Numerical Solutions to ODEs

CMPT 882

Jan. 23

Numerical Solutions of ODEs

- In general, $\dot{x} = f(x, u)$ does not have a closed-form solution
  - Instead, we usually compute numerical approximations to simulate system behaviour
  - Done through discretization: $t^k = kh,\ u^k \coloneqq u(t^k)$
    - h represents size of time step
  - Goal: compute $y^k \approx x(t^k)$

- Key considerations
  - Consistency: Does the approximation satisfy the ODE as $h \to 0$?
  - Accuracy: How fast does the solution converge?
  - Stability: Do approximation error remain bounded over time?
  - Convergence: Does the solution converge the true solution as $h \to 0$?

Euler Methods

- ODE: $\dot{x} = f(x, u)$
  - Discretization: $t^k = kh,\ u^k := u(t^k)$
  - Want: Approximate solution: $y^k \approx x(kh)$

$$\dot{x} = f(x, u)$$
$$\frac{x(t^{k+1}) - x(t^k)}{h} \approx f(x(t^k), u^k)$$
$$\frac{y^{k+1} - y^k}{h} = f(y^k, u^k)$$

- Forward Euler
  - Most naïve method (explicit method)

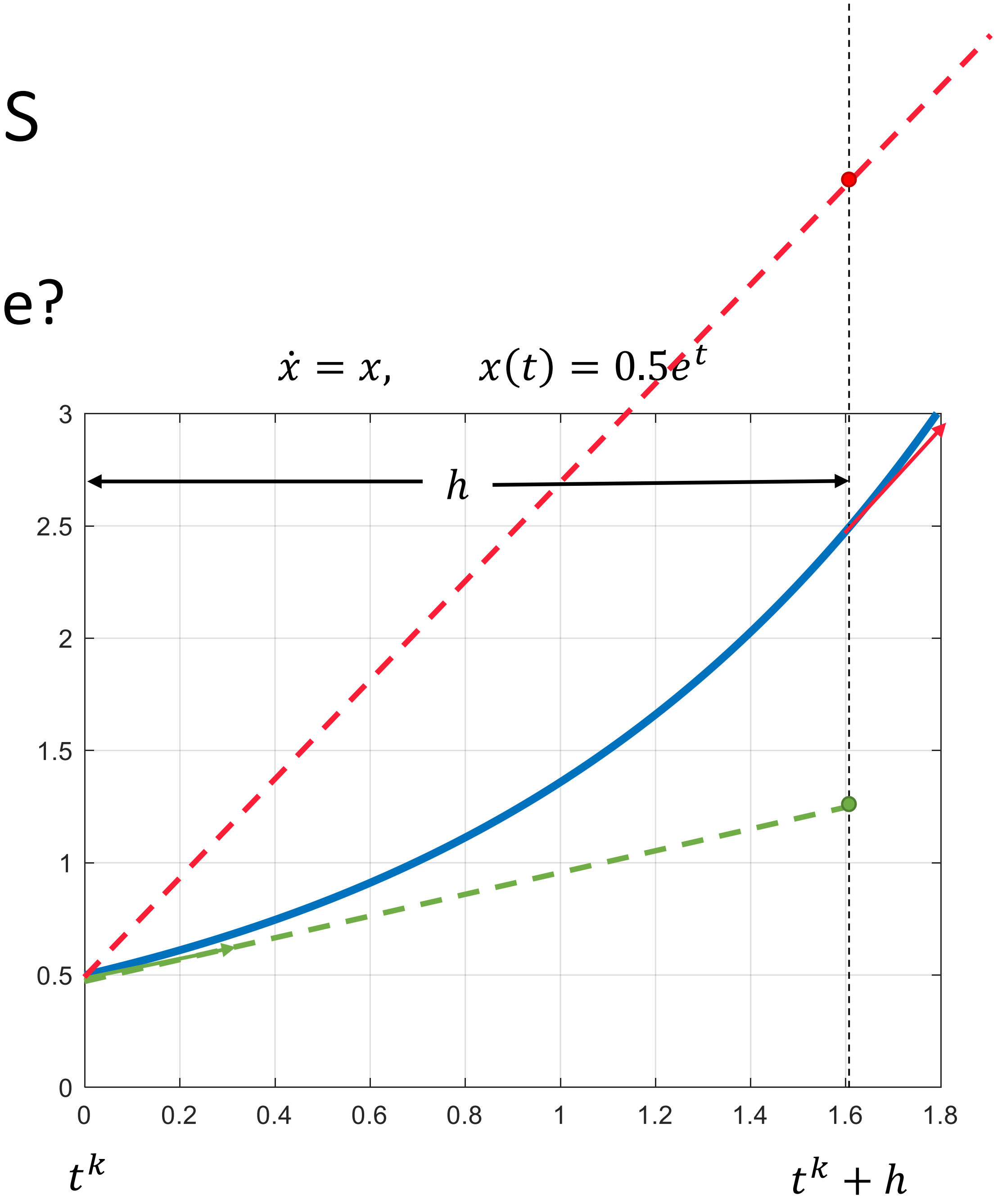
$$\frac{y^{k+1} - y^k}{h} = f(y^k, u^k) \Rightarrow y^{k+1} = y^k + hf(y^k, u^k)$$

