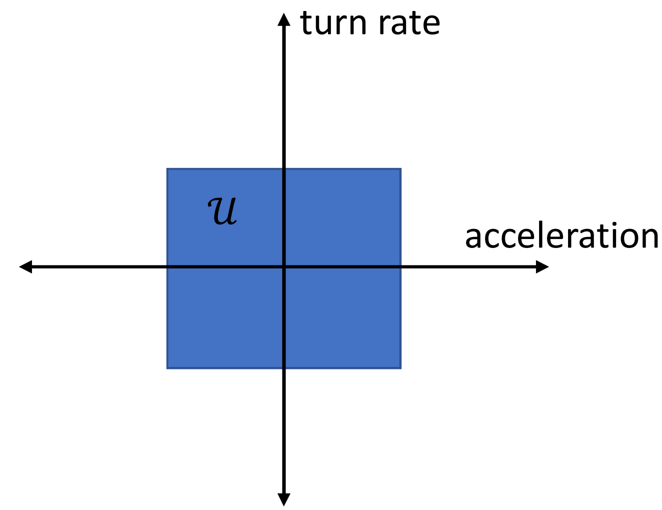
- **Disturbance:** $d(t) \in \mathcal{D}$
  - Examples: wind, input noise, another agent



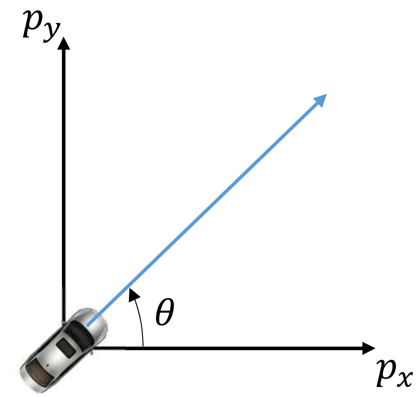
Differential Equations

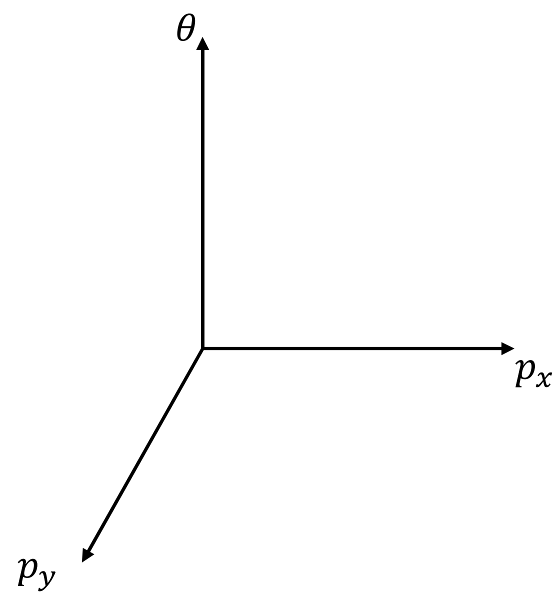
- Example: Simple car, $x = \left(p_x, p_y, \theta\right), t \geq 0$

$$\dot{p}_x = v\cos\theta$$
$$\dot{p}_y = v\sin\theta$$
$$\dot{\theta} = u$$

$$f(x,u) = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ u \end{bmatrix}$$

  - $x(t) \in \mathbb{R}^3$
  - $u \in [-1,1]$
  - v is constant





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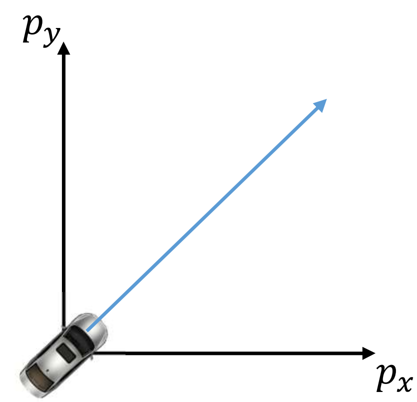

