# Nonlinear Systems I

CMPT 882

Jan. 16

# Nonlinear Systems Roadmap

- Introduction
- Analysis
- Control
- Numerical solutions

# Nonlinear Systems Roadmap: Today

- Introduction
- Analysis
  - Equilibrium points

  - Limit cycles

# Nonlinear systems

- $\dot{x} = f(x, u)$
  - **State:** $x(t) \in \mathbb{R}^n, x(t_0) = x_0$
  - **Control:** $u(t) \in \mathcal{U}$

- Existence and uniqueness of solutions
  - $f$ is a nonlinear function
  - Lipschitz continuous in $x$
    $$\exists L > 0, \forall u, \|f(x_1, u) - f(x_2, u)\| \leq L\|x_1 - x_2\|$$
  - $u(\cdot)$ is piecewise continuous

# Study of Nonlinear Systems

- In general, no closed form solutions
- Numerical approximations of solutions can be helpful
  - Widely used for simulations to predict system behaviour
- Analysis involves studying
  - equilibrium points
  - stability
  - limit cycles
  - bifurcations

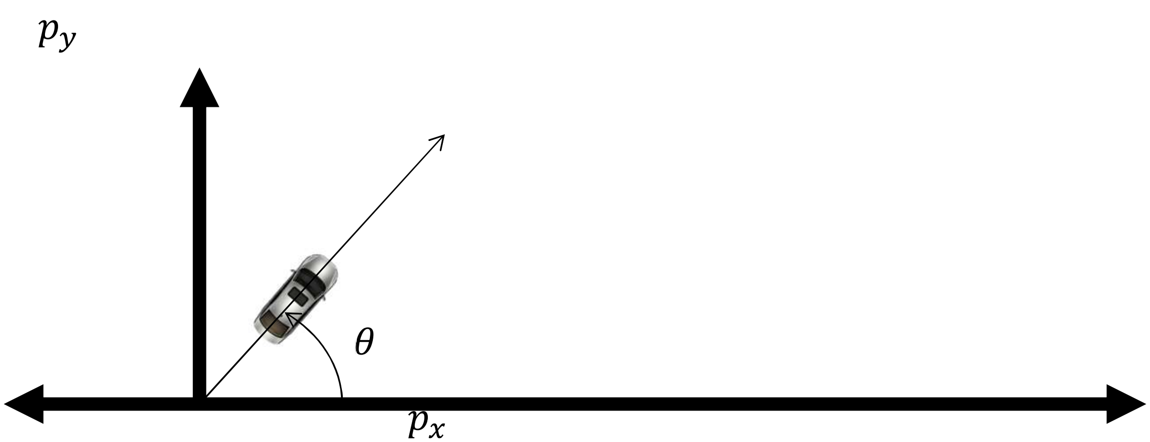
# Features of nonlinear systems

- Almost all real-world robots are modelled by nonlinear systems

# Examples of Nonlinear Systems

- Dubins Car

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = u$$



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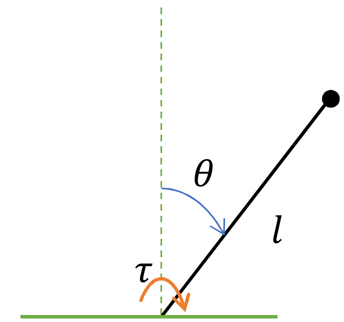


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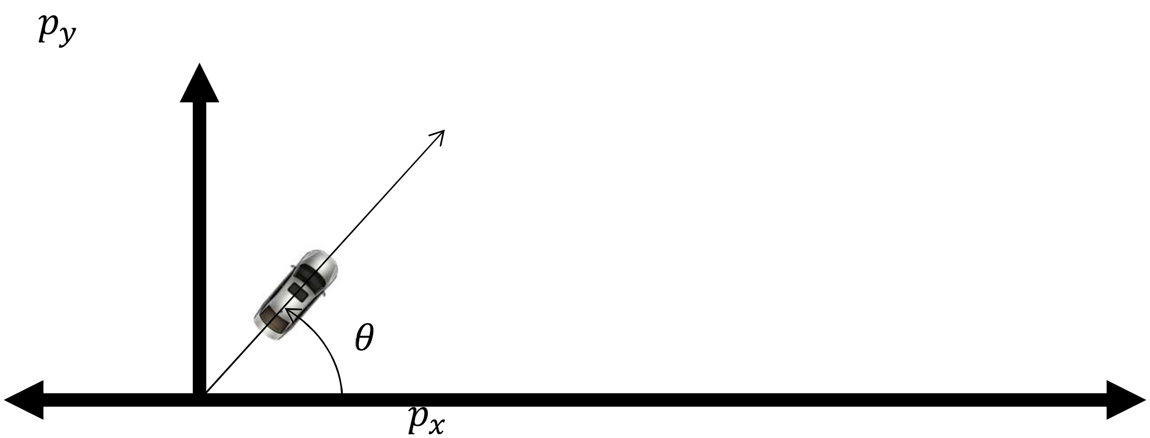
- Inverted pendulum

$$\ddot{\theta} - \frac{g}{l}\sin\theta = 0$$



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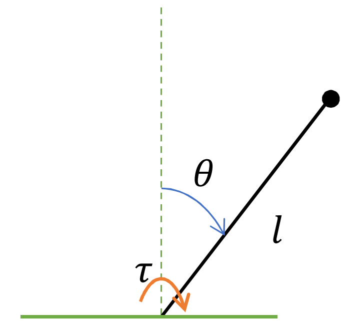


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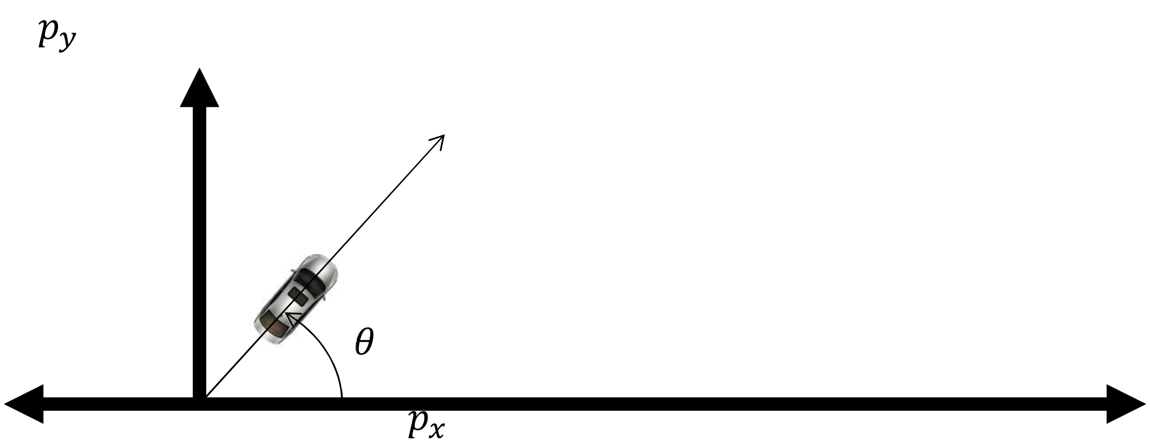