- Backward Euler

$$\frac{y^{k+1} - y^k}{h} = f(y^{k+1}, u^k) \Rightarrow \text{solve for } y^{k+1} \text{ implicitly}$$

Visualizing Euler Methods

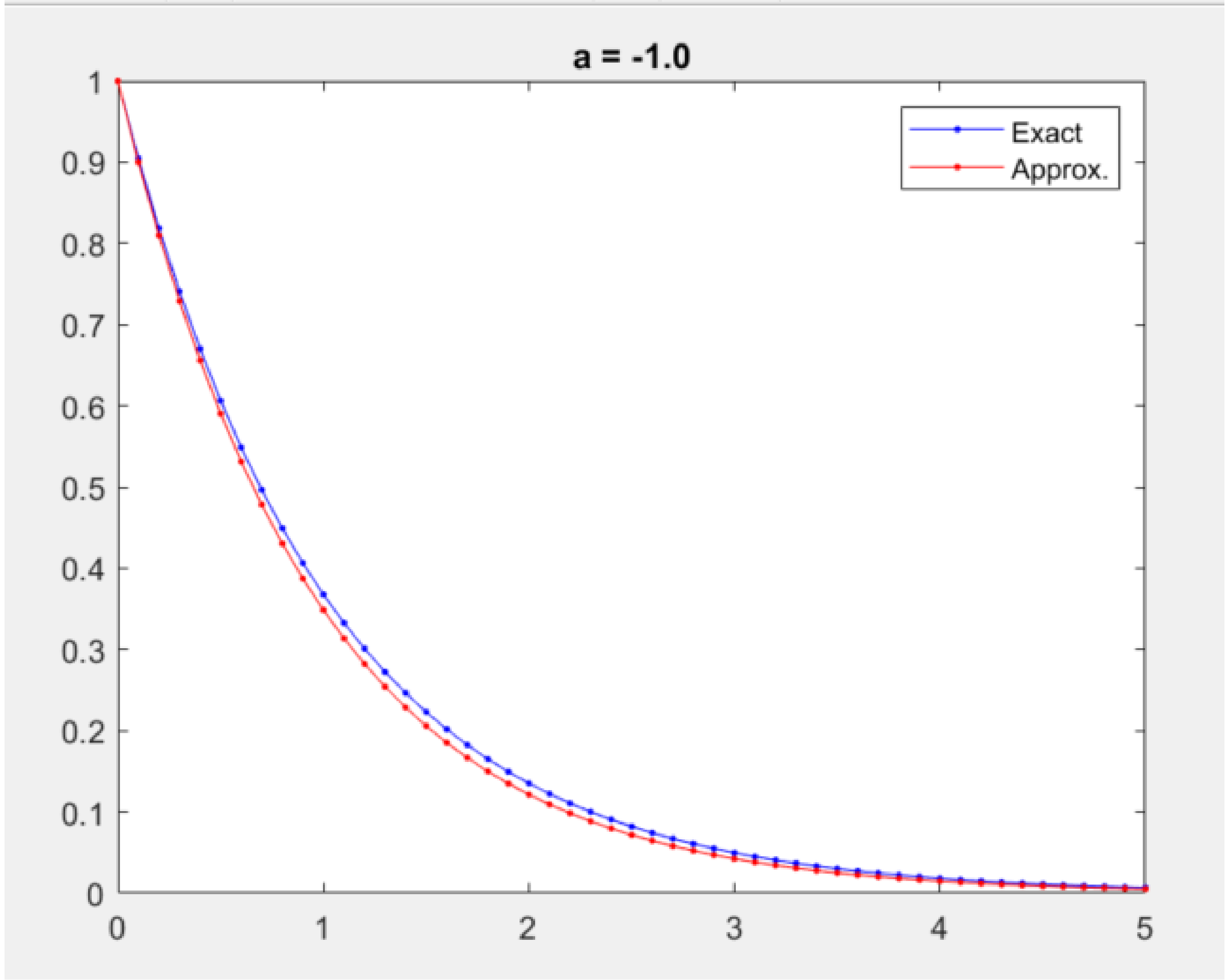
- Main consideration: what slope to use?
  - Forward Euler: slope at beginning
    $$y^{k+1} = y^k + hf\left(y^k, u^k\right)$$
  - Backward Euler: slope at the end
    $$y^{k+1} = y^k + hf\left(y^{k+1}, u^k\right)$$



