Nonlinear Systems II

CMPT 882

Jan. 19

Nonlinear Systems Roadmap

- Introduction
- Analysis
- Control
- Numerical solutions

Nonlinear Systems Roadmap: Today

- Analysis
  - Bifurcations

- Control
  - Lyapunov functions
  - Linearization by State Feedback

Bifurcations

- Parameters in ODE models of systems that determine key behaviours of the system
  - Sometimes, a small change in the parameter leads to big changes in system behaviour

- Example: $\dot{x} = \mu x - x^3$, μ is a parameter
  - Equilibrium points:

$$\begin{aligned} 0 &= \dot{x} \\ &= \mu x - x^3 \\ &= x(\mu - x^2) \\ x &= 0, \pm\sqrt{\mu} \end{aligned}$$

Bifurcations

- Example: $\dot{x} = \mu x - x^3$, μ is a parameter
  - Equilibrium points: $x = 0, \pm\sqrt{\mu}$
  - Stability: $\frac{\partial f}{\partial x} = \mu - 3x^2$

$x = 0$:

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \mu$$

Equilibrium always exists

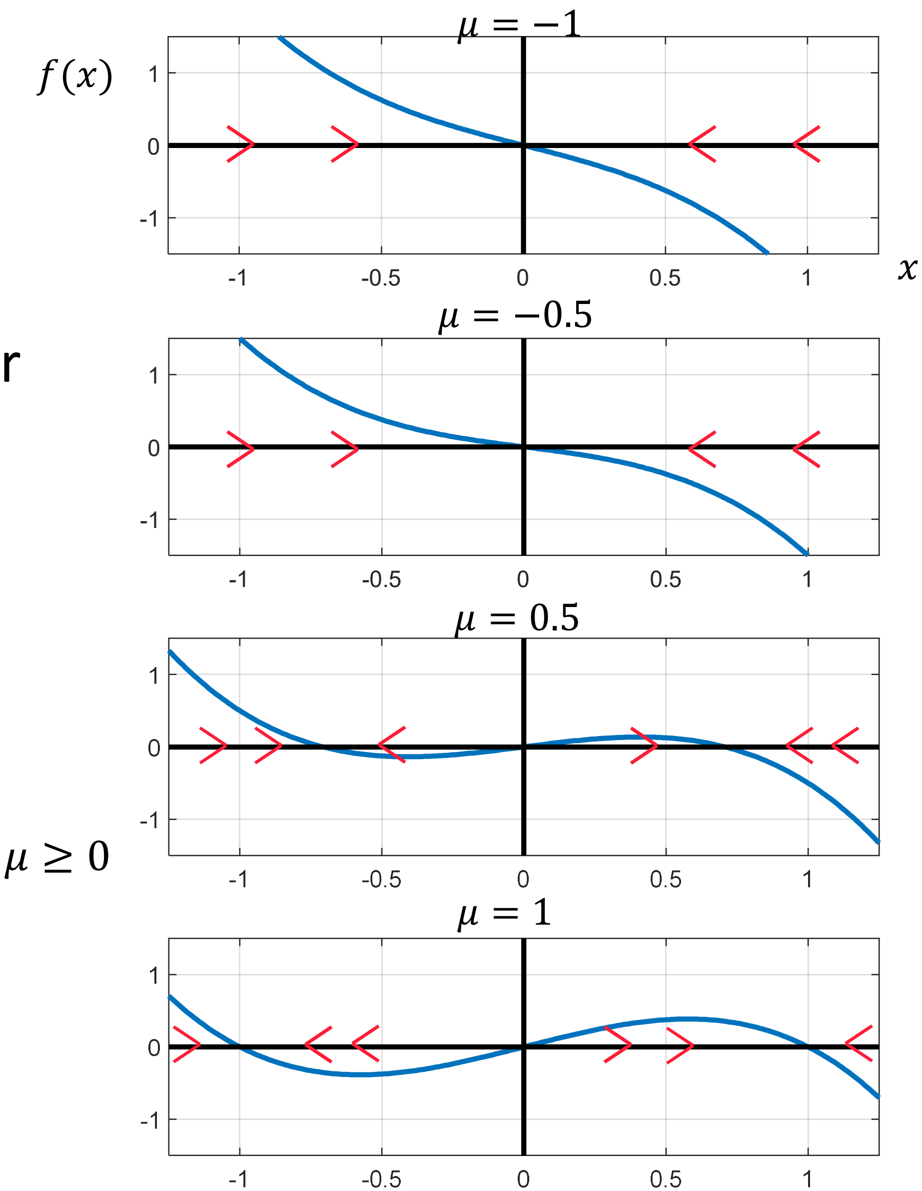
- Stable when $\mu < 0$
- Unstable when $\mu > 0$

$x = \pm\sqrt{\mu}$:

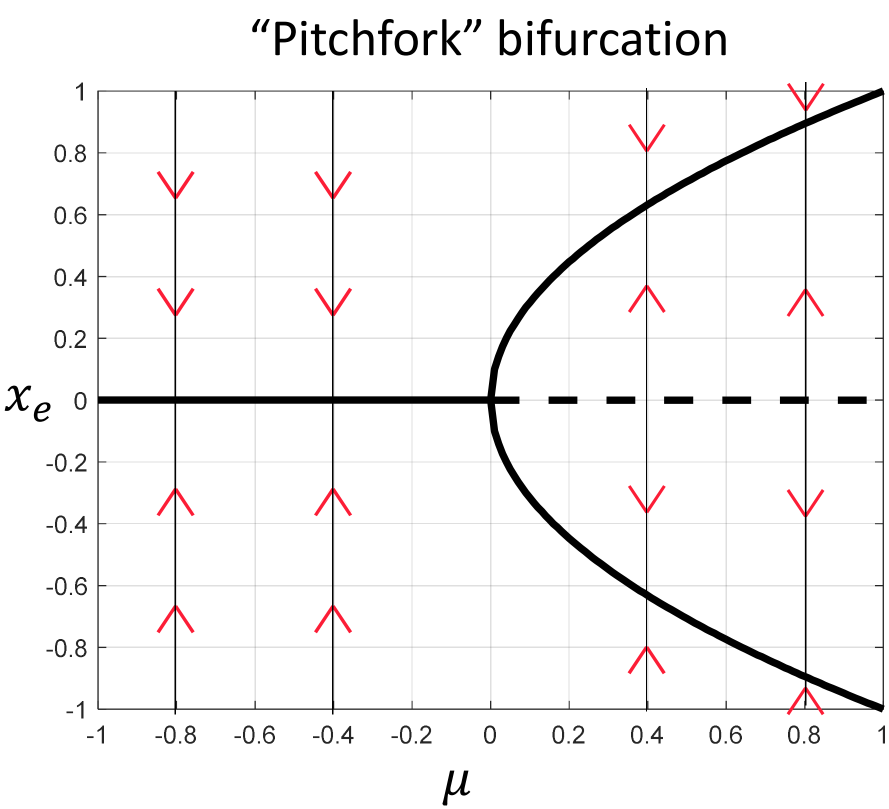
$$\left.\frac{\partial f}{\partial x}\right|_{x=\pm\sqrt{\mu}} = -2\mu$$

