Guest Lecture
Ian Mitchell, UBC

Date: Jan. 23
Time: 12:30
Location: AQ 5030

Title: Improvements in robust model checking for human-in-the-loop shared control of continuous state systems.

Abstract: Recent demonstrations of continuous state reachability in thousands of dimensions are impressive, but to maximize control authority while ensuring safety for human-in-the-loop robotic or cyber-physical systems we need not just to identify the existence of a safe trajectory, but to characterize the set of safe controls. In the first part of the talk I will describe two algorithms for constructing under-approximations of robust controlled invariant or viable sets of uncertain linear systems. Time permitting, I will also describe a class of methods that account for runtime state uncertainty. Examples are drawn from the domains of automated anesthesia, quadrotors and smart wheelchairs.

Bio: Ian M. Mitchell completed his doctoral work in engineering at Stanford University in 2002, spent a year as a postdoctoral researcher at the University of California at Berkeley, and is now a Professor of Computer Science at the University of British Columbia in Vancouver. He is the author of the Toolbox of Level Set Methods, the first publicly available high accuracy implementation of solvers for dynamic implicit surfaces and the time dependent Hamilton-Jacobi equation that works in arbitrary dimension. His current research emphasizes control and planning in cyber-physical and robotic systems with a focus on safety of human-in-the-loop designs. He also studies development of algorithms and software for differential equations, formal verification, assistive technology and reproducible research.

Lyapunov Stability

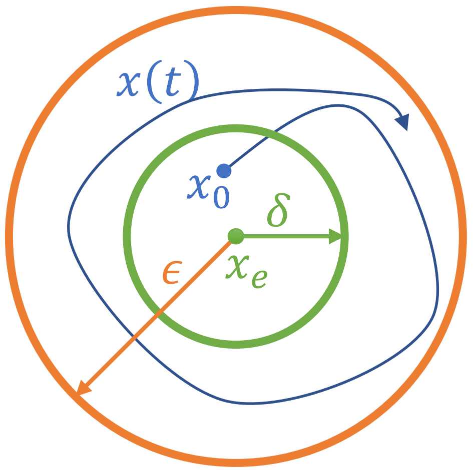
CMPT 882

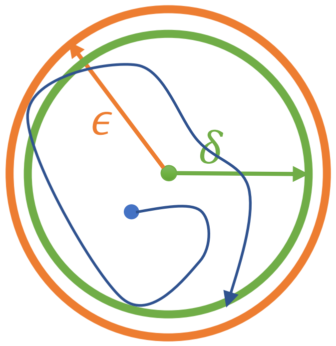
Jan. 21

Lyapunov Stability

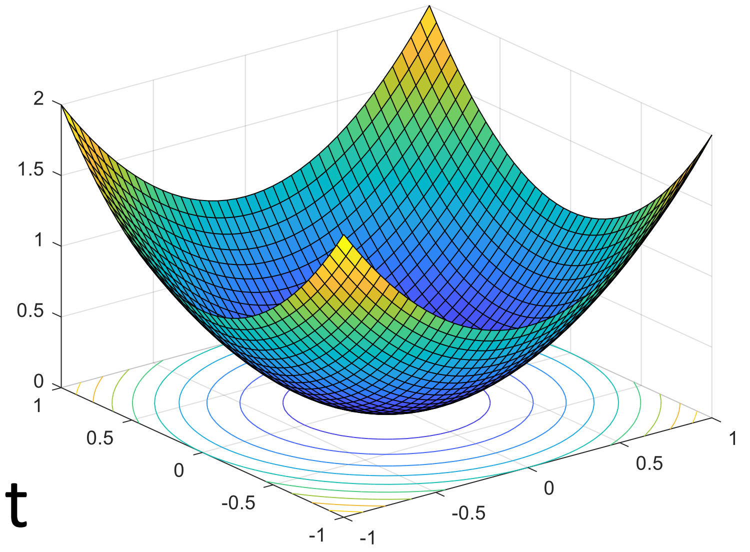
- A system is **stable in the sense of Lyapunov** if $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that

$$\|x_0 - x_e\| < \delta(\epsilon) \Rightarrow \forall t \geq t_0, \|x(t) - x_e\| < \epsilon$$