Example

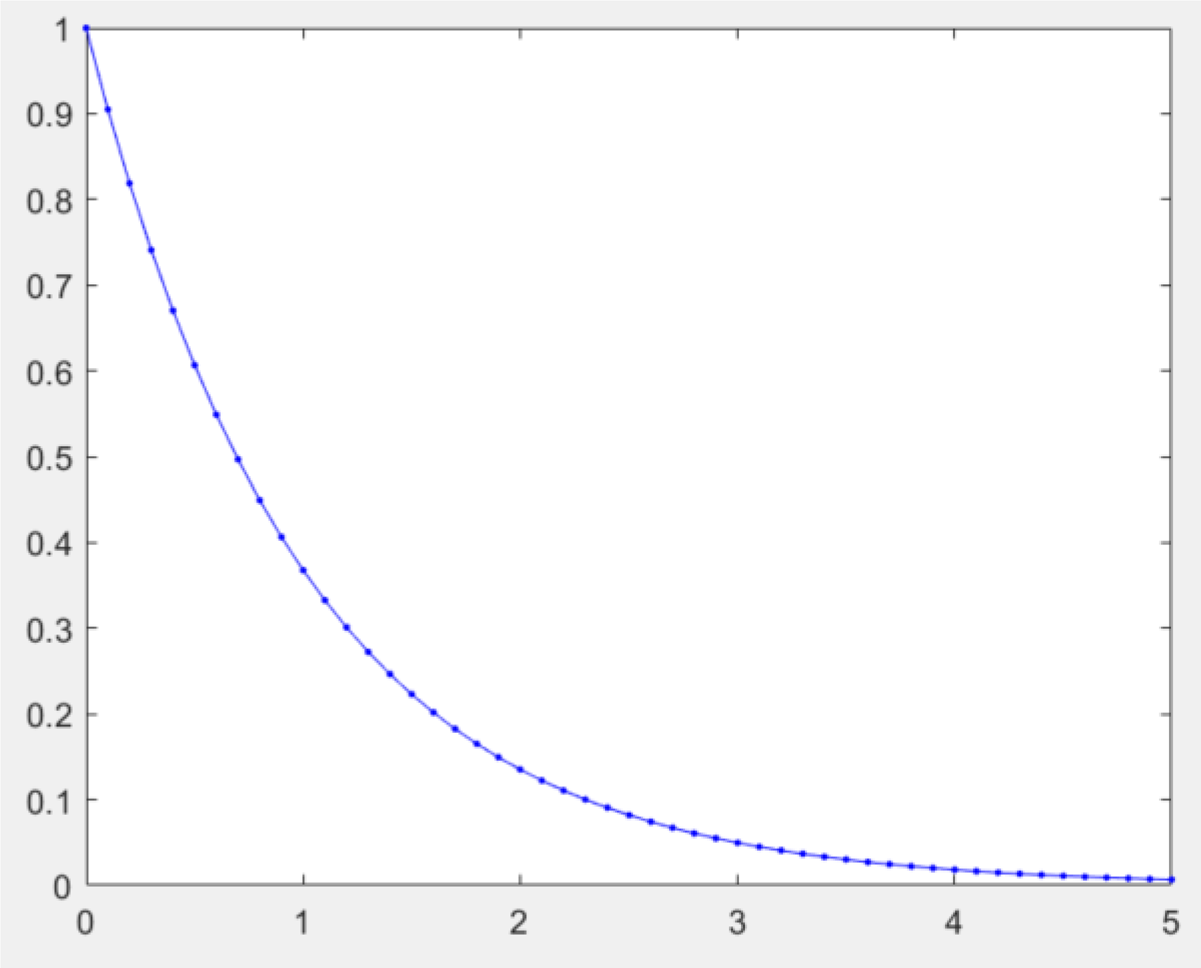
- $\dot{x} = ax, \;\; x(0) = x_0$
  - Analytic solution: $x(t) = x_0 e^{at}$

- Forward Euler
  - $y^{k+1} = y^k + hf\left(y^k, u^k\right)$
  - $y^{k+1} = y^k + hay^k$
  - $y^{k+1} = (1 + ha)y^k$