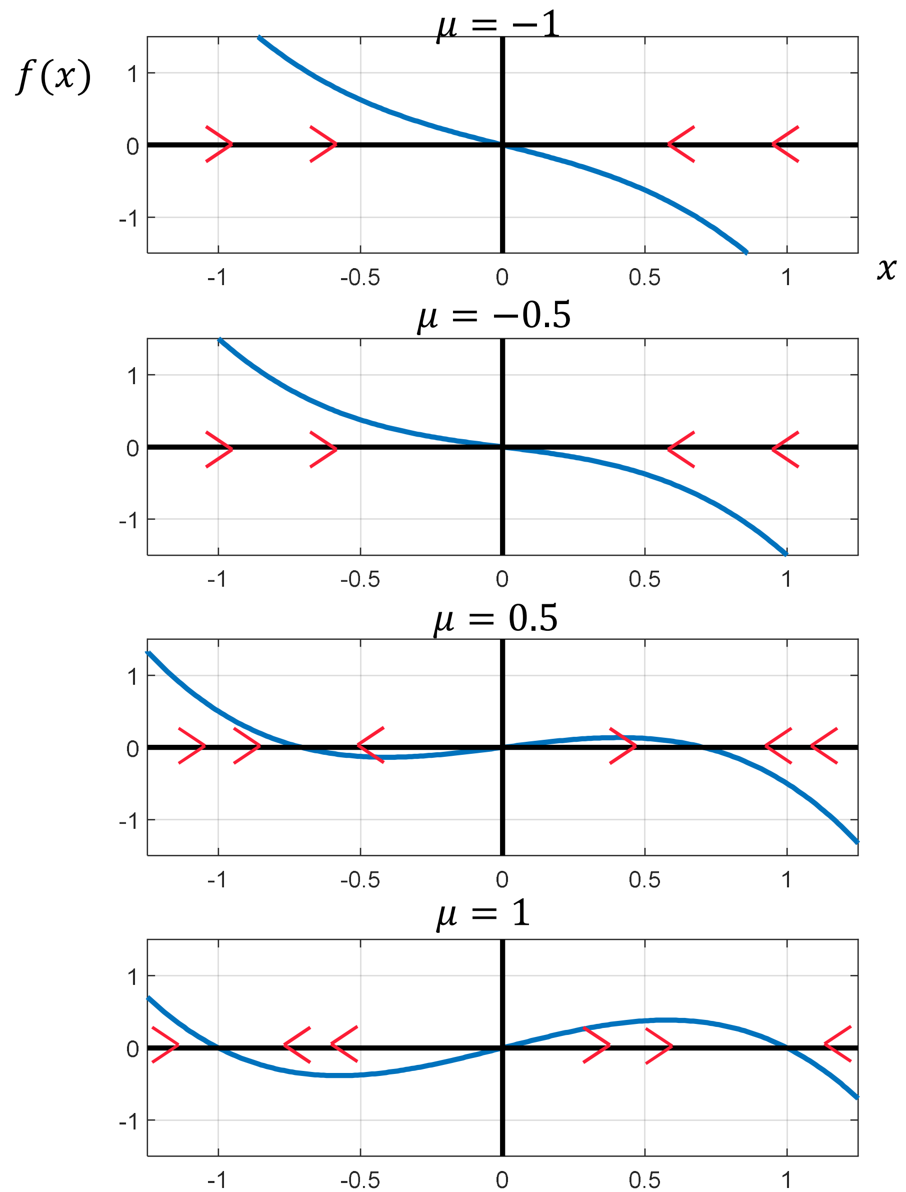
Equilibria exist only when $\mu \geq 0$

- Stable when they exist



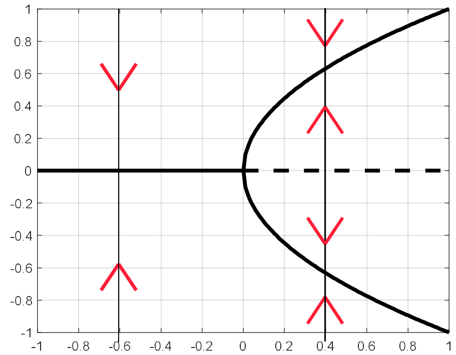
Bifurcations



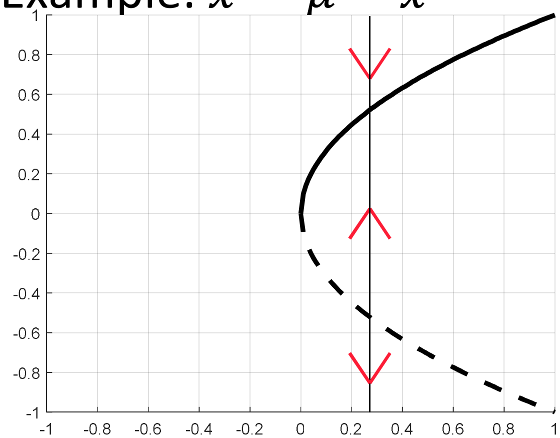


Some Bifurcation Types

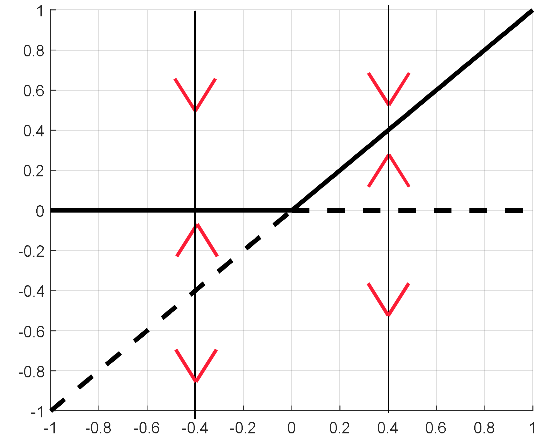
- Pitchfork bifurcation



- Fold bifurcation
  - Example: $\dot{x} = \mu - x^2$



- Transcritical bifurcation
  - Example: $\dot{x} = \mu x - x^2$



- Hopf bifurcation (2D, in your homework)

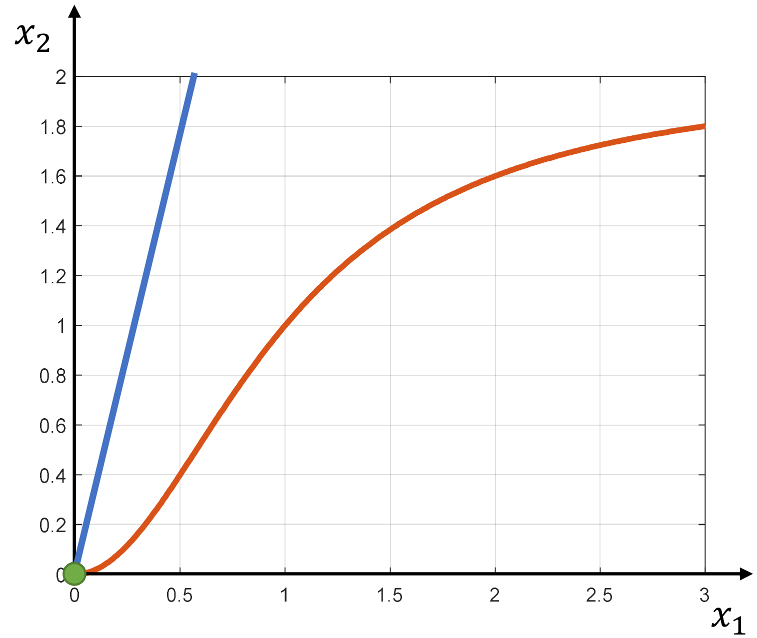
Bifurcation: 2D Example

- Example 2:

$$\dot{x}_1 = -ax_1 + x_2$$
$$\dot{x}_2 = \frac{x_1^2}{1 + x_1^2} - \frac{1}{2}x_2$$

  - Equilibrium points:

$$\dot{x}_2 = 0 \Rightarrow x_2 = \frac{2x_1^2}{1 + x_1^2}$$
$$\dot{x}_1 = 0 \Rightarrow x_2 = ax_1$$



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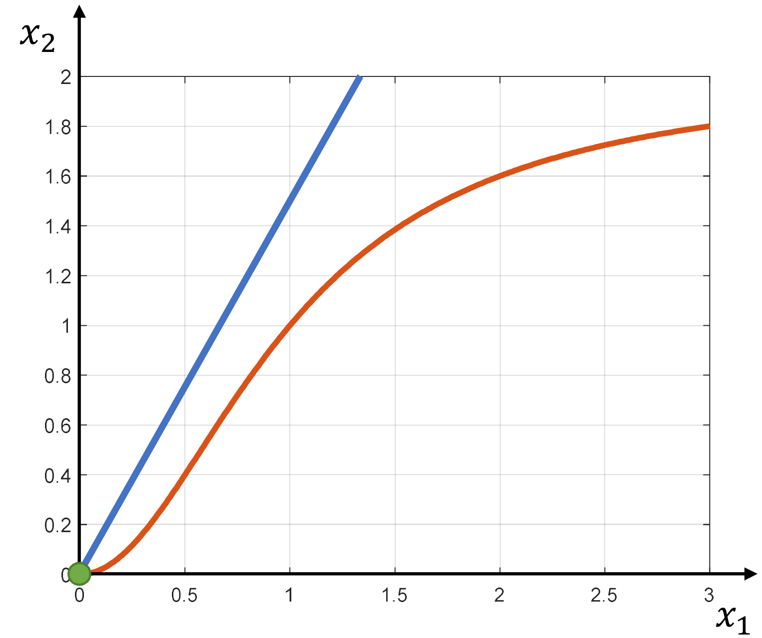

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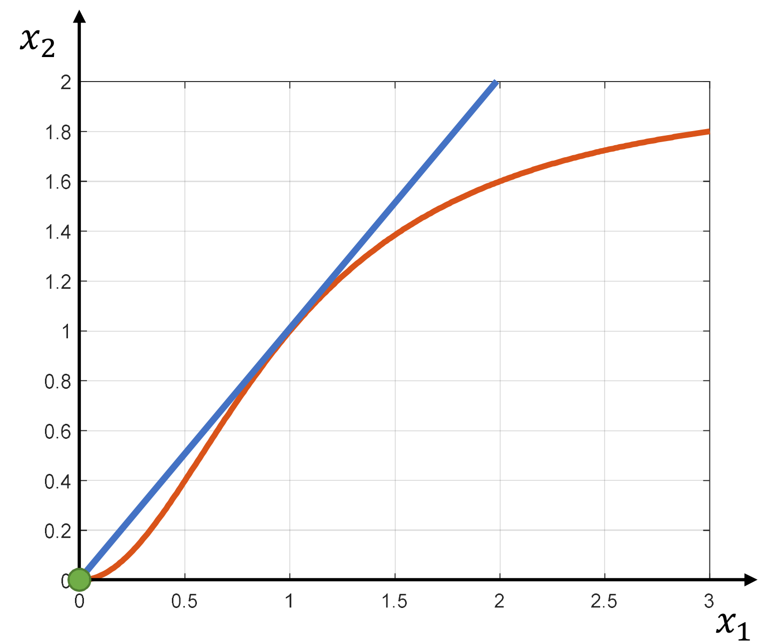

Bifurcation: 2D Example

- Example 2:
$$\dot{x}_1 = -ax_1 + x_2$$
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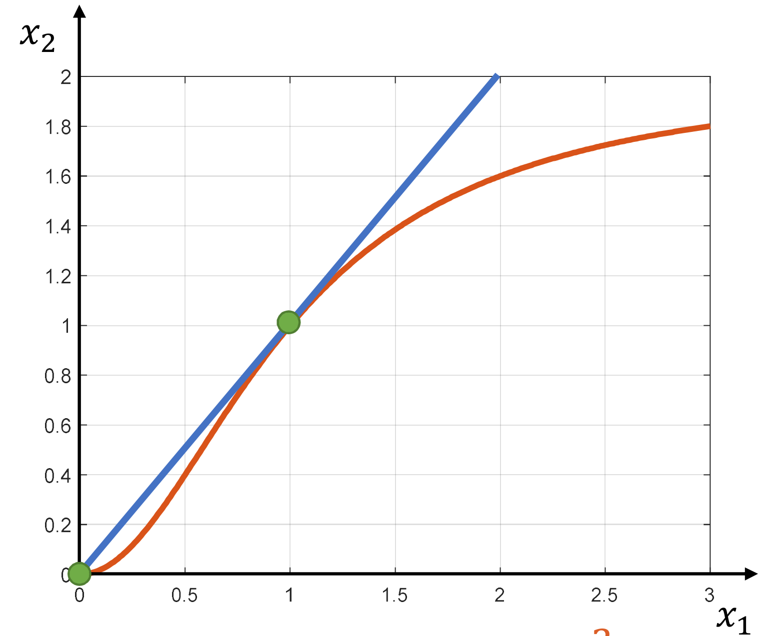
  - Starting at large values of a, there is only one equilibrium point at the origin
  - As a decreases, eventually another equilibrium point spawns