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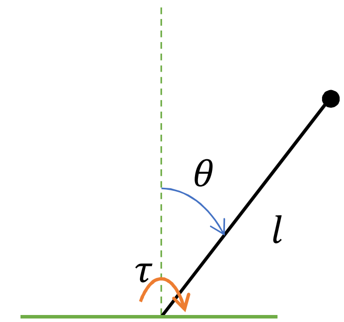


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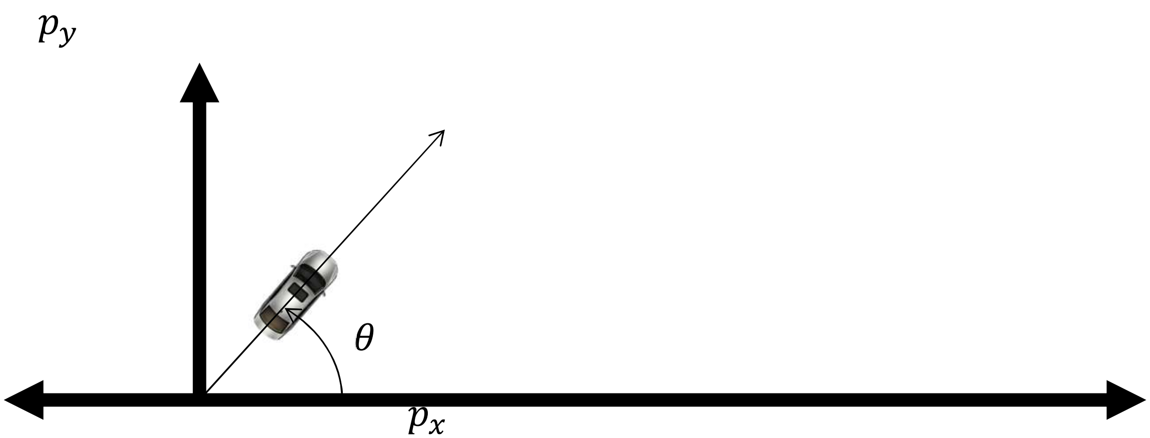


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$$\dot{x}_1 = x_2$$

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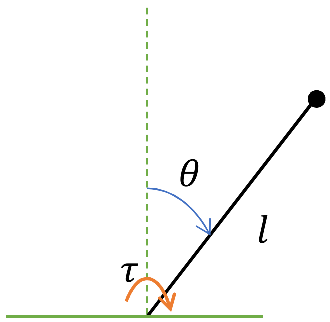


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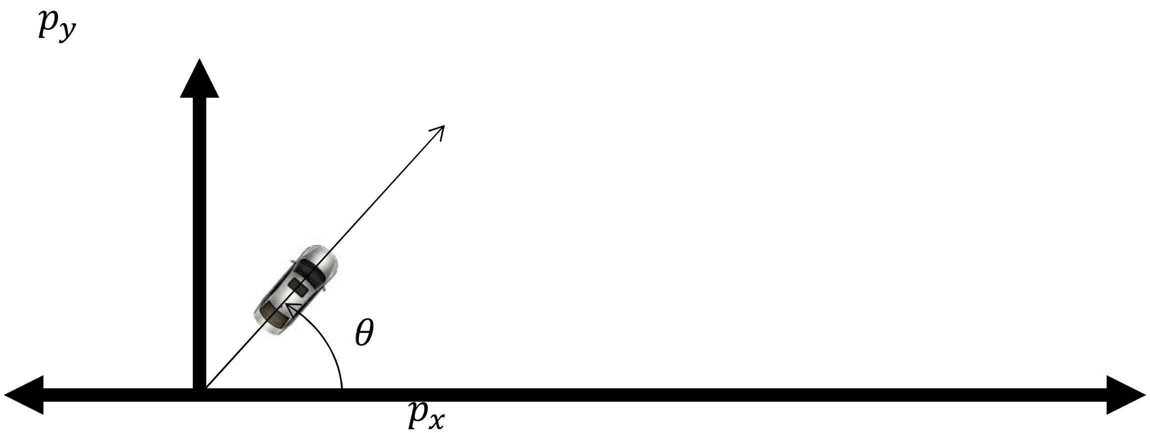


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$$\dot{x}_2 = \frac{g}{l}\sin x_1$$

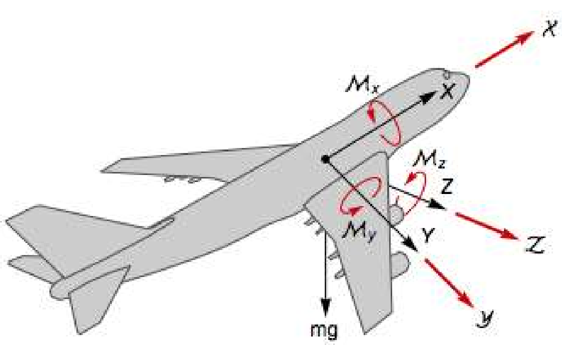
# Examples of Nonlinear Systems

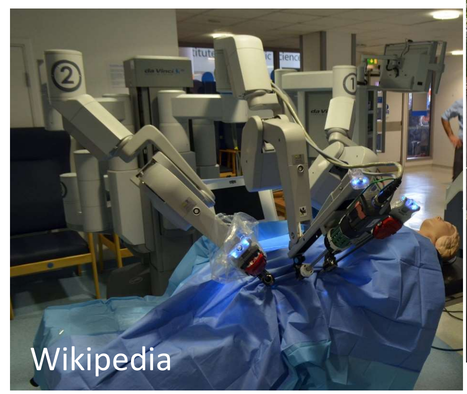
- Bicycle

$$\dot{x} = v_x$$
$$\dot{v}_x = \omega v_y + u_1$$
$$\dot{y} = v_y$$
$$\dot{v}_y = -\omega v_x + \frac{2}{m}\left(F_{c,f}\cos u_2 + F_{c,r}\right)$$
$$\dot{\psi} = \omega$$
$$\dot{\omega} = \frac{2}{I_z}\left(l_f F_{c,f} - l_r F_{c,r}\right)$$
$$\dot{X} = v_x \cos\psi - v_y \sin\psi$$
$$\dot{Y} = v_x \sin\psi + v_y \cos\psi$$

# Examples of Nonlinear Systems






# Features of Nonlinear Systems

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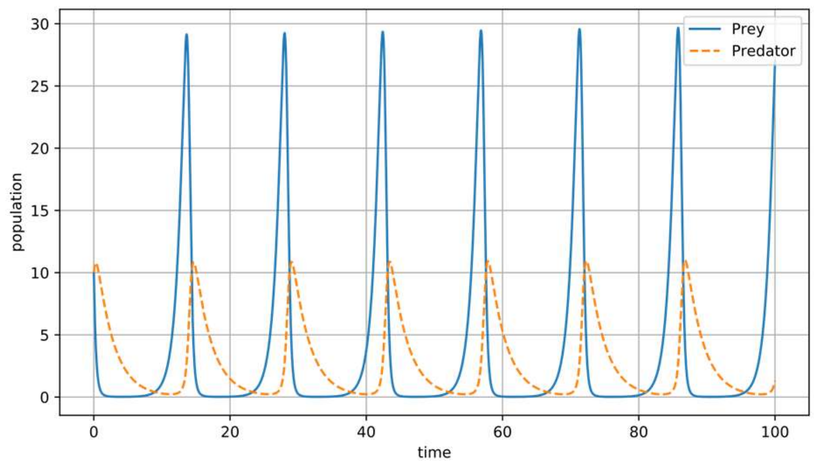
- Limit cycles