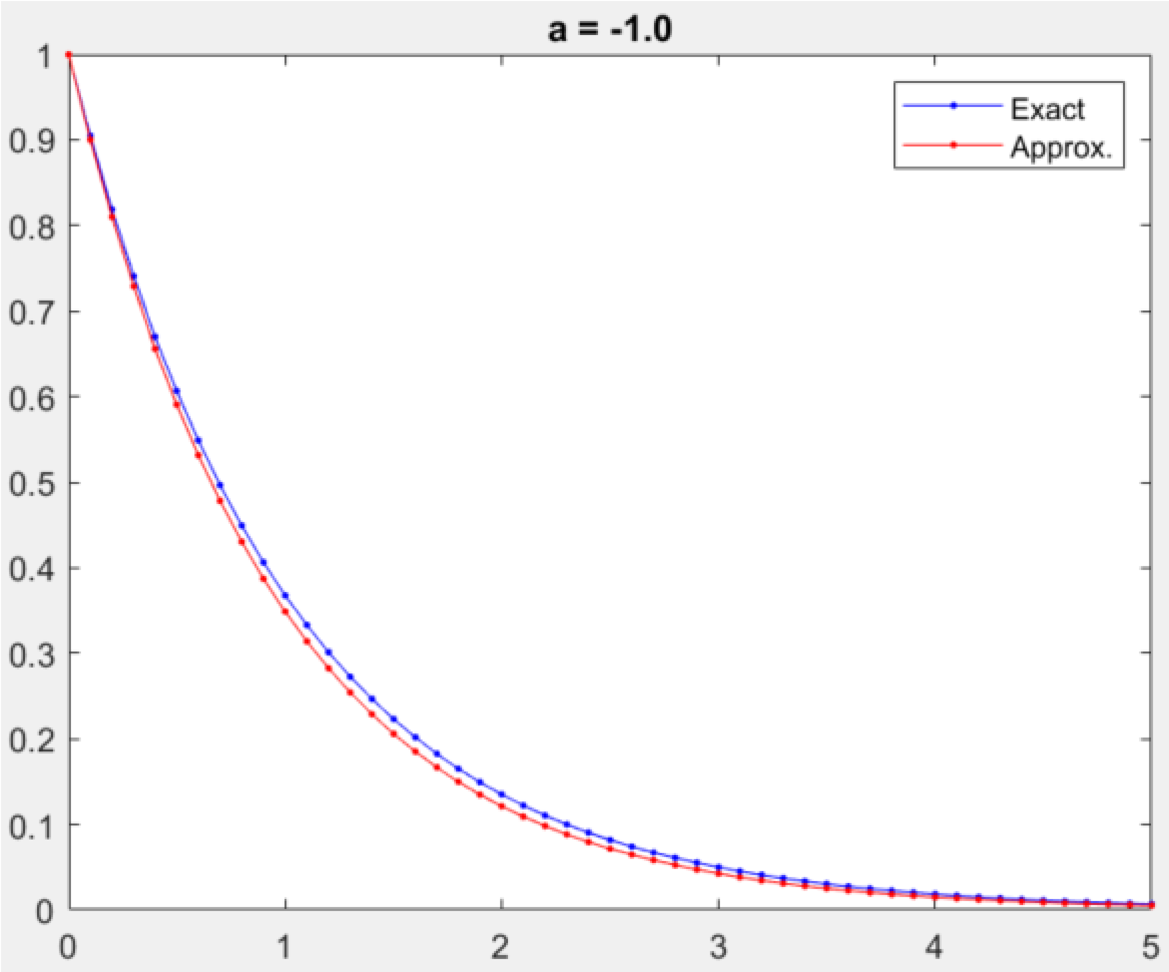
Example

```
%% Problem setup
x0 = 1;
a = -1;
h = 0.1;
T = 5;
tau = 0:h:T;

%% Exact solution
x_exact = @(t) exp(a*t);

figure
plot(tau, x_exact(tau), 'b.-')
```



Example

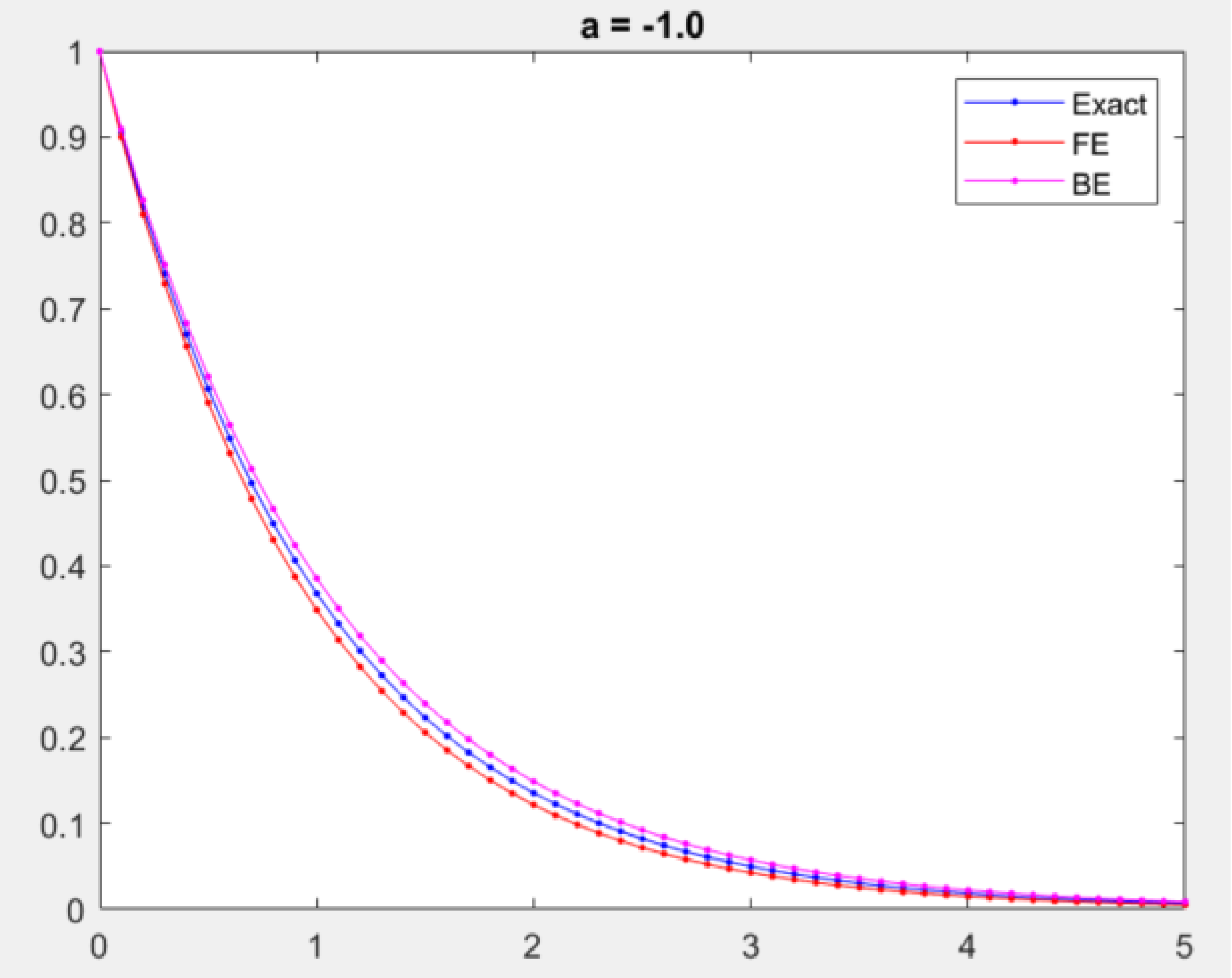
```
%% Forward Euler
f = @(x) a*x;
y_approx = -ones(size(tau));
y_approx(1) = x0;

% Initialize vector
for i = 2:length(tau)
  y_approx(i) = y_approx(i-1)*(1+h*a);
end

hold on
plot(tau, y_approx, 'r.-')
title(sprintf('a = %.1f', a))
legend('Exact', 'Approx.')
```