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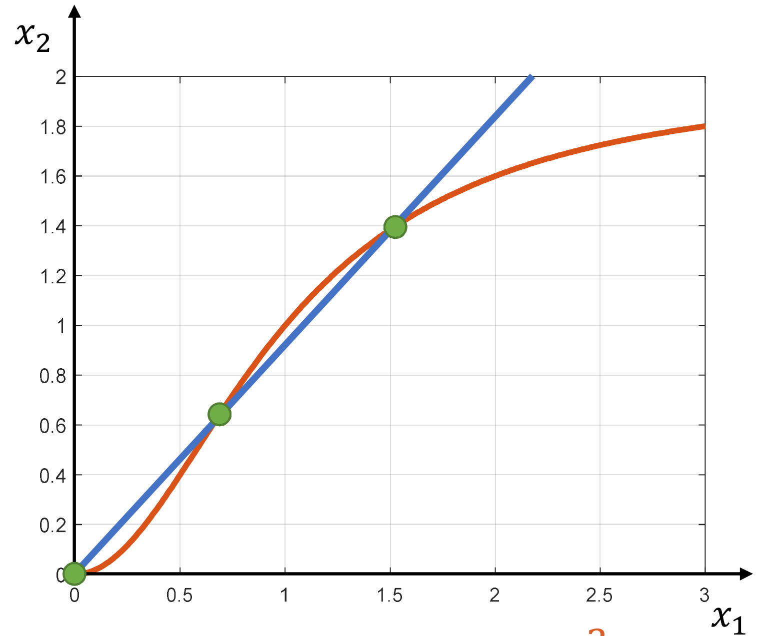
Bifurcation: 2D Example

- Example 2:
$$\dot{x}_1 = -a x_1 + x_2$$
$$\dot{x}_2 = \frac{x_1^2}{1 + x_1^2} - \frac{1}{2} x_2$$

  - Starting at large values of a, there is only one equilibrium point at the origin
  - As a decreases, eventually another equilibrium point spawns
  - For even smaller values of a, there are three equilibrium points in total



$$\dot{x}_2 = 0 \Rightarrow x_2 = \frac{2x_1^2}{1 + x_1^2}$$
$$\dot{x}_1 = 0 \Rightarrow x_2 = a x_1$$

More Complete Analysis

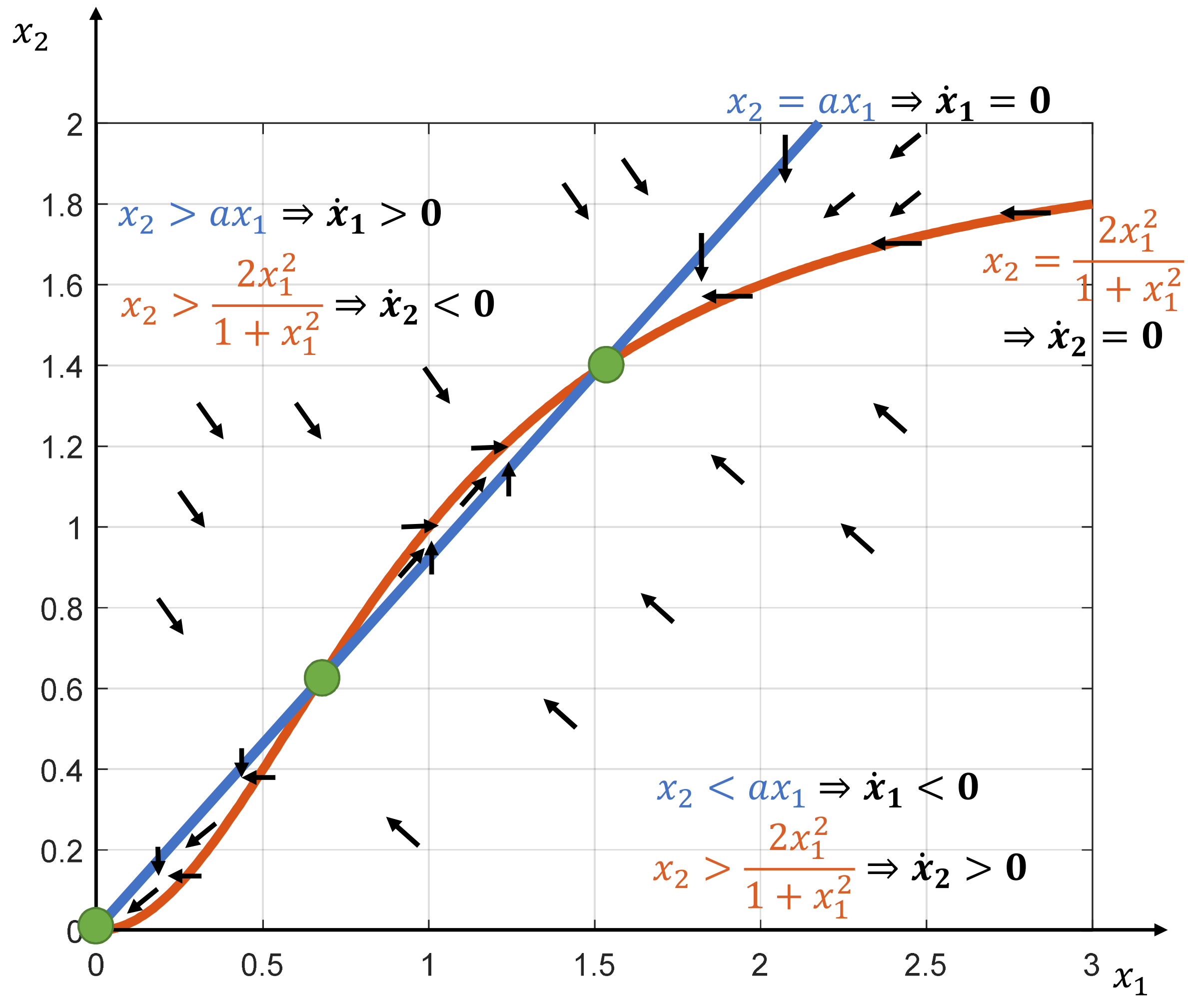
$$\dot{x}_1 = -ax_1 + x_2$$
$$\dot{x}_2 = \frac{x_1^2}{1+x_1^2} - \frac{1}{2}x_2$$