# Predator-Prey Model

- Predator-prey model: $x$ is number of preys, $y$ is number of predators

$$\dot{x} = \alpha x - \beta xy$$
$$\dot{y} = \delta xy - \gamma y$$

  - $\alpha$: prey natural growth rate



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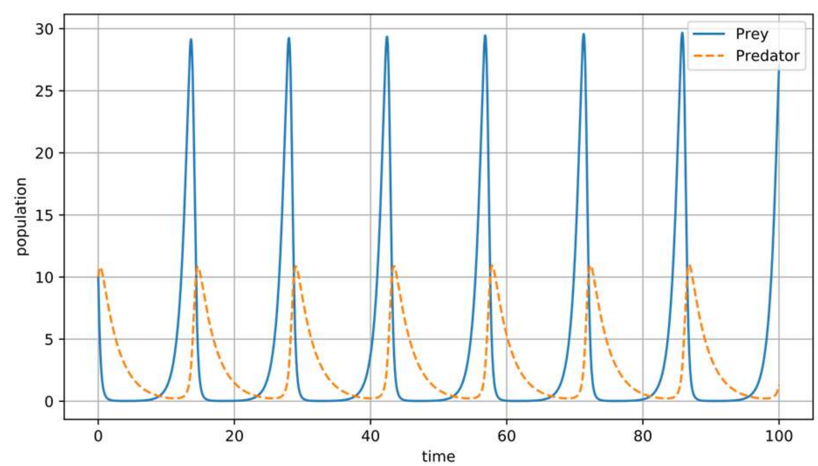

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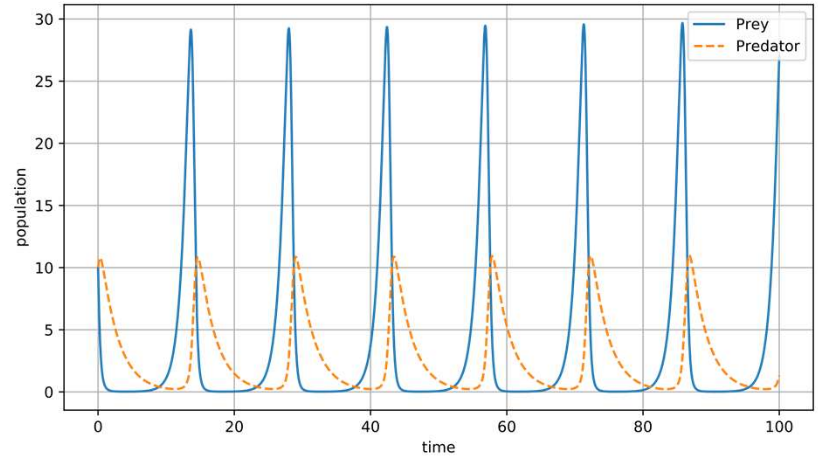

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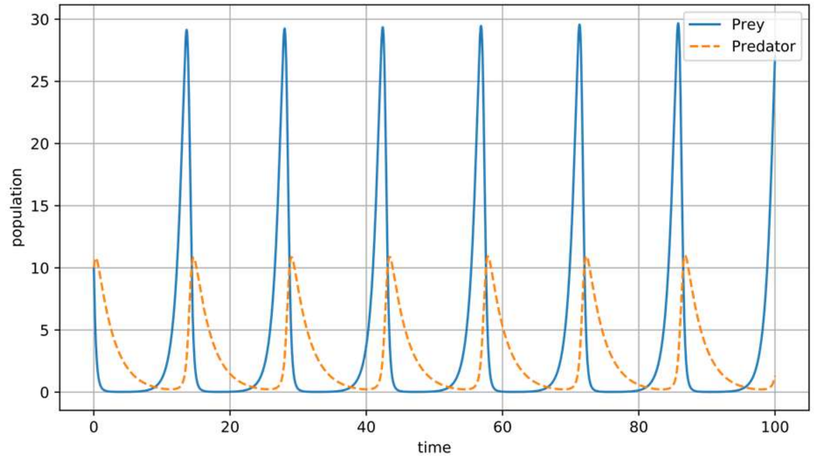

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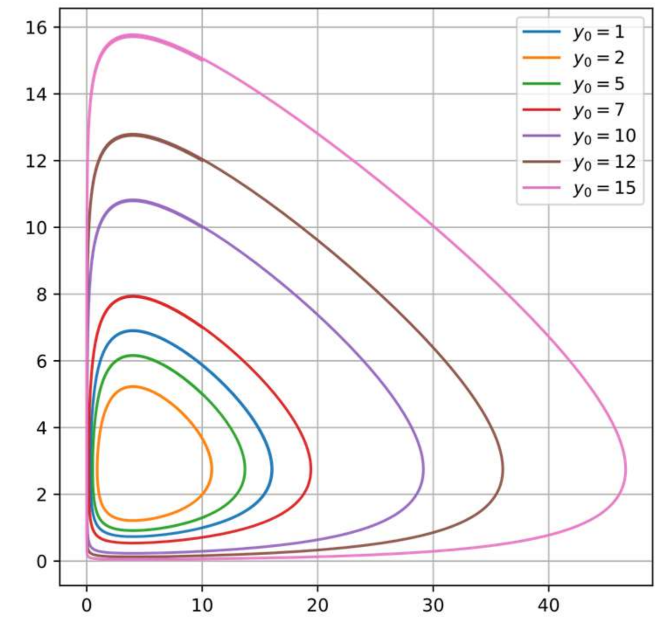
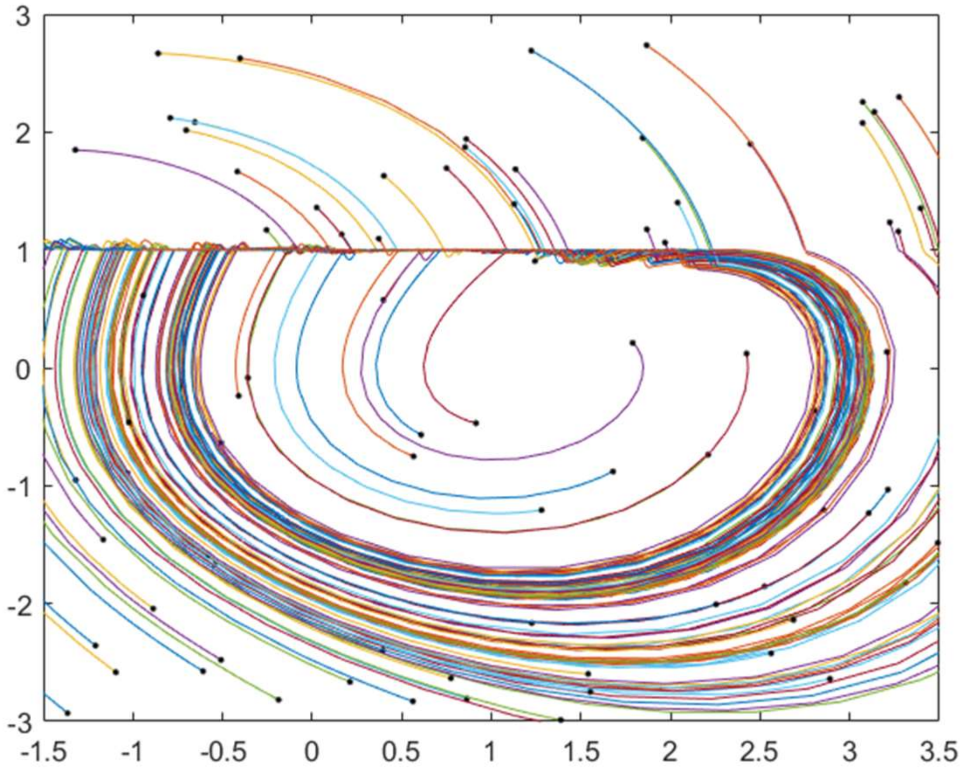