Example

- $\dot{x} = ax, \;\; x(0) = x_0$
  - Analytic solution: $x(t) = x_0 e^{at}$

- Backward Euler
  - $y^{k+1} = y^k + hf\left(y^{k+1}\right)$
  - $y^{k+1} = y^k + hay^{k+1}$
  - $y^{k+1} - hay^{k+1} = y^k$
  - $(1 - ha)y^{k+1} = y^k$
  - $y^{k+1} = \frac{y^k}{1-ha}$



Numerical Consistency: Forward Euler

- **Consistency:** ODE is satisfied as $h \to 0$
  - Forward Euler: $y^{k+1} = y^k + hf\left(y^k, u^k\right)$ $\qquad \dfrac{y^{k+1} - y^k}{h} = f\left(y^k, u^k\right)$

- **Local truncation error**: Analysis requires $\dfrac{\left\|e^k\right\|}{h} \to 0$ as $h \to 0$
  - $\left\|e^k\right\|$: Error induced during one step, assuming perfect previous information
  - Forward Euler approximate solution:

$$y^{k+1} = x\left(t^k\right) + hf\left(x\left(t^k\right), u^k\right)$$

  - True solution:

$$x\left(t^{k+1}\right) = x\left(t^k + h\right) = x\left(t^k\right) + h\frac{dx}{dt}\left(t^k\right) + \frac{h^2}{2}\frac{d^2x}{dx^2}\left(t^k\right) + O(h^3)$$

$$= x\left(t^k\right) + hf\left(x\left(t^k\right), u^k\right) + \frac{h^2}{2}\frac{d^2x}{dx^2}\left(t^k\right) + O(h^3)$$

Numerical Consistency: Forward Euler

- Local truncation error:

$$
\begin{aligned}
e^k &= x(t^{k+1}) - y^{k+1} \\
&= x(t^k) + hf(x(t^k), u^k) + \frac{h^2}{2}\frac{d^2x}{dx^2}(t^k) + O(h^3) - \left(x(t^k) + hf(x(t^k), u^k)\right) \\
&= \frac{h^2}{2}\frac{d^2x}{dx^2}(t^k) + O(h^3) \\
&= O(h^2)
\end{aligned}
$$

- Consistency requires $\frac{\|e^k\|}{h} \to 0$ as $h \to 0$

$$
\frac{\|e^k\|}{h} = \frac{\left|\frac{h^2}{2}\frac{d^2x}{dx^2}(t^k) + O(h^3)\right|}{h} = \left|\frac{h}{2}\frac{d^2x}{dx^2}(t^k) + O(h^2)\right| \to 0
$$

- If $\frac{\|e^k\|}{h} = O(h^p)$, then the numerical method is “order p”.
  - Forward Euler is an order 1 method, or first order method

Numerical Consistency

- More generally: $y^{k+1} = \sum_{n=k_1}^{k} \alpha_i y^i + h \sum_{n=k_2}^{k} \beta_i f\left(y^i, u^i\right)$

- Truncation error:

$$e^k \coloneqq x\left(t^{k+1}\right) - \sum_{n=k_1}^{k} \alpha_n x(nh) - h \sum_{n=k_2}^{k} \beta_i f\left(x(nh), u^i\right)$$

- Consistency requires $\frac{\|e^k\|}{h} \to 0$ as $h \to 0$
- If $\frac{\|e^k\|}{h} = O(h^p)$, then the numerical method is “order p”.