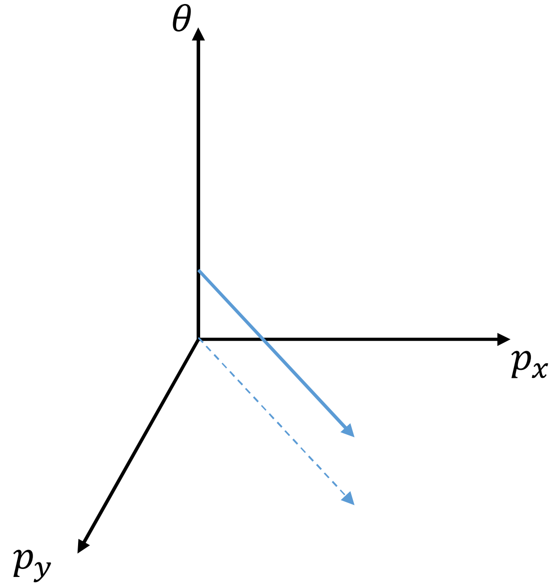

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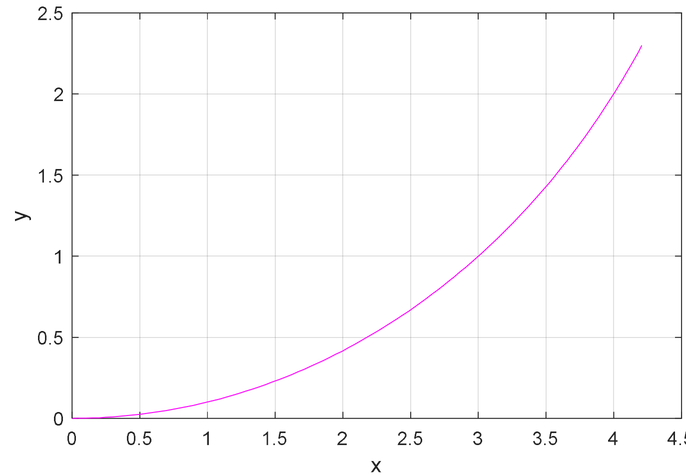

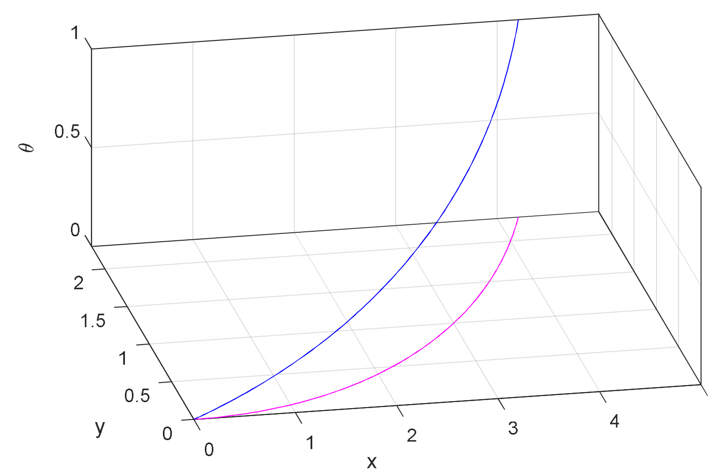

Existence and Uniqueness of Solutions

- $\dot{x} = f(t, x, u, d), t \geq t_0$
  - **State:** $x(t) \in \mathbb{R}^n, x(t_0) = x_0$
  - **Control:** $u(t) \in \mathcal{U}$
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- Conditions for existence and uniqueness of solution $x(t)$
  - f is piecewise continuous in t
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    $$\forall u, d, x_1, x_2, t, \|f(t, x_1, u) - f(t, x_2, u)\| \leq L(t)\|x_1 - x_2\|$$