# Rayleigh’s Model of Violin String

- See assignment 1



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- Limit cycles

- Multiple isolated equilibrium points

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- More complex model of oscillators compared to the simple harmonic oscillator, which is a linear system

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$$\dot{x} = y$$
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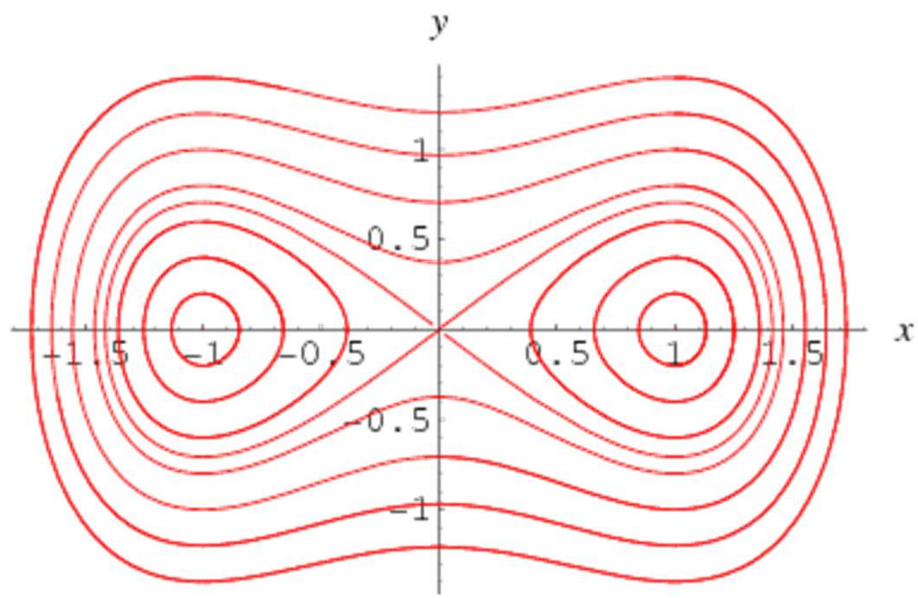


# Features of Nonlinear Systems

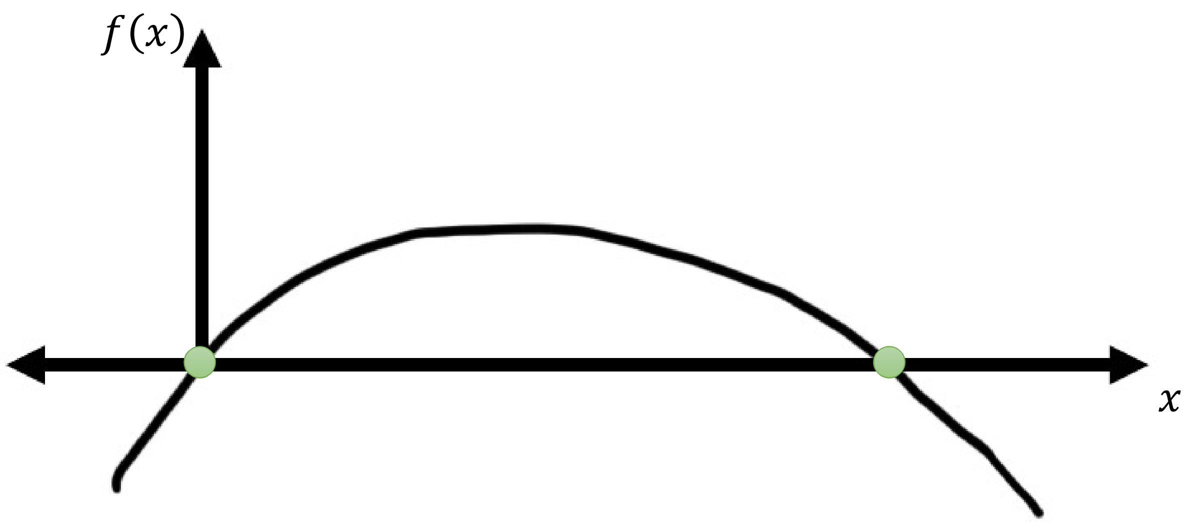
- Almost all real-world robots are modelled by nonlinear systems
- Limit cycles
- Multiple isolated equilibrium points
- Bifurcations

# Bifurcations

- Sudden changes in system behaviour for small changes in parameter

# Equilibrium Points and Stability: 1D

- General nonlinear system: $\dot{x} = f(x)$
  - If $f(x) = 0$, then $x$ is an equilibrium point, denoted $x_e$
- Example: $f(x) = -x(x-1)$



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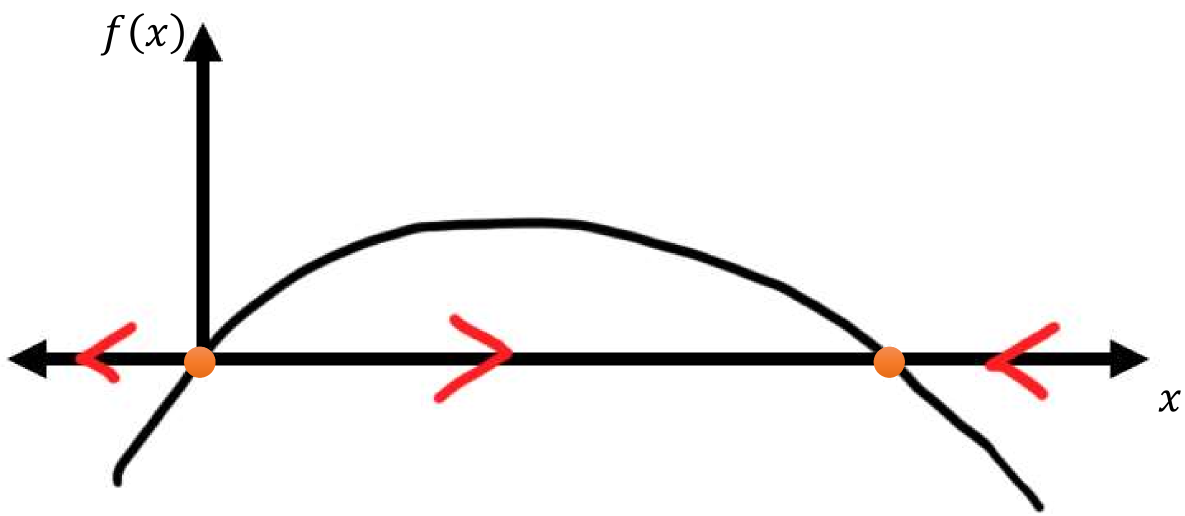