Numerical Stability: Forward Euler

- $y^{k+1} = y^k + hf\left(y^k, u^k\right)$
  - A map from y^k to y^{k+1}
  - Stability means y^k does not "blow up" when the true solution $x\left(t^k\right)$ is bounded
  - Usually, stability requires that the time step h cannot be too large

- Example: $\dot{x} = ax, a < 0$
  - $y^{k+1} = (1 + ah)y^k$
  - Stability requires $|1 + ah| \leq 1 \Leftrightarrow -ah \leq 2$
  - For $a = -10$, we have $|1 - 10h| \leq 1 \Leftrightarrow h \leq 0.2$

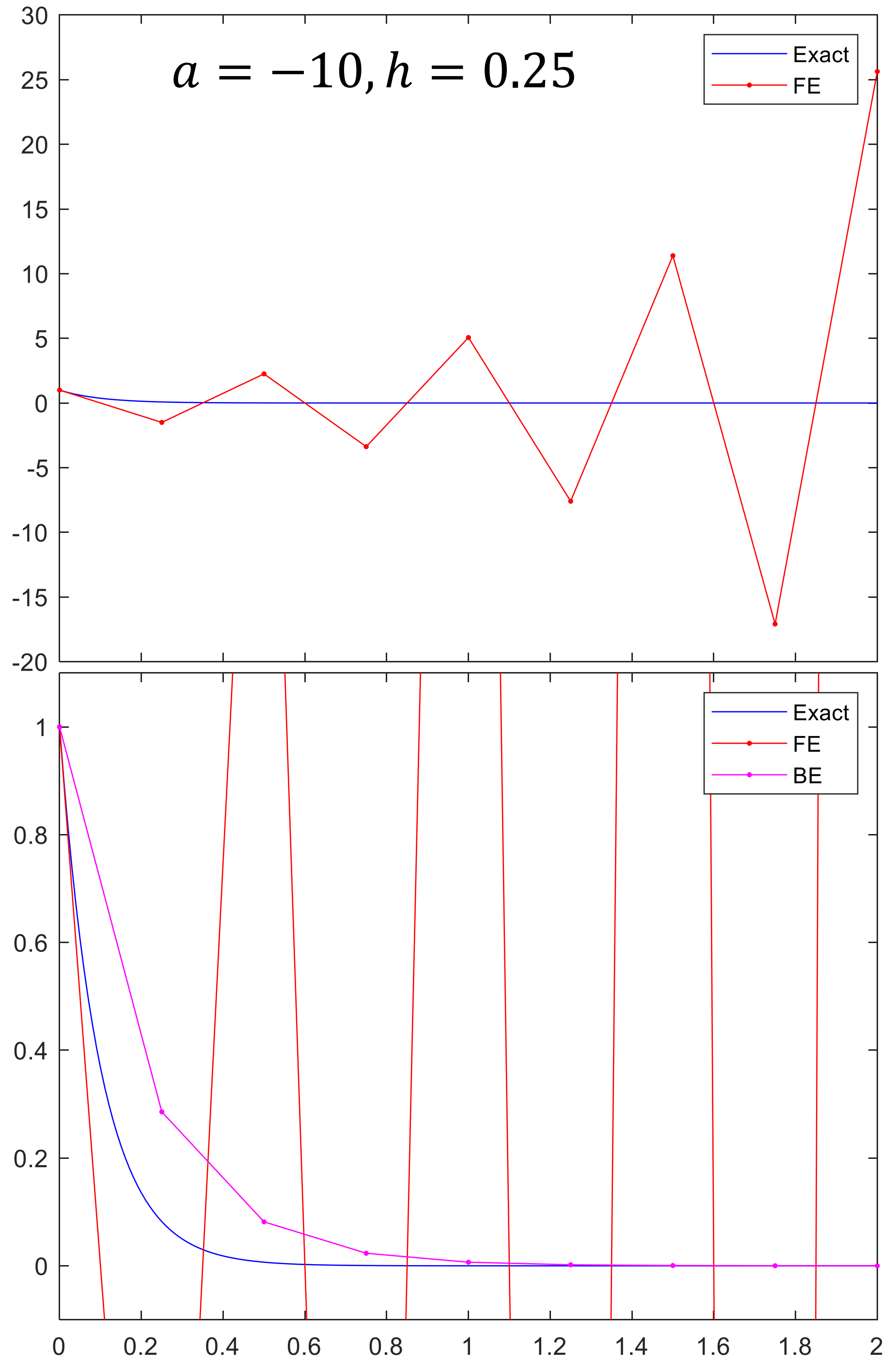
Numerical Stability: Backward Euler

- $y^{k+1} = y^k + hf\left(y^{k+1}, u^k\right)$
  - A map from y^k to y^{k+1}
  - Stability means y^k does not "blow up" when the true solution $x\left(t^k\right)$ is bounded
  - Usually, stability requires that the time step h cannot be too large

- Example: $\dot{x} = ax, a < 0$
  - $y^{k+1} = \frac{y^k}{1+ah}$
  - Stability requires $\left|\frac{1}{1-ah}\right| \leq 1$
  - No restrictions on h, for any a!