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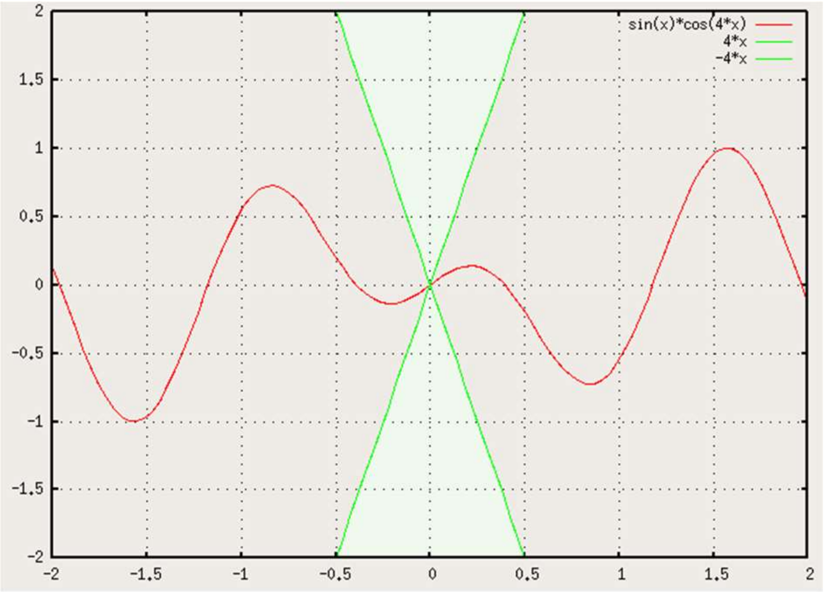
  - $u(\cdot)$ and $d(\cdot)$ are piecewise continuous

Lipschitz Continuity

- A Lipschitz continuous function $f\colon \mathbb{R} \to \mathbb{R}$
  - $f(x) = \sin x \cos 4x$
  - $\forall x_1, x_2, |f(x_1) - f(x_2)| \leq 4|x_1 - x_2|$

- Graph shows $x_2 = 0$
  - All of red curve is outside of green cones

- Translate the green lines anywhere along the red curve
  - All of red curve is still outside green cones



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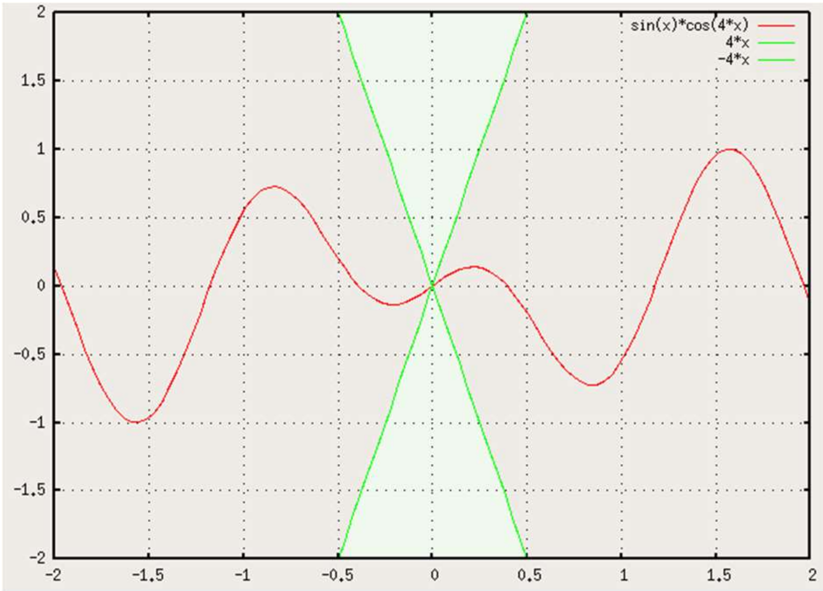