$\dot{x}_1 = 0 \Rightarrow x_2 = ax_1$

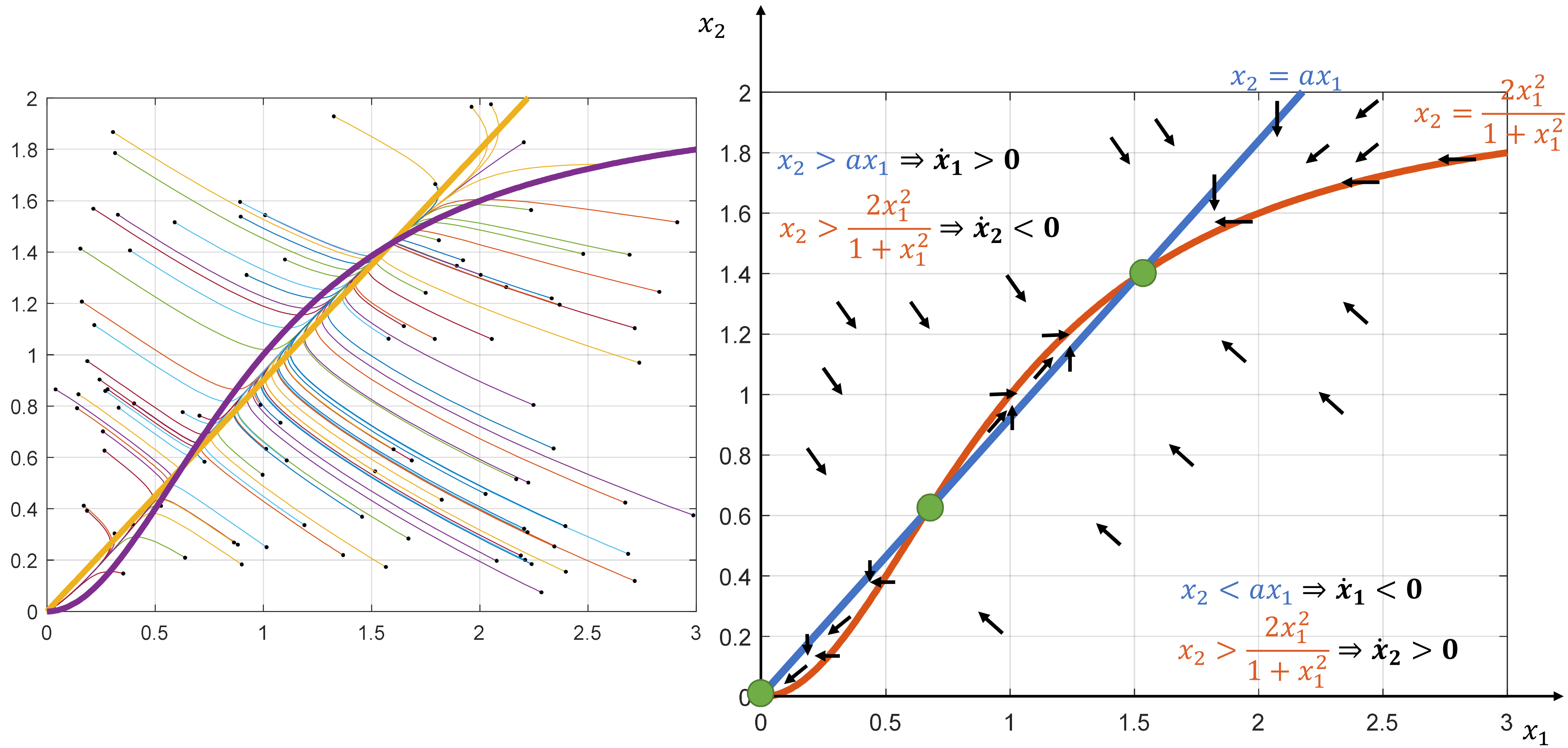
- Vertical flow field

$\dot{x}_2 = 0 \Rightarrow x_2 = \dfrac{2x_1^2}{1+x_1^2}$

- Horizontal flow field



More Complete Analysis

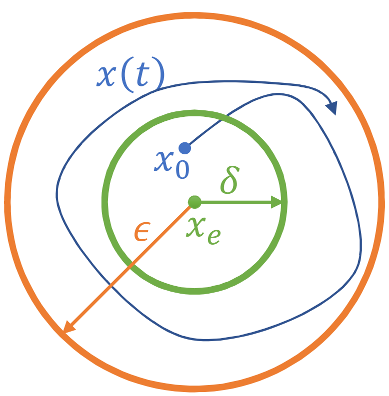


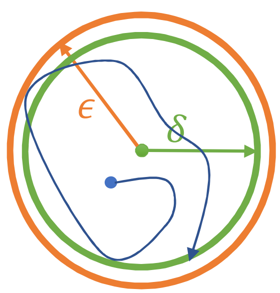
Lyapunov Stability

- General stability theory for nonlinear systems
  - No need to solve ODE
  - No need to linearize: direct analysis of nonlinear systems

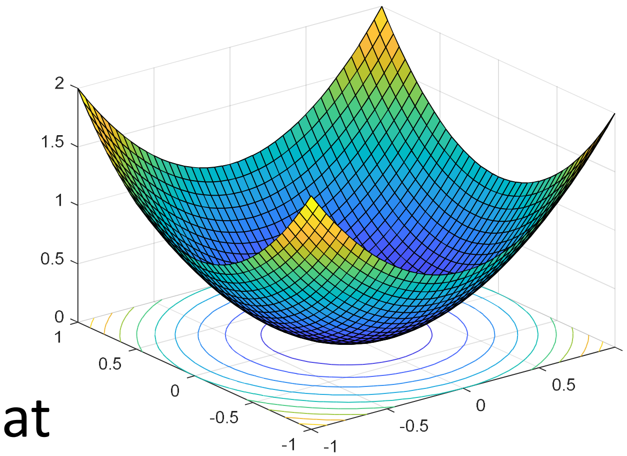
A system is **stable in the sense of Lyapunov** if $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that

$$\|x_0 - x_e\| < \delta(\epsilon) \Rightarrow \forall t \geq t_0, \|x(t) - x_e\| < \epsilon$$





Lyapunov Stability Main Result

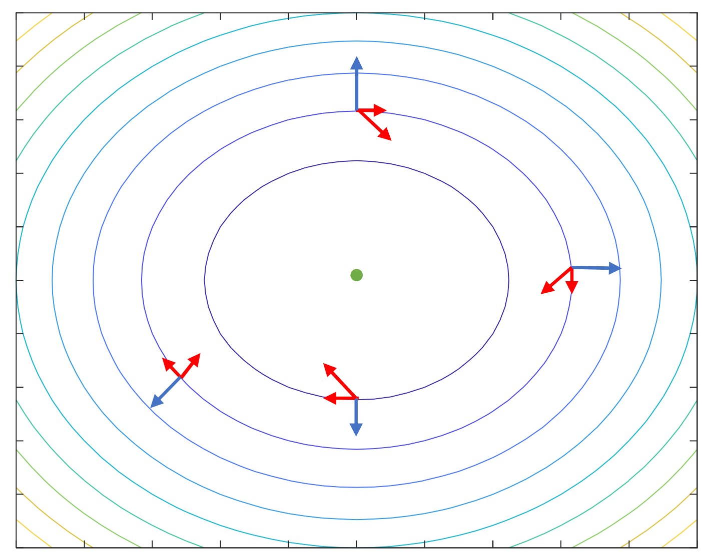


- Let $x = 0$ be an equilibrium point
- Suppose there is a function $V(x): \mathbb{R}^n \rightarrow \mathbb{R}$ such that

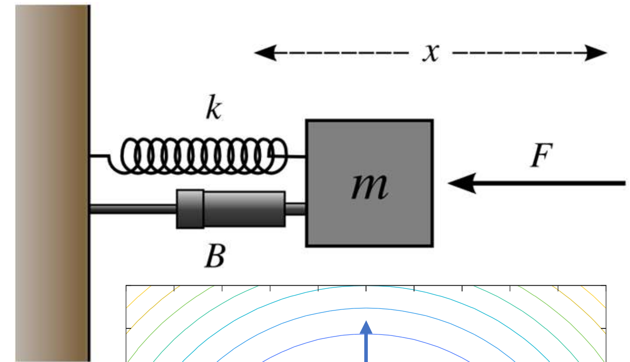
  $V(x) = 0$ if and only if $x = 0$,

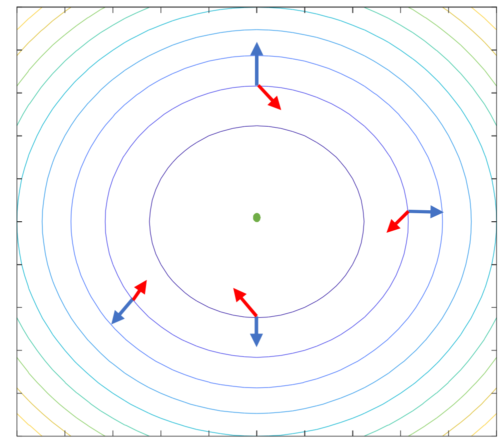
  $V(x) > 0$ if and only if $x \neq 0$.

  - If for all $x \neq 0, \dot{V}(x) = \nabla V^\top f(x) \leq 0$, then $x = 0$ is **stable in the sense of Lyapunov**
  - If for all $x \neq 0, \dot{V}(x) = \nabla V^\top f(x) < 0$, then $x = 0$ is **asymptotically stable**
- $V(x)$ is called a **Lyapunov function**



Lyapunov Stability Example in $\mathbb{R}^2$



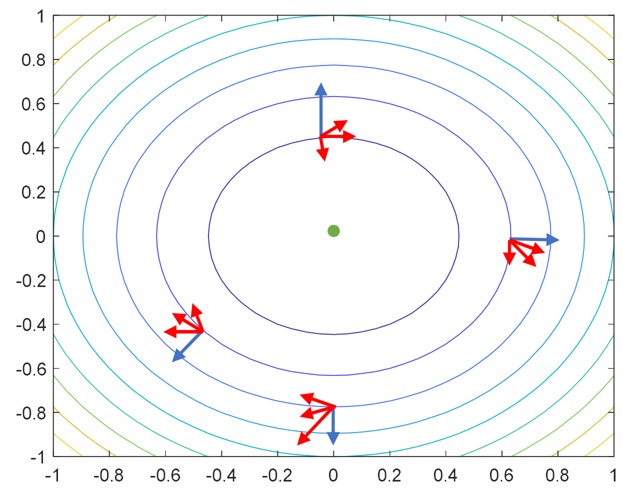


- Damped spring system: $\ddot{x} + b\dot{x} + kx = 0, b \geq 0$
  - Intuition: The system should be stable due to friction
  - Let $x_1 = x, x_2 = \dot{x}$ $\Rightarrow$ $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 - bx_2 \end{aligned}$

- Let $V(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$
  - Potential energy plus kinetic energy

$$\Rightarrow \quad \begin{aligned} \dot{V}(x_1, x_2) &= \nabla V^\top f(x) \\ &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &= kx_1\dot{x}_1 + x_2\dot{x}_2 \\ &= kx_1x_2 - kx_1x_2 - bx_2^2 \\ &= -bx_2^2 \\ &< 0 \text{ for all } x \neq (0,0) \end{aligned}$$

Lyapunov Stability: Discussion



- What if there is control? $\dot{x} = f(x, u)$
  - Need at least one control that makes V non-increasing

- Advantages
  - Direct nonlinear analysis
  - “Global” result
  - “Region of attraction”

- How to find a Lyapunov function?
  - Intuition → Guess something that works
  - Computational techniques
    - Optimization
    - Optimal control

Feedback Stabilization

- Given control affine dynamics $\dot{x} = f(x) + g(x)u$, design control policy $u = \alpha(x)$ such that $x = 0$ is asymptotically stable.