Numerical Stability

- Example: $\dot{x} = ax$ with forward Euler
  - If $a = -10$, $h \leq 0.2$ is required for stability
- Example 2: $\dot{x} = ax$ with backward Euler
  - No restrictions on h, for any a



Numerical Stability

- More generally: $y^{k+1} = \sum_{n=k_1}^{k} \alpha_i y^i + h \sum_{n=k_2}^{k} \beta_i f(y^i, u^i)$
  - Desired property: the approximation y^k does not "blow up" when the true solution $x(t^k)$ is bounded
  - Usually, this means time step h cannot be too large

- Specifically, one typically considers $\dot{x} = ax, a < 0$.
  - A stable numerical approximation to $\dot{x} = ax, a < 0$ has the property that $y^k \to 0$

Numerical Convergence

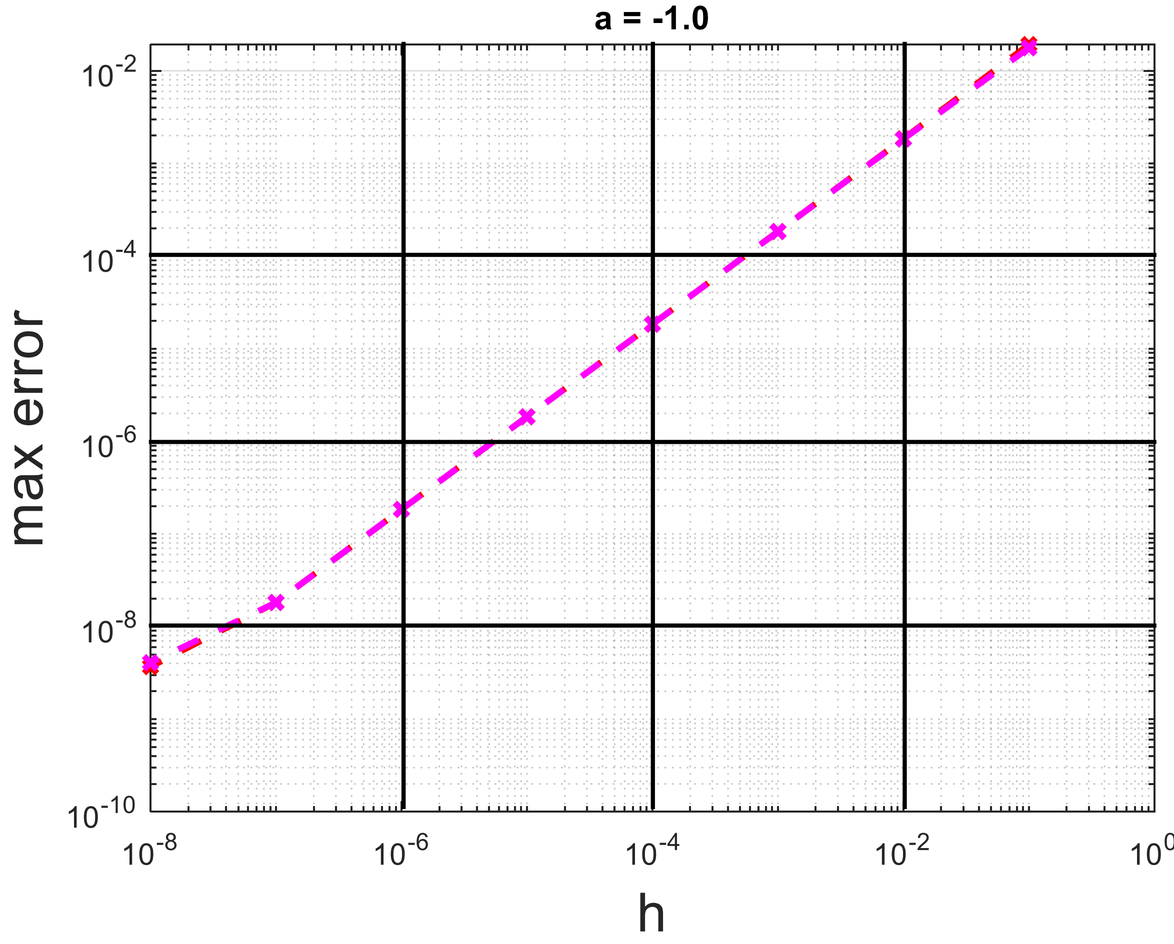
- **Convergence**: $\max_k \left\| x\left(t^k\right) - y^k \right\| \rightarrow 0$ as $h \rightarrow 0$
  - Maximum error goes to zero as time step goes to 0