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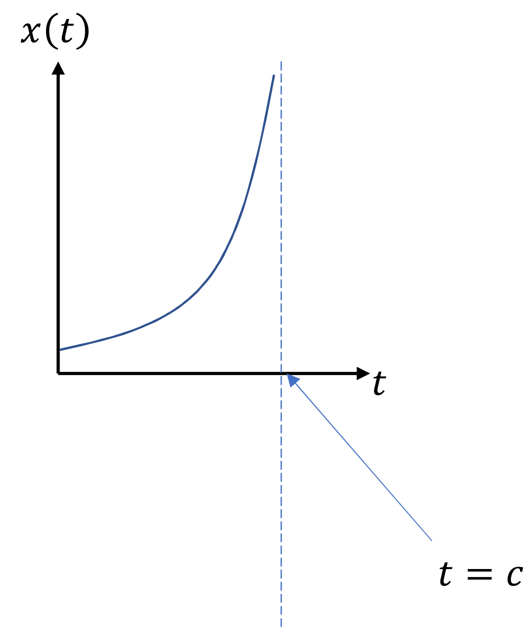
$$\frac{dx}{dt} = x^2 \Rightarrow \frac{dx}{x^2} = dt \Rightarrow \int \frac{dx}{x^2} = \int dt$$

$$-\frac{1}{x} + c = t$$

$$x(t) = \frac{1}{c - t}$$



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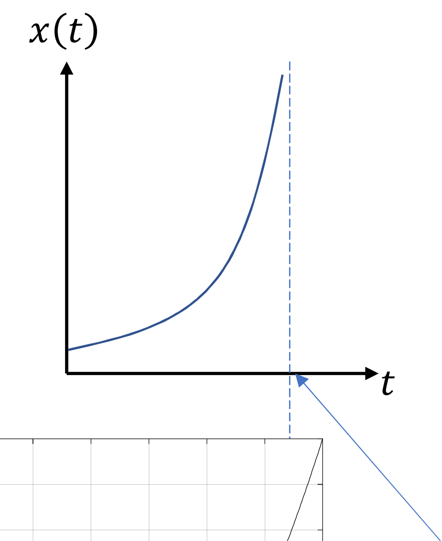
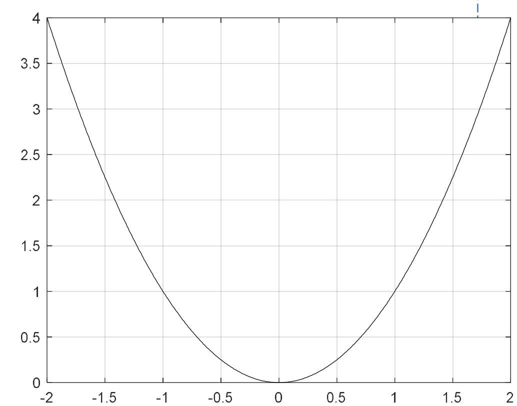
$$-\frac{1}{x} + c = t$$

$$x(t) = \frac{1}{c-t}$$

  - Here, $f(x) = x^2$, not Lipschitz continuous in x





Checking Lipschitz Continuity

- Example: $f(x) = x^2$
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  - $\Leftrightarrow \exists L > 0, \forall x_1, x_2, |(x_1 - x_2)(x_1 + x_2)| \leq L|x_1 - x_2|$
  - $\Leftrightarrow \exists L > 0, \forall x_1, x_2, (x_1 + x_2)|x_1 - x_2| \leq L|x_1 - x_2|$

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  - $\exists L > 0, \forall x_1, x_2, \|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$
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  - If $f(x)$ is differentiable, then f is Lipchitz continuous with Lipchitz constant L if and only if $\forall x, |f'(x)| < L$

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LTI Systems

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- Linear time-invariant (LTI) systems: $\dot{x} = Ax + Bu$
  - Damped mass spring systems
  - Circuits involving resistors, capacitors, inductors
  - Approximations of nonlinear systems