- Take a Lyapunov approach
  - Suppose we have a stabilizing control policy and Lyapunov function for
$$\dot{X} = F(X) + G(X)u,$$
with $u = \alpha(X)$ and $\bar{V}(X)$ such that $\dot{\bar{V}}(X) = \frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X)\big) < 0$
  - Given this, consider the special case where we need to come up with a stabilizing policy for
$$\dot{X} = F(X) + G(X)\bar{x}$$
$$\dot{\bar{x}} = u$$

Feedback Stabilization

Suppose we have a stabilizing policy $u = \alpha(X)$ for $\dot{X} = F(X) + G(X)u$, with $\bar{V}(X)$ such that

$$\dot{\bar{V}}(X) = \frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X)\big) < 0$$

- Consider the special case

$$\begin{aligned}\dot{X} &= F(X) + G(X)\bar{x} \\ \dot{\bar{x}} &= u\end{aligned} \xrightarrow[\bar{x} = z + \alpha(X)]{\text{Change of variables} \quad z := \bar{x} - \alpha(X)} \begin{aligned}\dot{X} &= F(X) + G(X)z + G(X)\alpha(X) \\ \dot{z} &= u - \dot{\alpha}(X)\end{aligned}$$

- Lucky guess: $V(X, z) = \bar{V}(X) + \frac{1}{2}z^2$

$$\begin{aligned}\dot{V}(X,z) &= \dot{\bar{V}}(X) + z\dot{z} \\ &= \frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X) + G(X)z\big) + z\big(u - \dot{\alpha}(X)\big) \\ &= \underbrace{\frac{\partial \bar{V}}{\partial X}\big(F(X) + G(X)\alpha(X)\big)}_{<\,0\text{, by assumption}} + \underbrace{z\left(\frac{\partial \bar{V}}{\partial X}G(X) + u - \dot{\alpha}(X)\right)}_{<\,0 \text{ if } u = \dot{\alpha}(X) - \frac{\partial \bar{V}}{\partial X}G(X) - kz,\ k > 0}\end{aligned}$$

$$\dot{\alpha}(X) = \frac{\partial \alpha}{\partial X}\big(F(X) + G(X)\bar{x}\big)$$

Feedback Stabilization

- Example:
  - $\dot{x}_1 = x_1^2 + x_2$
  - $\dot{x}_2 = u$
- Treat x_2 as a "virtual" control in $\dot{x}_1$:
  - $\dot{x}_1 = x_1^2 + u$
  - This is easy to stabilize and find Lyapunov function:

$$u = \alpha(x_1) = -x_1^2 - \bar{k}x_1, \bar{k} > 0; \quad \bar{V}(x_1) = \frac{1}{2}x_1^2$$

$$\begin{aligned}\dot{\bar{V}}(x_1) &= x_1\dot{x}_1 \\ &= x_1(x_1^2 + u) \\ &= x_1\left(x_1^2 - x_1^2 - \bar{k}x_1\right) \\ &= x_1(-\bar{k}x_1) \\ &= -\bar{k}x_1^2\end{aligned}$$

- Apply previous result:
  - $u = \dot{\alpha}(x_1) - \frac{\partial \bar{V}}{\partial x_1} G(x_1) - kz$
  - $u = \left(-2x_1 - \bar{k}\right)(x_1^2 + x_2) - x_1 - k\left(x_2 + x_1^2 + \bar{k}x_1\right)$

$$\dot{\alpha}(x_1) = \frac{\partial \alpha}{\partial x_1}(x_1^2 + x_2) = \left(-2x_1 - \bar{k}\right)(x_1^2 + x_2)$$

$$\frac{\partial \bar{V}}{\partial x_1} = x_1, G(x_1) = 1$$

$$z = x_2 - \alpha(x_1) = x_2 + x_1^2 + \bar{k}x_1$$

Numerical Solutions of ODEs

- Discretization: $t^k = kh,\ u^k := u(t^k)$
- Approximate solution: $y^k \approx x(kh)$

$$\dot{x} = f(x,u)$$
$$\frac{x\big((k+1)h\big) - x(kh)}{h} = f\big(x(kh), u^k\big)$$
$$\frac{y^{k+1} - y^k}{h} = f\big(y^k, u^k\big)$$

- Simplest methods:
  - Forward Euler
$$\frac{y^{k+1} - y^k}{h} = f\big(y^k, u^k\big) \Rightarrow y^{k+1} = y^k + hf\big(y^k, u^k\big)$$
  - Backward Euler
$$\frac{y^{k+1} - y^k}{h} = f\big(y^{k+1}, u^k\big) \Rightarrow \text{solve for } y^{k+1} \text{ implicitly}$$

Example

- $\dot{x} = \lambda x, \; x(0) = x_0$
    - Analytic solution: $x(t) = x_0 e^{\lambda t}$

Consistency

- ODE is satisfied as $h \to 0$
  - Forward Euler: $$\frac{y^{k+1} - y^k}{h} = f\left(y^k, u^k\right)$$
  - More generally: $y^{k+1} = \sum_{n=\underline{k}}^{k} \alpha_i y^i + h \sum_{n=\underline{k}}^{k} \beta_i f\left(y^i, u^i\right)$

- **Truncation error**:
  - induced during one step, assuming perfect information
$$e^k := y^{k+1} - \sum_{n=\underline{k}}^{k} \alpha_n x(nh) - h \sum_{n=\underline{k}}^{k} \beta_i f\left(x(nh), u^i\right)$$
  - Consistency requires $\frac{\|e^k\|}{h} \to 0$ as $h \to 0$
  - If $\frac{\|e^k\|}{h} = O(h^p)$, then the numerical method is "order p".

Numerical stability

- $y^{k+1} = \sum_{n=\underline{k}}^{k} \alpha_i y^i + h \sum_{n=\underline{k}}^{k} \beta_i f\left(y^i, u^i\right)$
  - A map from $\left\{y^i\right\}_{n=\underline{k}}^{k}$ to y^{k+1}
  - Stability is desirable (at least for ODEs with stable solutions)

- Example: $\dot{x} = \lambda x$ with forward Euler
  - $y^{k+1} = y^k + h\lambda y^k$
  - $y^{k+1} = (1 + h\lambda) y^k$
  - Stability requires $|1 + h\lambda| \leq 1$
  - For $\lambda = -1$, we have $|1 - h| \leq 1 \Leftrightarrow h \leq 2$

Numerical convergence

- Definition: $\max_k \left\| x(kh) - y^k \right\| \to 0$ as $h \to 0$
  - Basic requirement for numerical solutions
- Dahlquist Equivalence Theorem
  - Consistency + stability $\Leftrightarrow$ convergence
- Convergence rate
  - Typically, for order p methods: $\max_k \left\| x(kh) - y^k \right\| \leq O(h^p)$
  - Forward and backward Euler: $p = 1$

Stiff equations

- ODEs with components that have very fast rates of change
  - Usually requires very small step sizes for stability

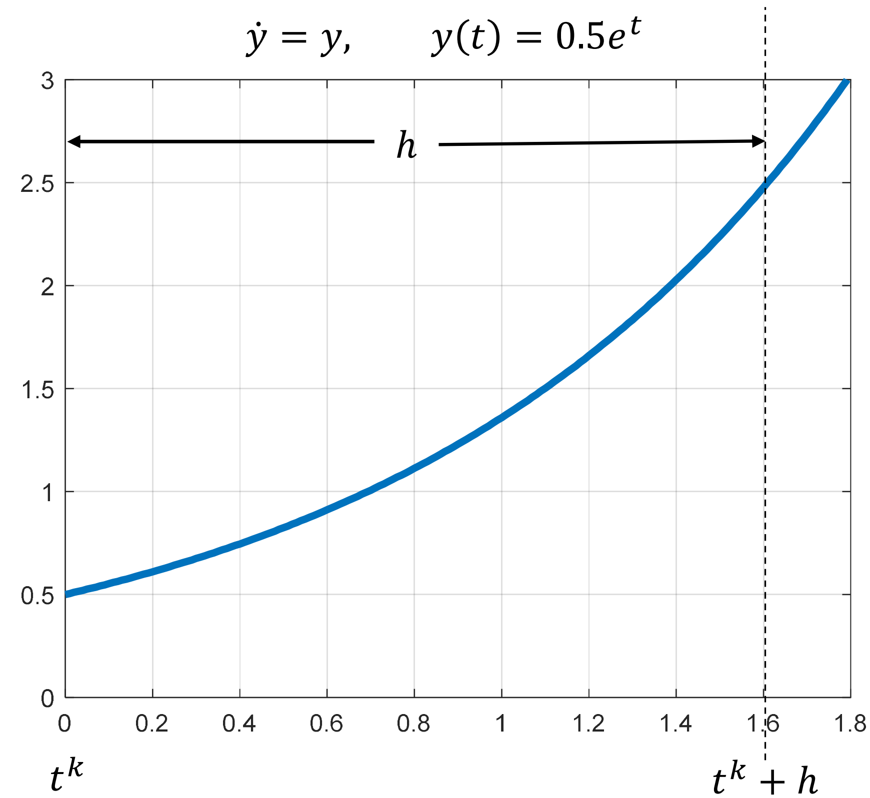
- Example: $\dot{x}_1 = \lambda x_1$ with forward Euler
  - Stability requires $|1 + h\lambda| \leq 1$
  - For $\lambda = -100$, we have $|1 - 100h| \leq 1 \Leftrightarrow h \leq 0.02$

- Small step size is required even if there are other slower changing components like $\dot{x}_2 = x_1$
  - Implicit methods are useful here (accuracy limited to order 2)

Classical Runge-Kutta Method (RK4)

- Main consideration: what slope to use?
  - Weighted average

- $y^{k+1} = y^k + \quad (\qquad\qquad\qquad\qquad)$



Classical Runge-Kutta Method (RK4)