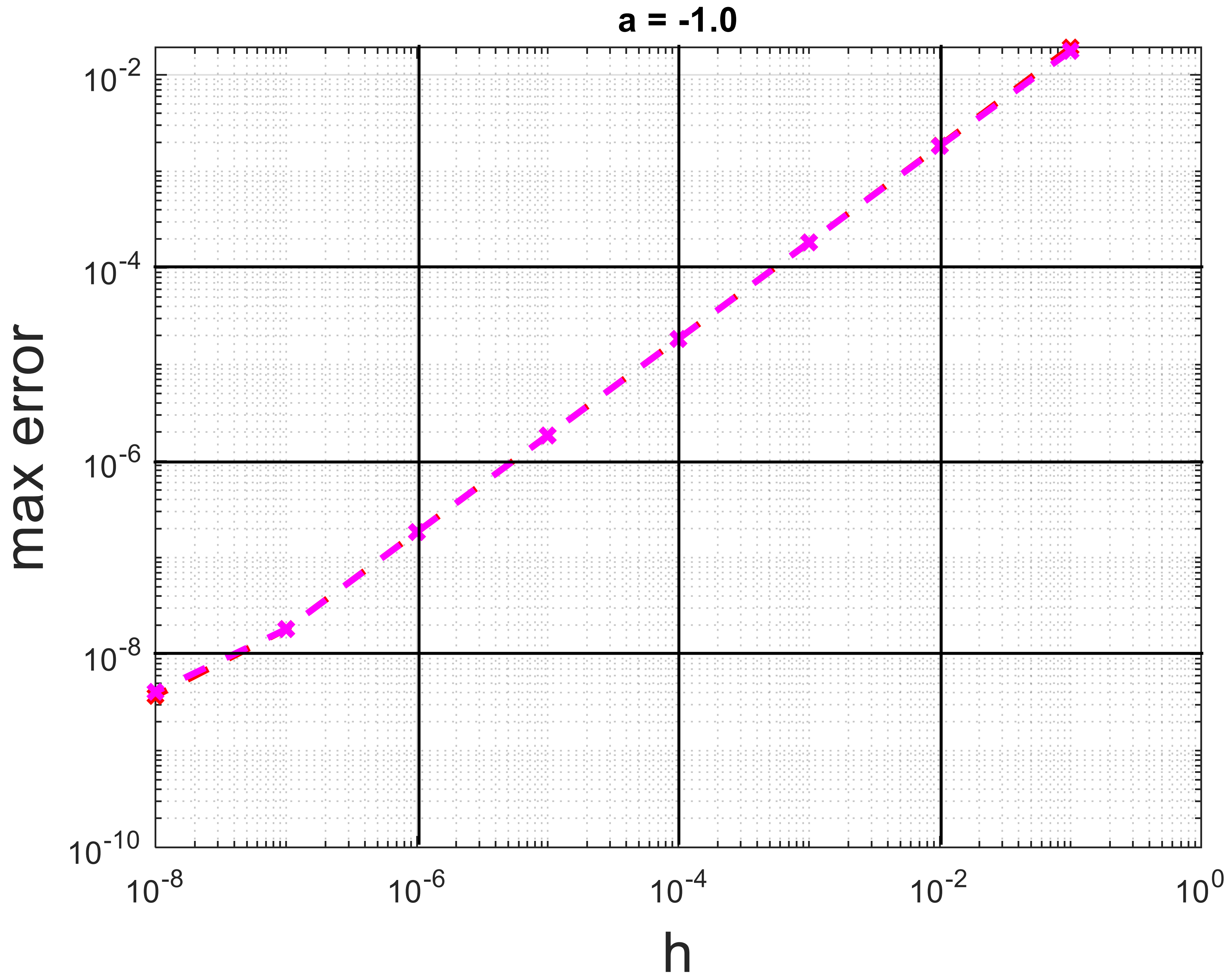
- Dahlquist Equivalence Theorem
  - Consistency + stability ⇔ convergence

- Convergence rate
  - For order p methods: $\max_k \left\| x\left(t^k\right) - y^k \right\| \leq O(h^{p+1})$
  - Forward and backward Euler: $p = 1$
    - If we half h, then the error also halves

Numerical Convergence

- Visualize convergence rate with Max error vs. h plot
- Forward and backward Euler are both 1st order
  - Half the size of h leads to half the error
- Usually, log-log plots are used to show a wide range of errors and h
  - Order p method has a slope of p (approximately).





a = -1.0
max error
h
10-2
10-4
10-6
10-8
10-10
10-8
10-6
10-4
10-2
100

Stiff equations

- ODEs with components that have very fast rates of change
  - Usually requires very small step sizes for stability

- Example: $\dot{x}_1 = ax_1$ with forward Euler
  - Stability requires $|1 + ha| \leq 1$
  - For $a = -100$, we have $|1 - 100h| \leq 1 \Leftrightarrow h \leq 0.02$

- Small step size is required even if there are other slower changing components like $\dot{x}_2 = x_1 - x_2$
  - Implicit methods (eg. backward Euler) are useful here

$$\dot{x}_1 = -100x_1$$
$$\dot{x}_2 = x_1 - x_2$$

$$\dot{x} = \begin{bmatrix} -100 & 0 \\ 1 & -1 \end{bmatrix} x$$