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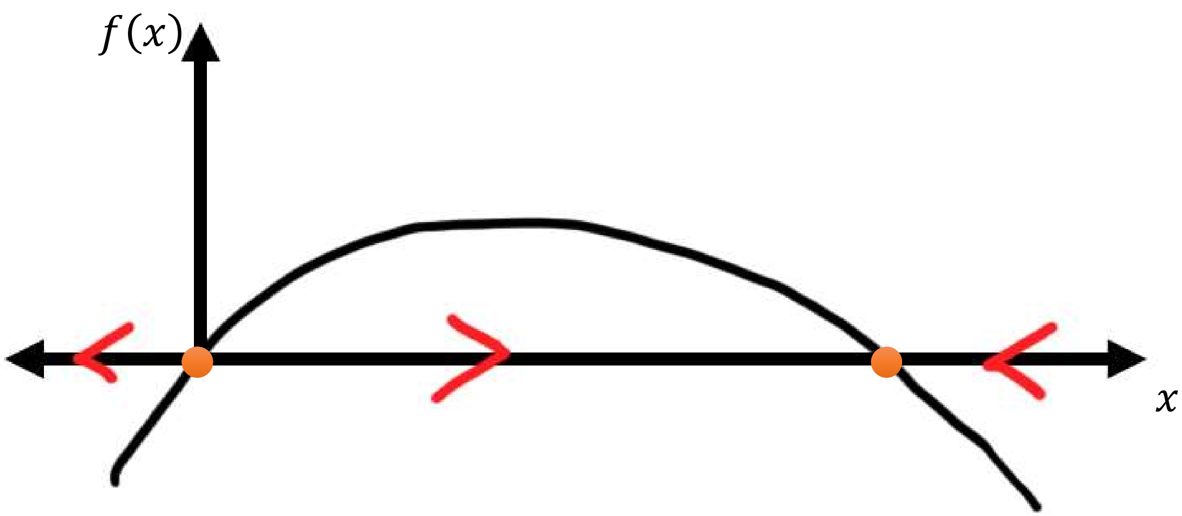


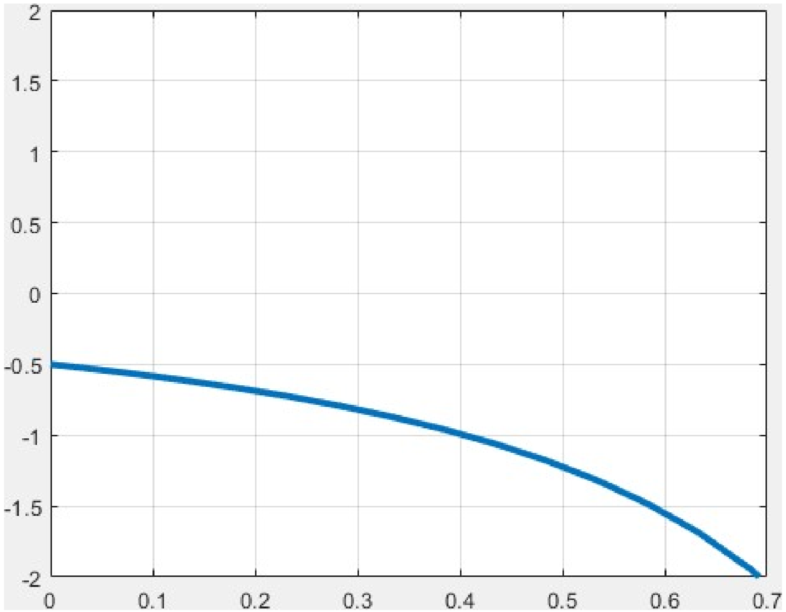

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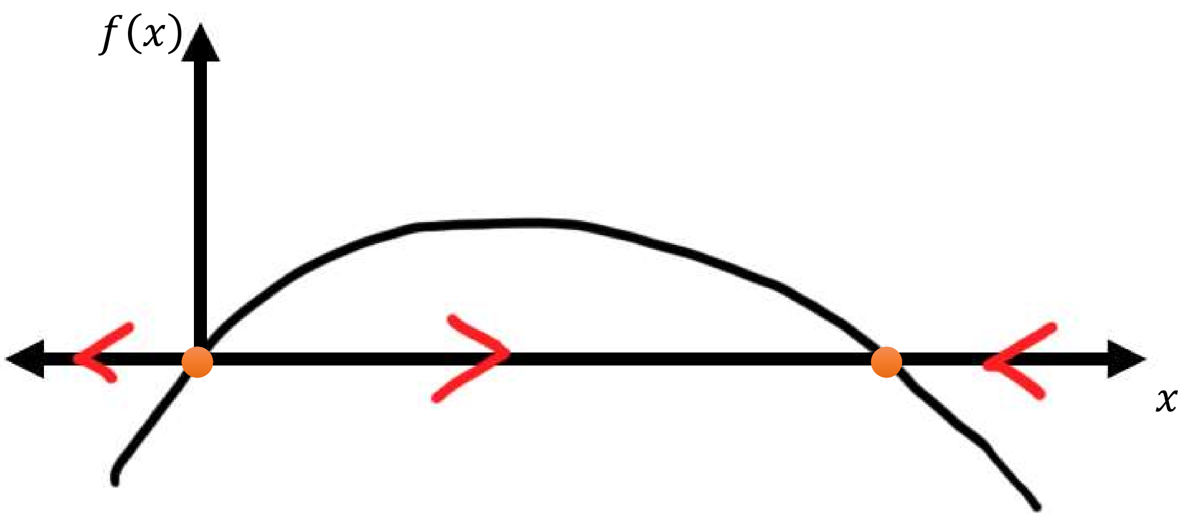