- Main consideration: what slope to use?
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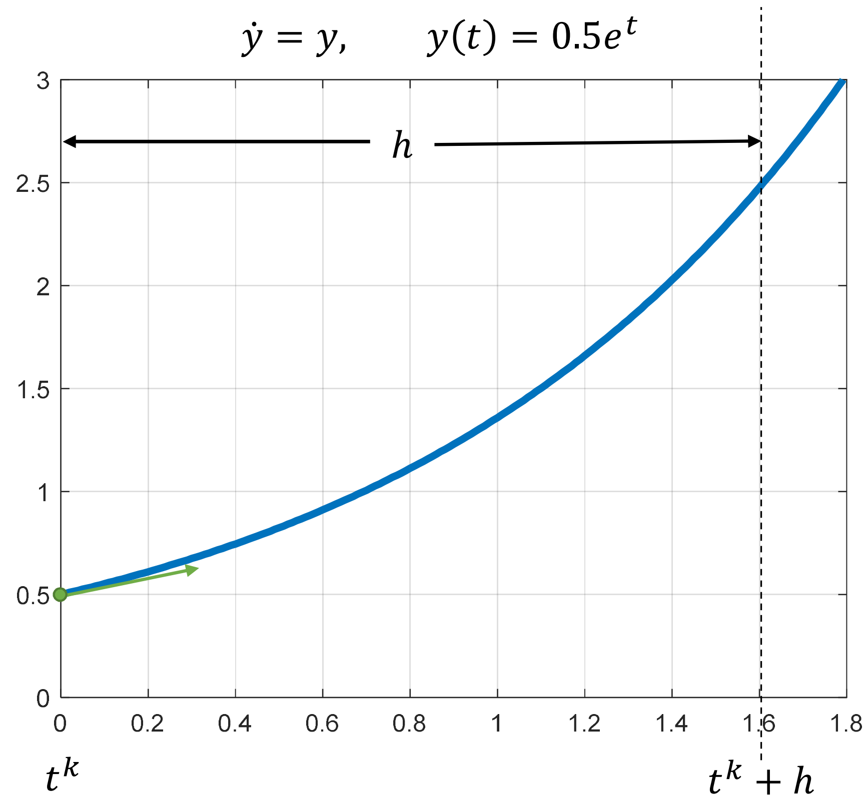
- $y^{k+1} = y^k + \quad (k_1 \qquad\qquad\qquad\qquad)$
  - $k_1 = hf\left(t^k, y^k\right)$



Classical Runge-Kutta Method (RK4)

- Main consideration: what slope to use?
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- $y^{k+1} = y^k + \quad (k_1 \qquad k_2 \qquad\qquad)$
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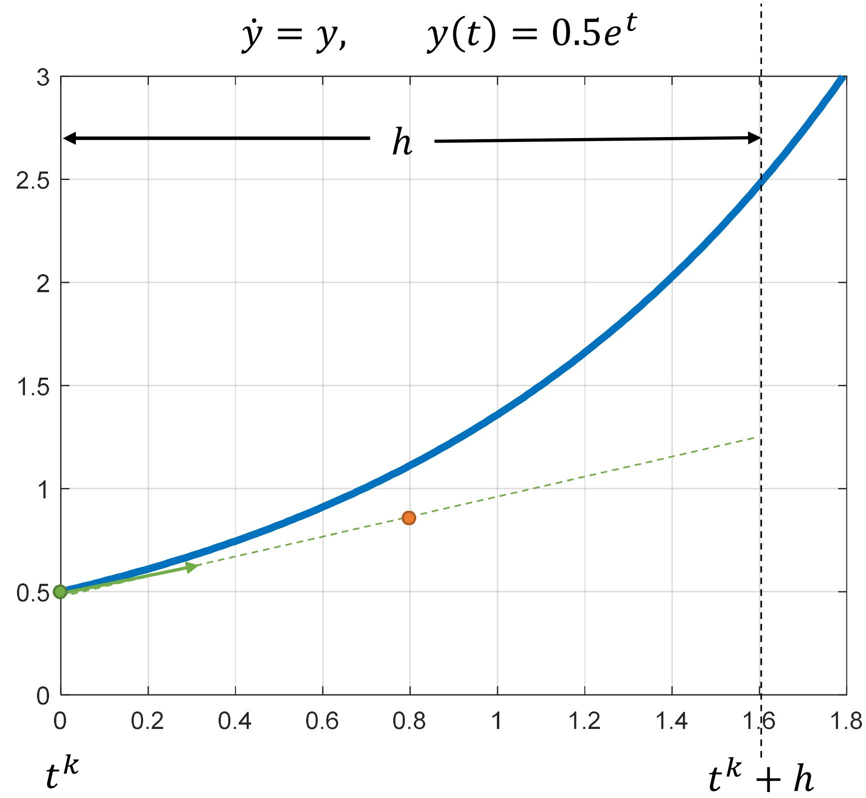
  - $k_2 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_1}{2}\right)$



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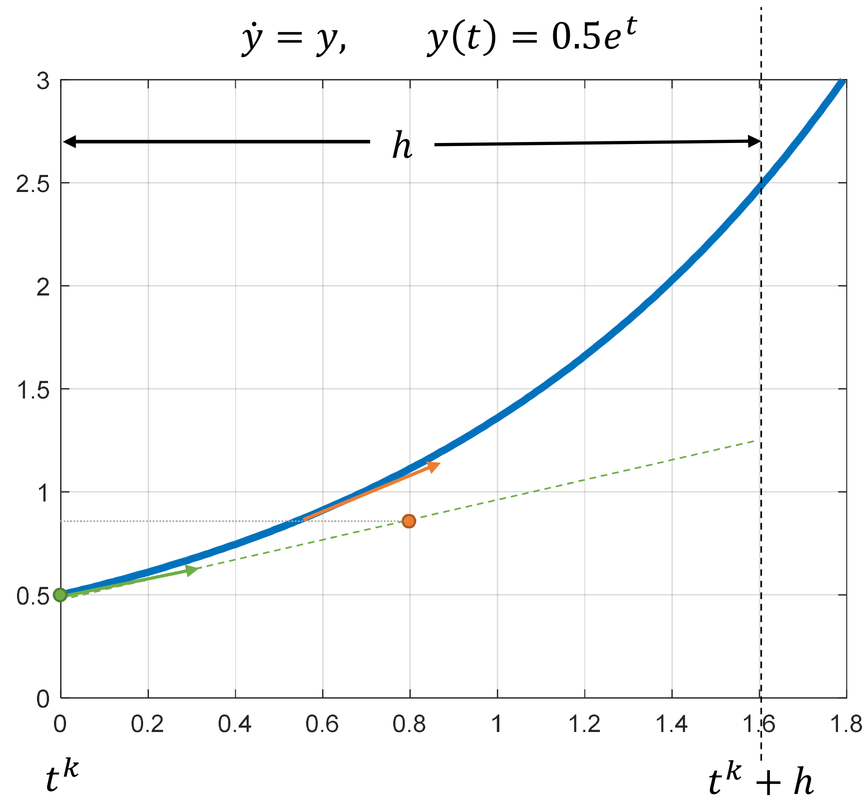

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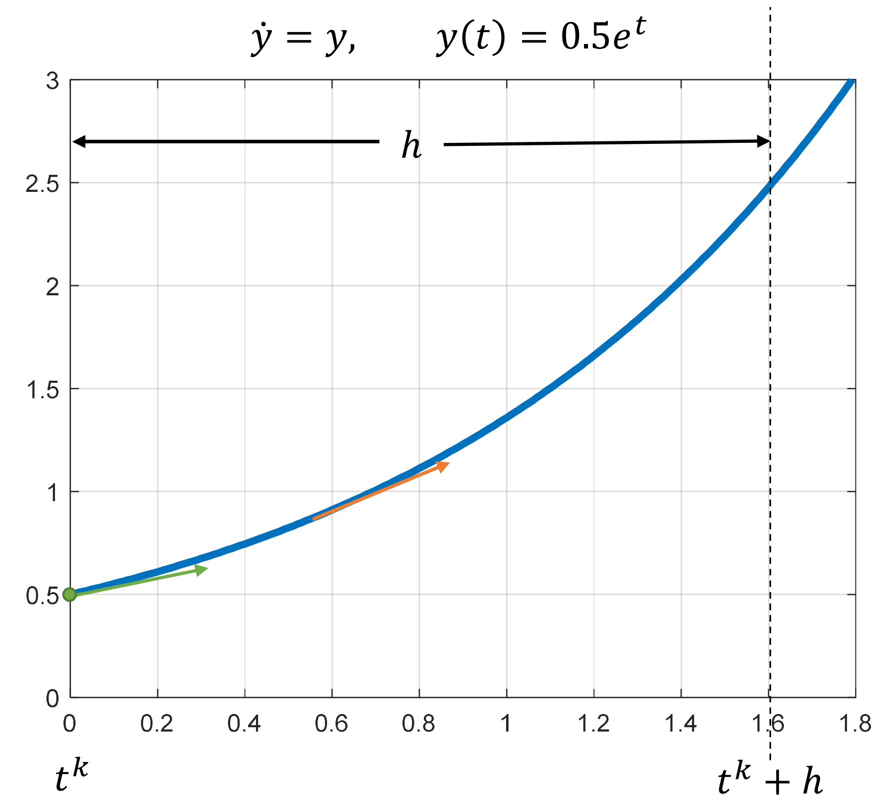
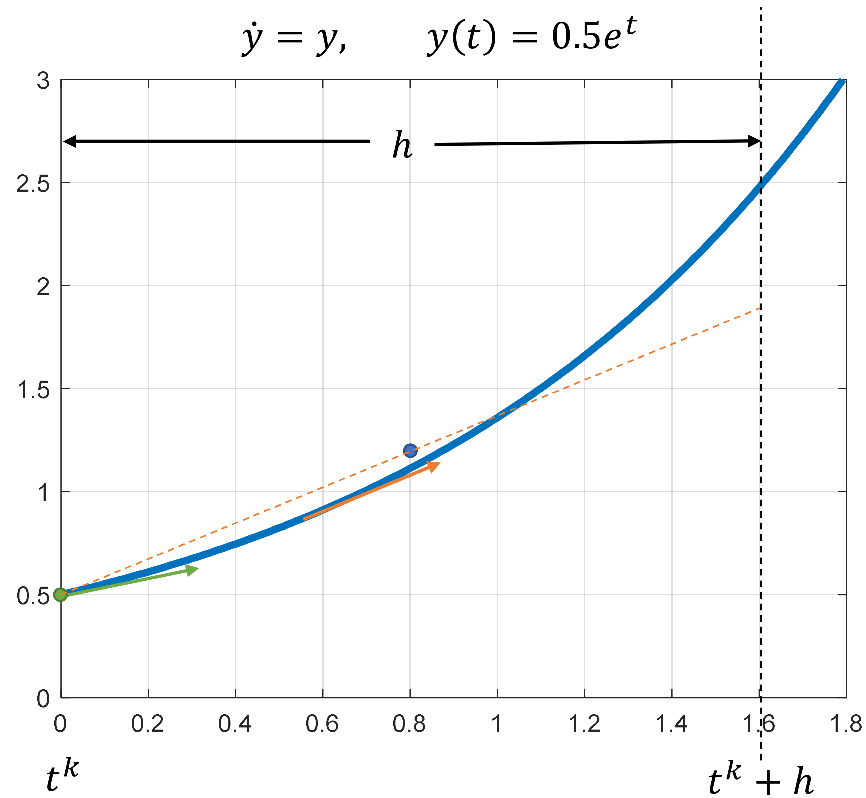