Lyapunov Stability Main Result

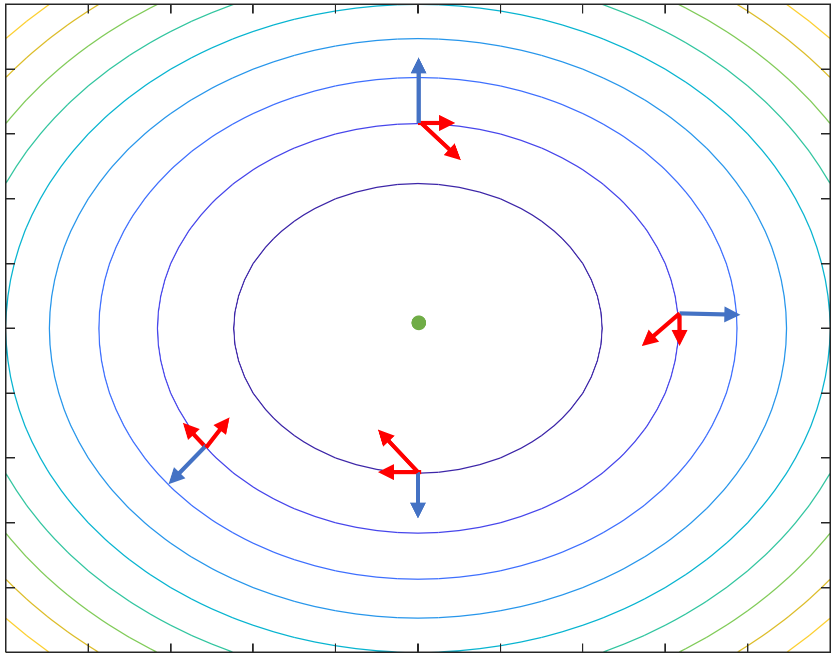


- Let $x = 0$ be an equilibrium point
- Suppose there is a function $V(x): \mathbb{R}^n \to \mathbb{R}$ such that

  $V(x) = 0$ if and only if $x = 0$,

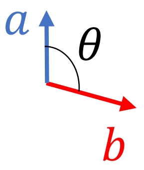
  $V(x) > 0$ if and only if $x \neq 0$.

  - If for all $x \neq 0, \dot{V}(x) = \nabla V^\top f(x) \leq 0$, then $x = 0$ is **stable in the sense of Lyapunov**
  - If for all $x \neq 0, \dot{V}(x) = \nabla V^\top f(x) < 0$, then $x = 0$ is **asymptotically stable**
- $V(x)$ is called a **Lyapunov function**



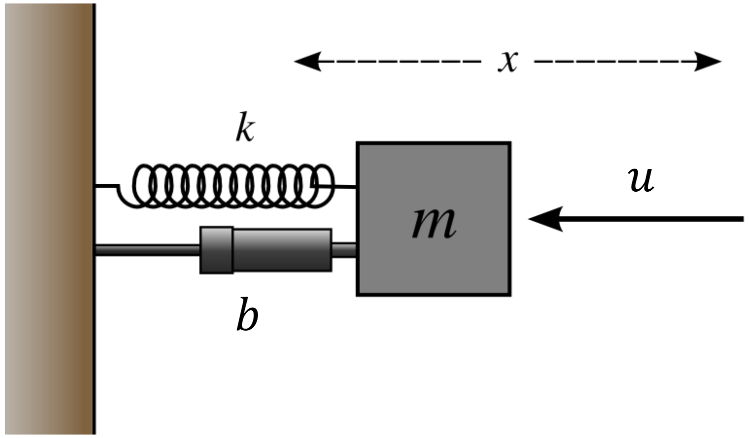
Key linear algebra fact: Given two vectors a and b, separated by an angle of θ,

$$a^\top b = \sum_{i}^{n} a_i b_i = \|a\|_2 \|b\|_2 \cos\theta$$



Proof: https://en.wikipedia.org/wiki/Dot_product

Lyapunov Stability Example in $\mathbb{R}^2$



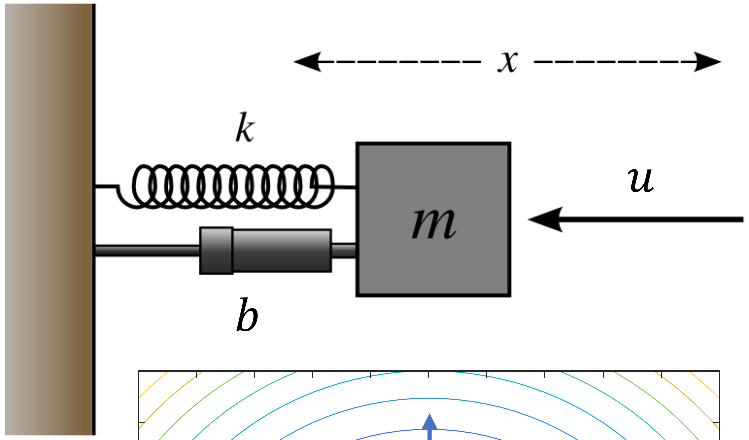
- Damped mass spring system
  - Newton's laws: $F = ma = m\ddot{x}$
  - Assume $m = 1, u = 0$
  - Define state space representation: Let $x_1 = x, x_2 = \dot{x}$