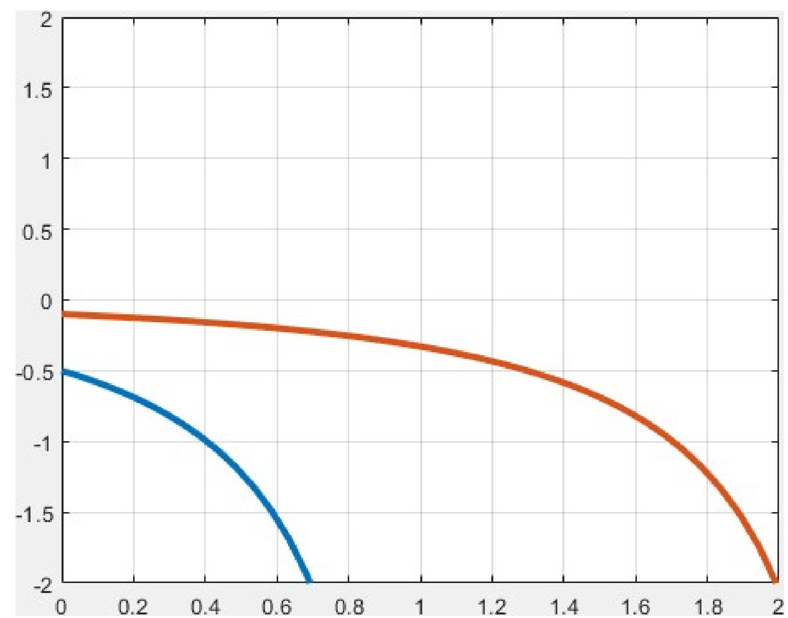

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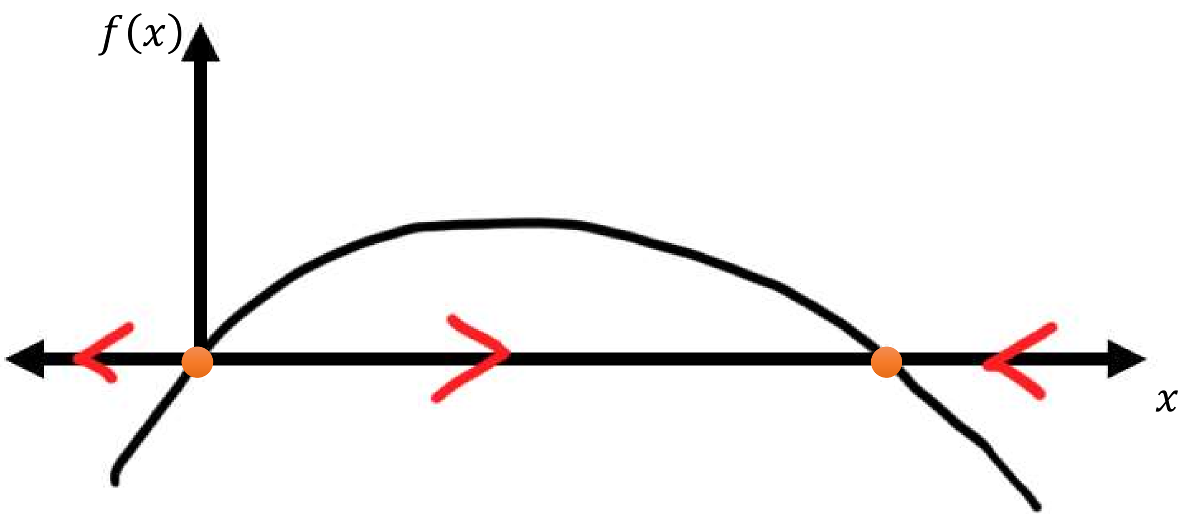


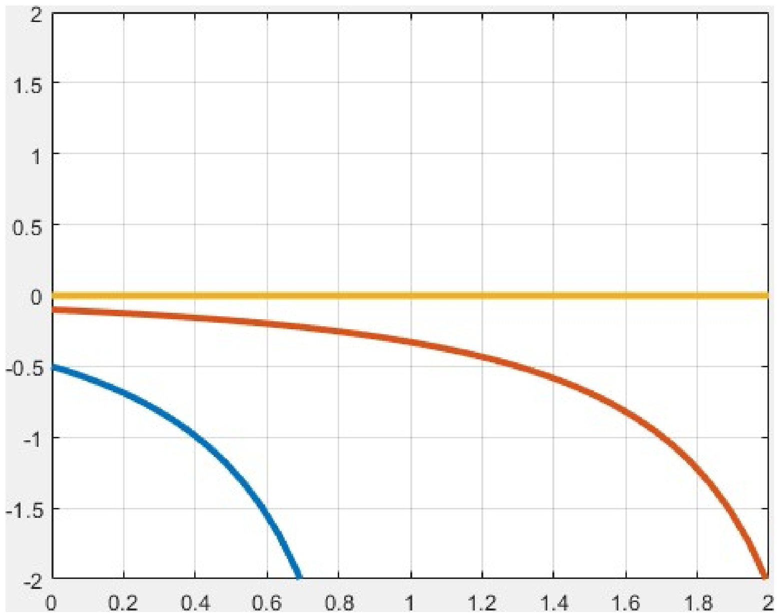

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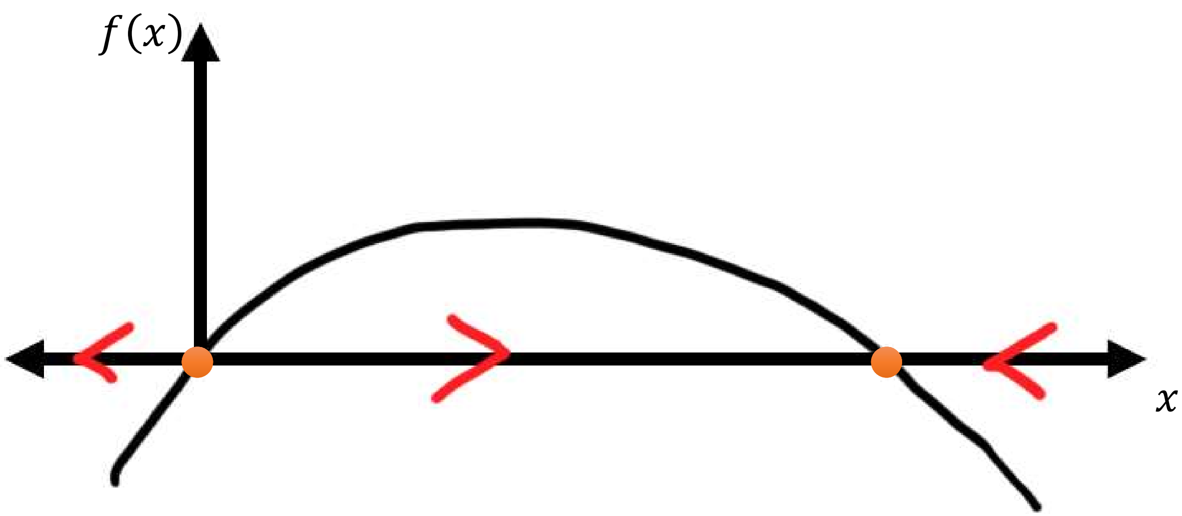