Classical Runge-Kutta Method (RK4)

- Main consideration: what slope to use?
  - Weighted average

- $y^{k+1} = y^k + \quad (k_1 \qquad k_2 \qquad k_3 \qquad)$
  - $k_1 = hf\left(t^k, y^k\right)$
  - $k_2 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_1}{2}\right)$
  - $k_3 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_2}{2}\right)$



Classical Runge-Kutta Method (RK4)

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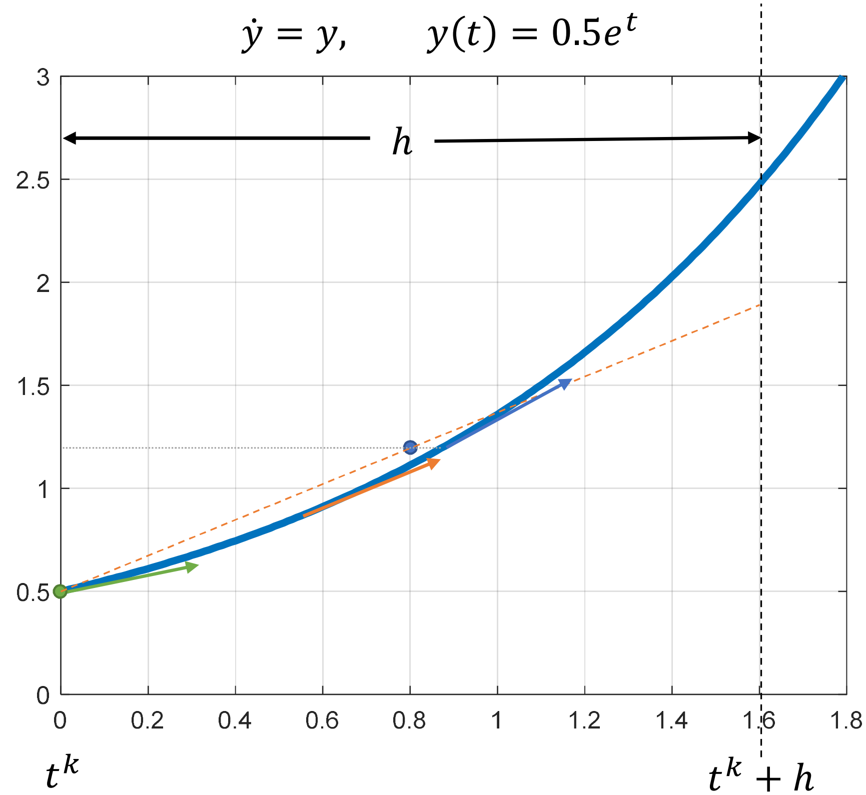

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- Main consideration: what slope to use?
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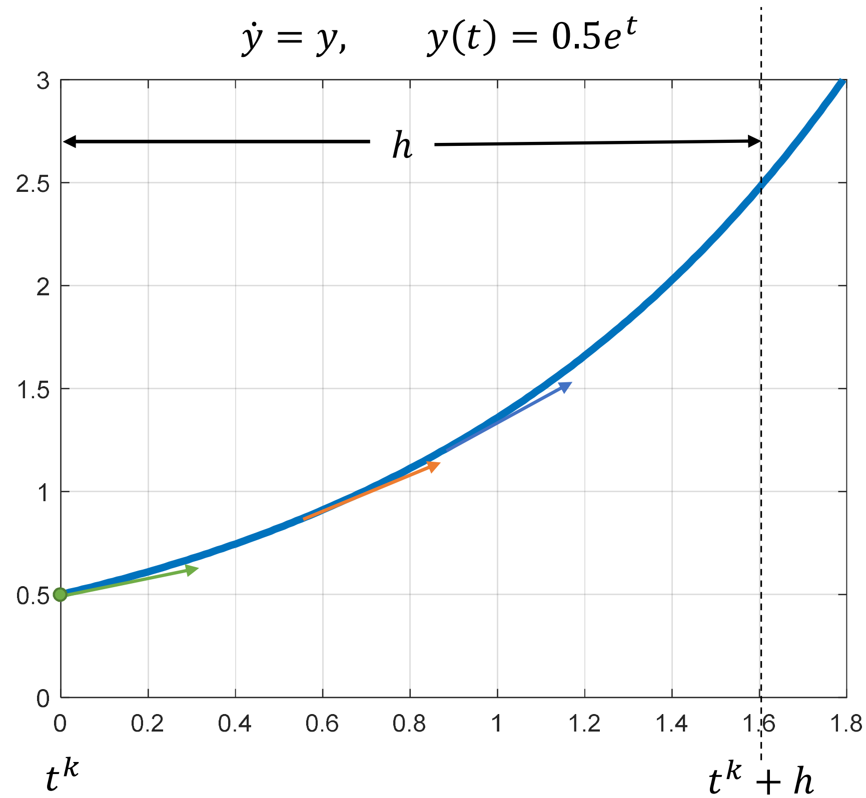
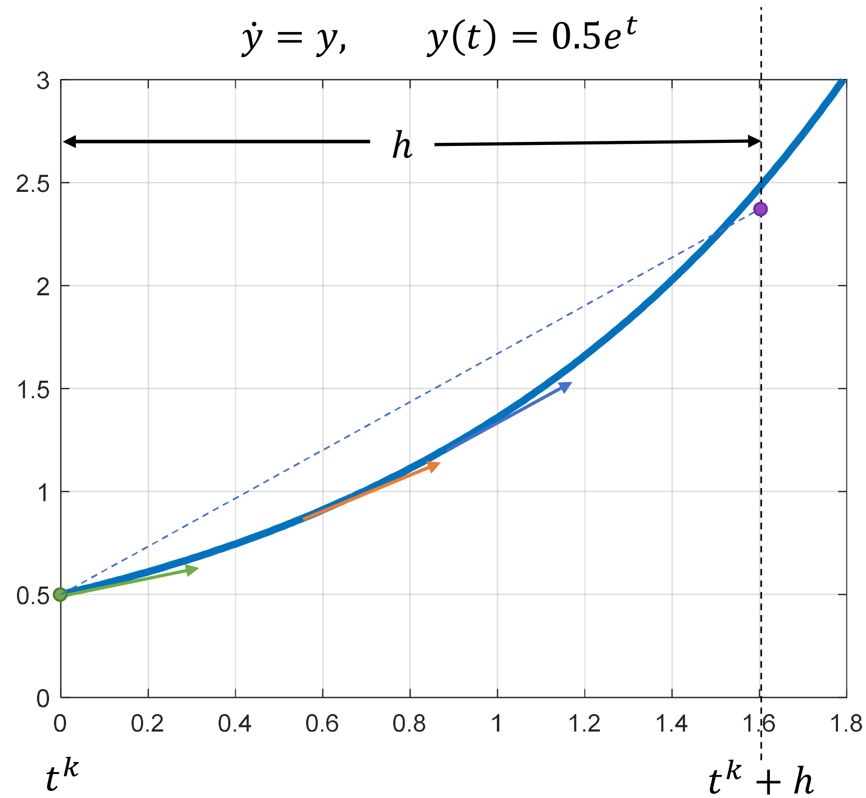
- $y^{k+1} = y^k + \quad (k_1 \qquad k_2 \qquad k_3 \qquad)$
  - $k_1 = hf\left(t^k, y^k\right)$
  - $k_2 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_1}{2}\right)$
  - $k_3 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_2}{2}\right)$



Classical Runge-Kutta Method (RK4)

- Main consideration: what slope to use?
  - Weighted average

- $y^{k+1} = y^k + \quad (k_1 \qquad k_2 \qquad k_3 \qquad k_4)$
  - $k_1 = hf\left(t^k, y^k\right)$
  - $k_2 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_1}{2}\right)$
  - $k_3 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_2}{2}\right)$
  - $k_4 = hf\left(t^k + h, y^k + k_3\right)$