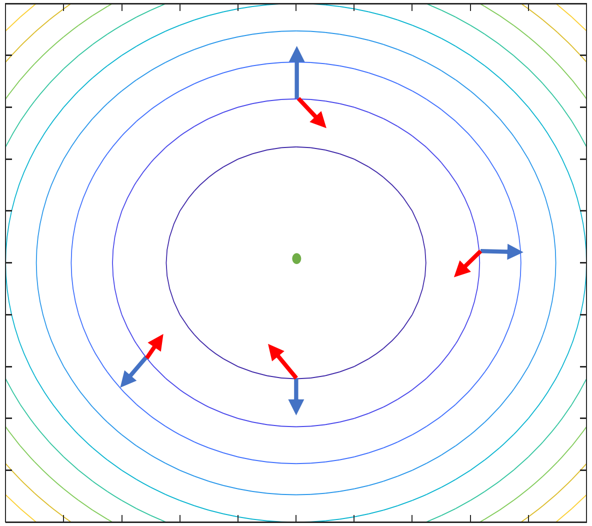
$$m\ddot{x} = -kx - b\dot{x} + u$$
$$\ddot{x} = -kx - b\dot{x}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -kx_1 - bx_2$$

- Equilibrium point: $\dot{x}_1 = 0 \Rightarrow x_2 = 0$
  $\dot{x}_2 = 0 \Rightarrow x_1 = 0$
  - Is this stable?
  - Intuition:
    - The origin is stable because there is friction
    - Friction causes the energy of the system to decrease, until no energy remains

Lyapunov Stability Example in $\mathbb{R}^2$



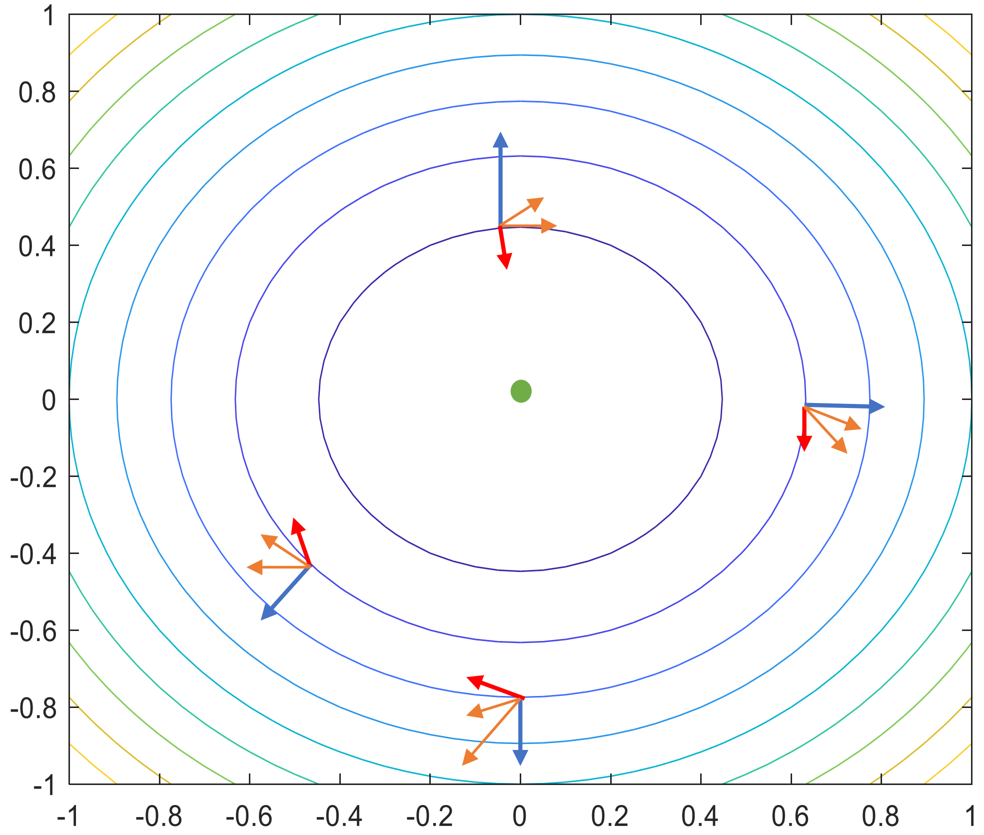


- Let $V(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$
  - Potential energy plus kinetic energy

$$
\begin{aligned}
\Rightarrow \quad \dot{V}(x_1, x_2) &= \nabla V^\top f(x) \\
&= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\
&= kx_1\dot{x}_1 + x_2\dot{x}_2 \\
&= kx_1x_2 - kx_1x_2 - bx_2^2 \\
&= -bx_2^2 \\
&< 0 \text{ for all } x \neq (0,0)
\end{aligned}
$$