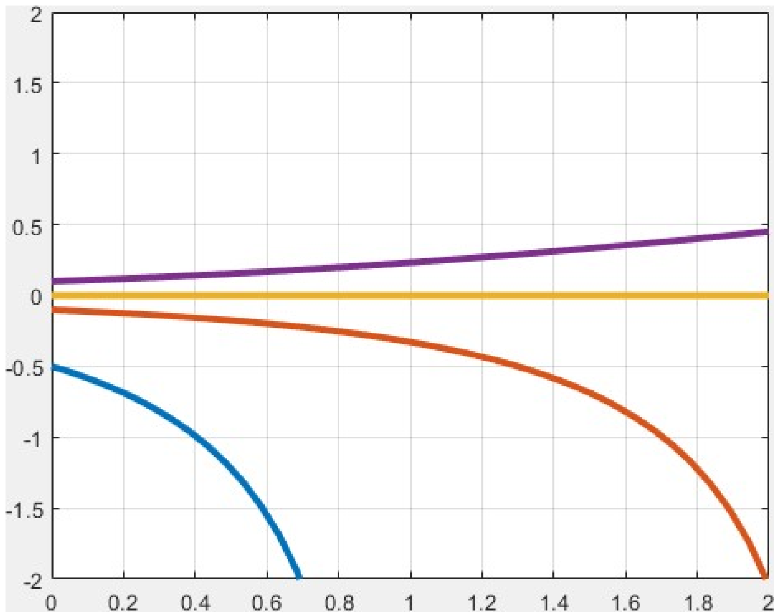

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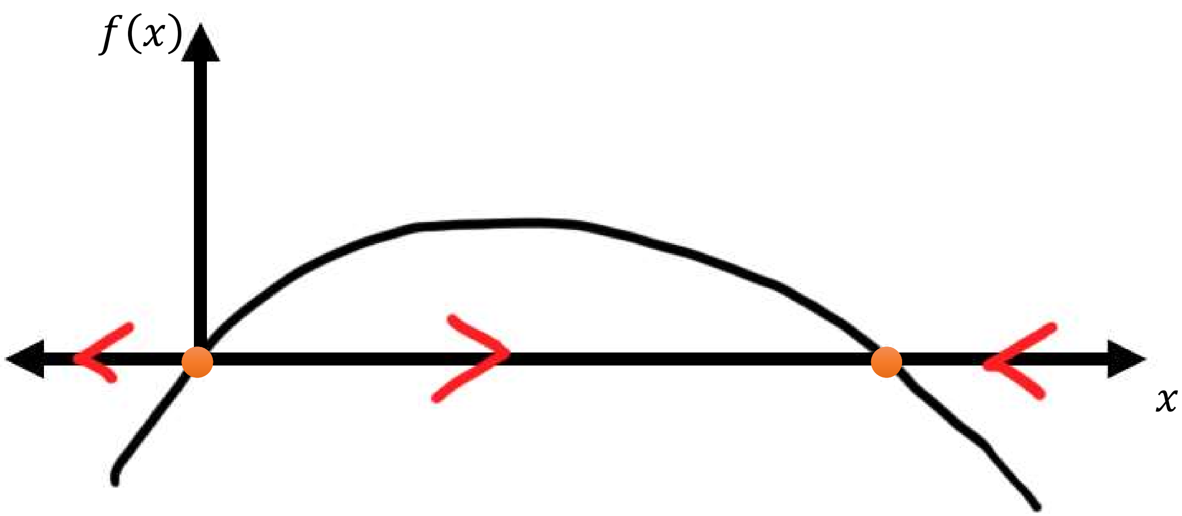


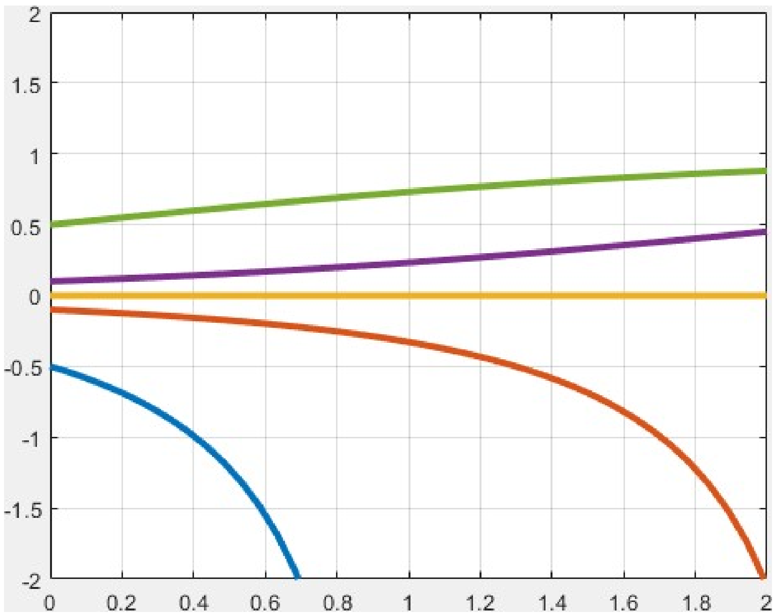

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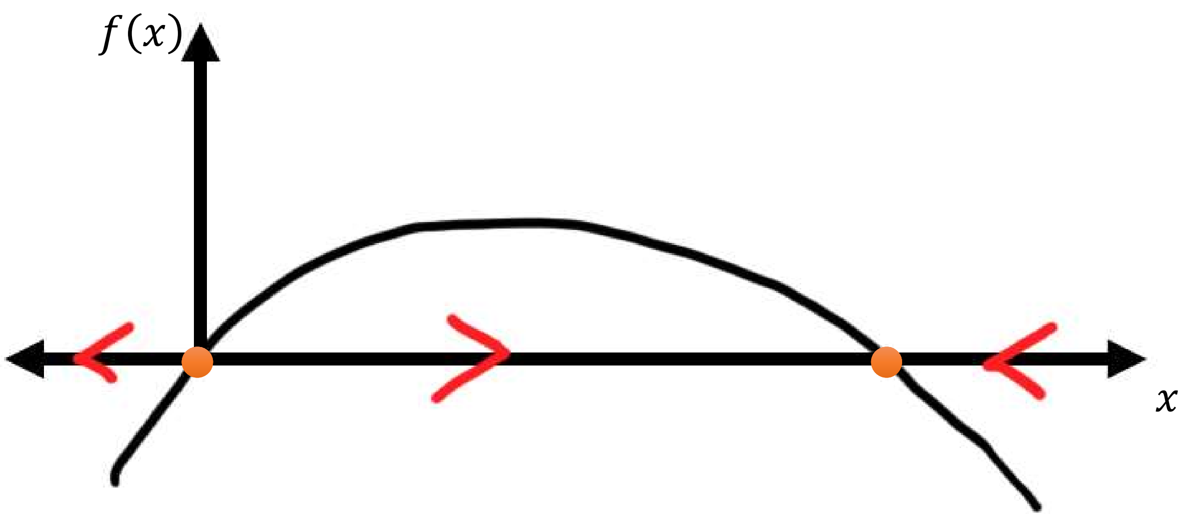