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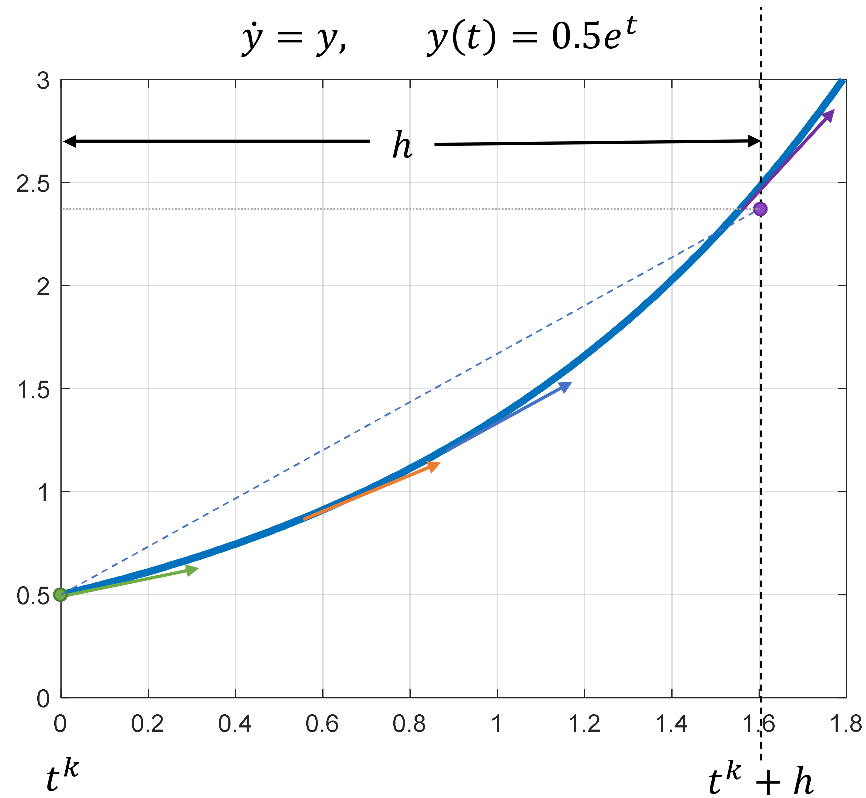

Classical Runge-Kutta Method (RK4)

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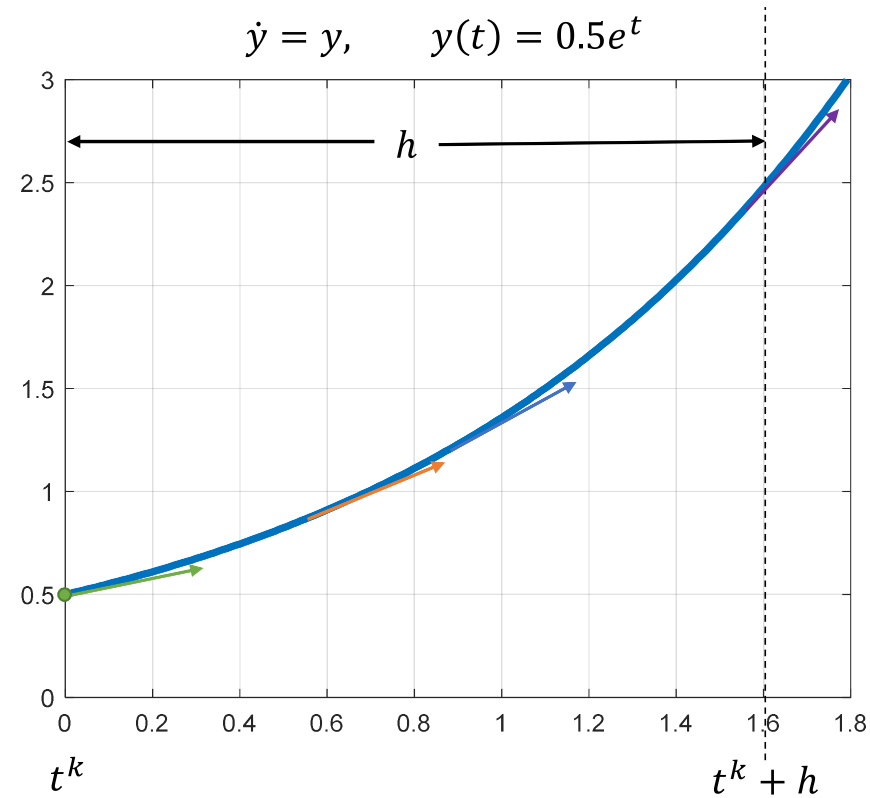
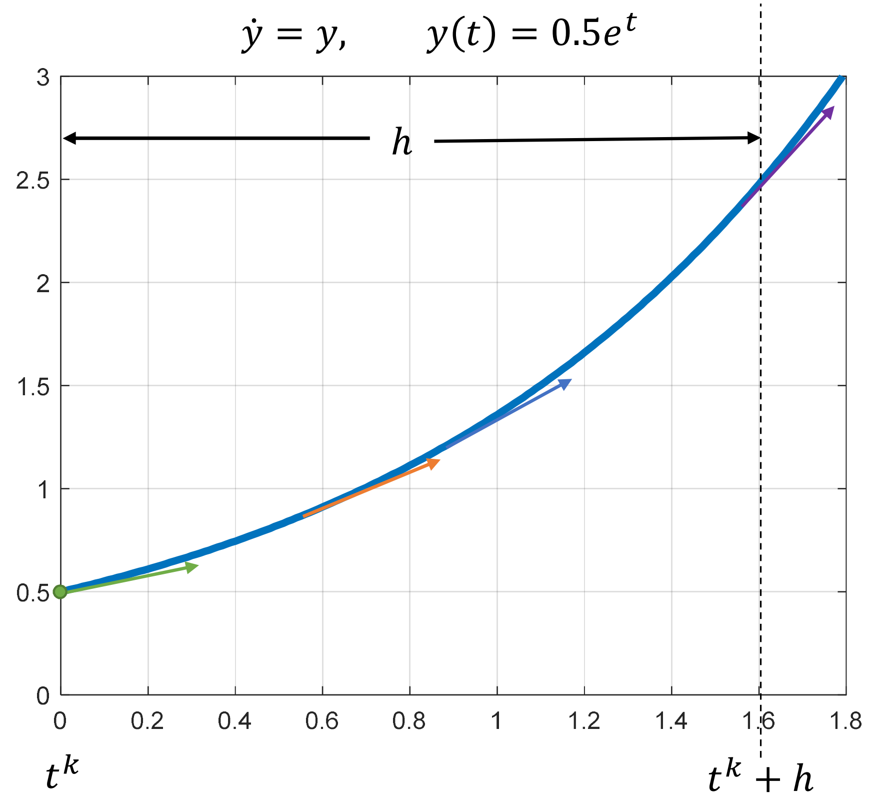


Classical Runge-Kutta Method (RK4)

- Main consideration: what slope to use?
  - Weighted average

- $y^{k+1} = y^k + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
  - $k_1 = hf\left(t^k, y^k\right)$
  - $k_2 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_1}{2}\right)$
  - $k_3 = hf\left(t^k + \frac{h}{2}, y^k + \frac{k_2}{2}\right)$
  - $k_4 = hf\left(t^k + h, y^k + k_3\right)$



Classical Runge-Kutta Method (RK4)

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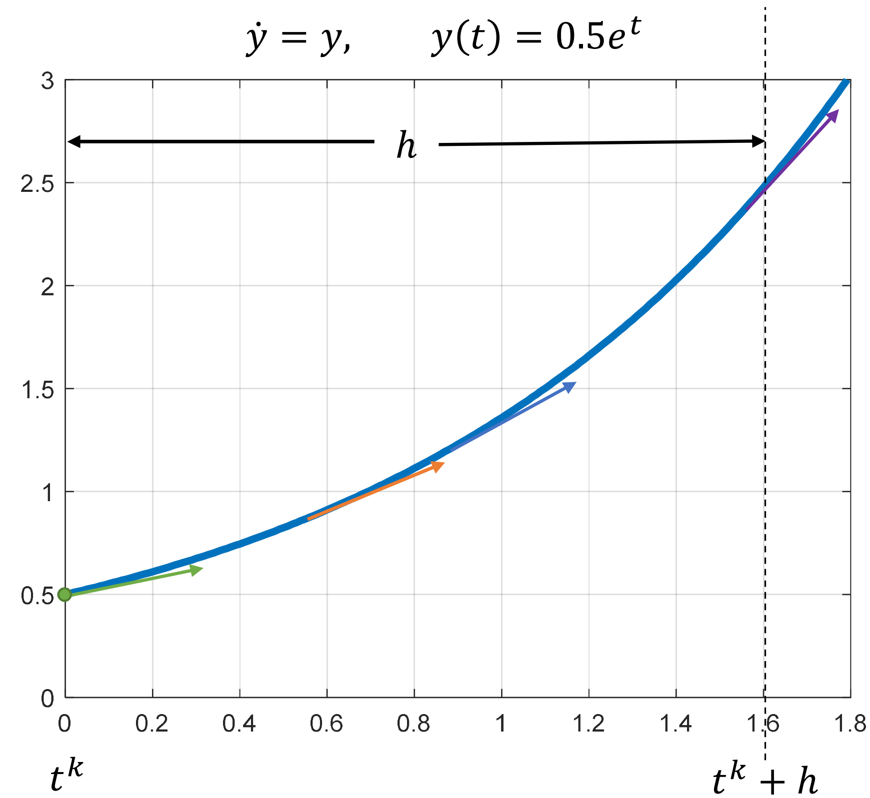
  - $k_4 = hf\left(t^k + h, y^k + k_3\right)$

- Properties
  - Equivalent to Simpson's rule
  - 4th order acccuracy



Numerical solutions: issues

- Stiff equations

- Approximation errors
  - Typically cannot be used to prove system properties