Lyapunov Stability: Discussion

- What if there is control? $\dot{x} = f(x, u)$
  - Need at least one control that makes V non-increasing
- Advantages
  - Direct nonlinear analysis
  - "Global" result
  - "Region of attraction": the region where $$\dot{V}(x) \leq 0$$
- How to find a Lyapunov function?
  - Intuition → Guess something that works
  - Computational techniques
    - Optimization
    - Optimal control



Feedback Stabilization by Backstepping

- Given control affine dynamics $\dot{x} = f(x) + g(x)u$, design control policy $u = \alpha(x)$ such that $x = 0$ is asymptotically stable.

- Take a Lyapunov approach
  - Suppose we have a stabilizing control policy and Lyapunov function for
  $$\dot{X} = F(X) + G(X)u,$$
  with $u = \alpha(X)$ and $\bar{V}(X)$ such that $\dot{\bar{V}}(X) = \frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X)\big) < 0$
  - Given this, consider the special case where we need to come up with a stabilizing policy for
  $$\dot{X} = F(X) + G(X)\bar{x}$$
  $$\dot{\bar{x}} = u$$

Backstepping

Suppose we have a stabilizing policy $u = \alpha(X)$ for $\dot{X} = F(X) + G(X)u$, with $\bar{V}(X)$ such that

$$\dot{\bar{V}}(X) = \frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X)\big) < 0$$

- Consider the special case

$$\begin{aligned}\dot{X} &= F(X) + G(X)\bar{x} \\ \dot{\bar{x}} &= u\end{aligned}$$

Change of variables

$z \coloneqq \bar{x} - \alpha(X)$

$\bar{x} = z + \alpha(X)$

$$\begin{aligned}\dot{X} &= F(X) + G(X)z + G(X)\alpha(X) \\ \dot{z} &= u - \dot{\alpha}(X)\end{aligned}$$

- Lucky guess: $V(X,z) = \bar{V}(X) + \frac{1}{2}z^2$

$$\begin{aligned}\dot{V}(X,z) &= \dot{\bar{V}}(X) + z\dot{z} \\ &= \frac{\partial \bar{V}}{\partial X}\left(F(X) + G(X)\alpha(X) + G(X)z\right) + z\big(u - \dot{\alpha}(X)\big) \\ &= \underbrace{\frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X)\big)}_{<\,0,\ \text{by assumption}} + \underbrace{z\left(\frac{\partial \bar{V}}{\partial X}G(X) + u - \dot{\alpha}(X)\right)}_{<\,0 \text{ if } u = \dot{\alpha}(X) - \frac{\partial \bar{V}}{\partial X}G(X) - kz,\ k > 0}\end{aligned}$$

$$\dot{\alpha}(X) = \frac{\partial \alpha}{\partial X}\left(F(X) + G(X)\bar{x}\right)$$

Backstepping

- Example:
  - $\dot{x}_1 = x_1^2 + x_2$
  - $\dot{x}_2 = u$
- Treat x_2 as a "virtual" control in $\dot{x}_1$:
  - $\dot{x}_1 = x_1^2 + u$
  - This is easy to stabilize and find Lyapunov function:

$$u = \alpha(x_1) = -x_1^2 - \bar{k}x_1, \bar{k} > 0; \quad \bar{V}(x_1) = \frac{1}{2}x_1^2$$

$$\begin{aligned}\dot{\bar{V}}(x_1) &= x_1\dot{x}_1 \\ &= x_1(x_1^2 + u) \\ &= x_1\left(x_1^2 - x_1^2 - \bar{k}x_1\right) \\ &= x_1\left(-\bar{k}x_1\right) \\ &= -\bar{k}x_1^2\end{aligned}$$

- Apply previous result:
  - $u = \dot{\alpha}(x_1) - \frac{\partial \bar{V}}{\partial x_1}G(x_1) - kz$
  - $u = \left(-2x_1 - \bar{k}\right)(x_1^2 + x_2) - x_1 - k\left(x_2 + x_1^2 + \bar{k}x_1\right)$

$$\dot{\alpha}(x_1) = \frac{\partial \alpha}{\partial x_1}(x_1^2 + x_2) = \left(-2x_1 - \bar{k}\right)(x_1^2 + x_2)$$

$$\frac{\partial \bar{V}}{\partial x_1} = x_1, G(x_1) = 1$$

$$z = x_2 - \alpha(x_1) = x_2 + x_1^2 + \bar{k}x_1$$