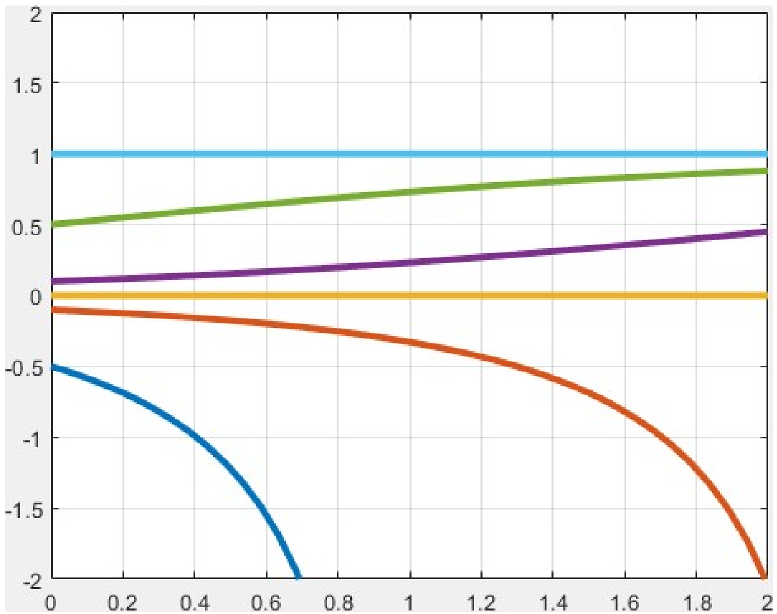

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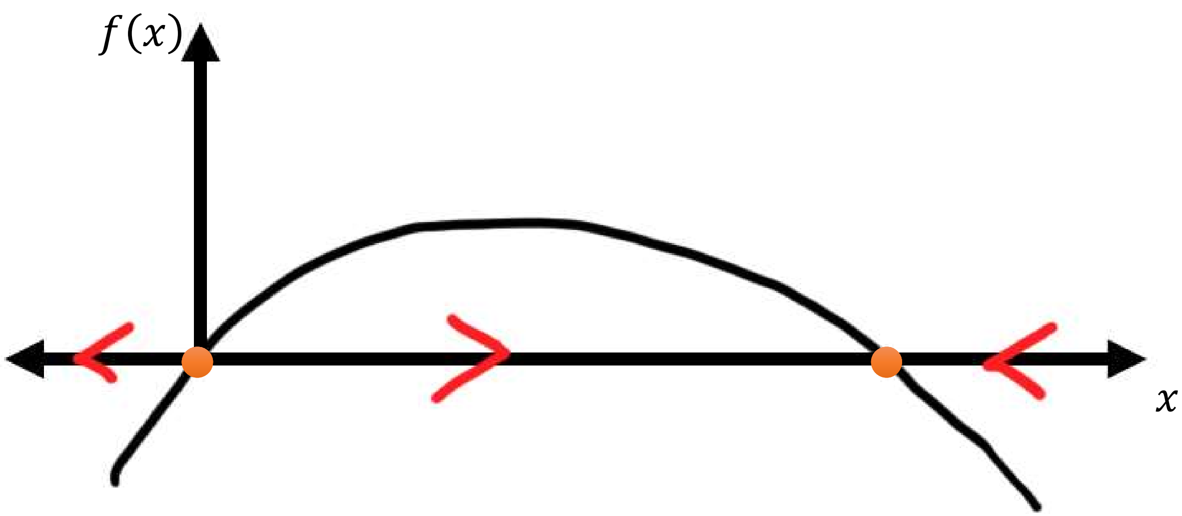


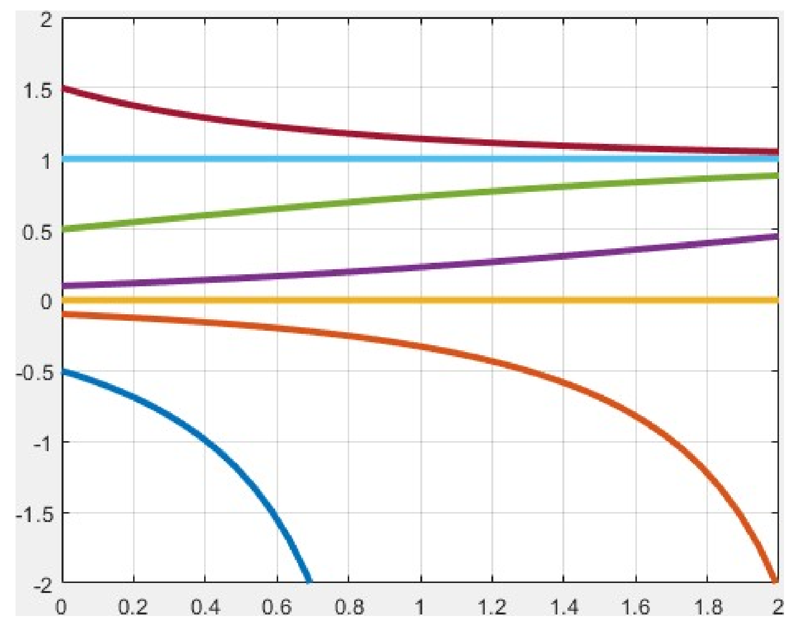

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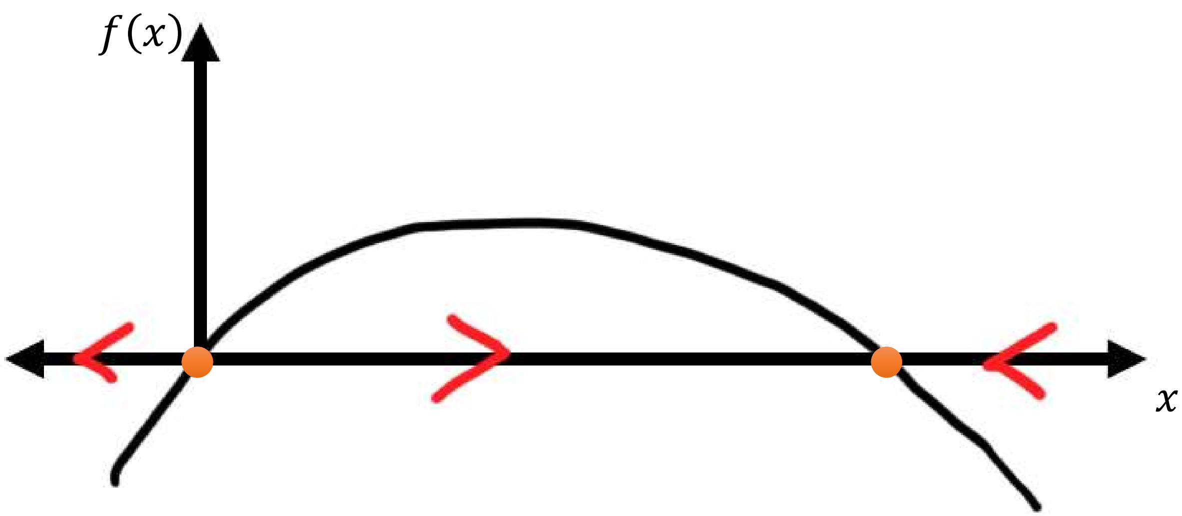


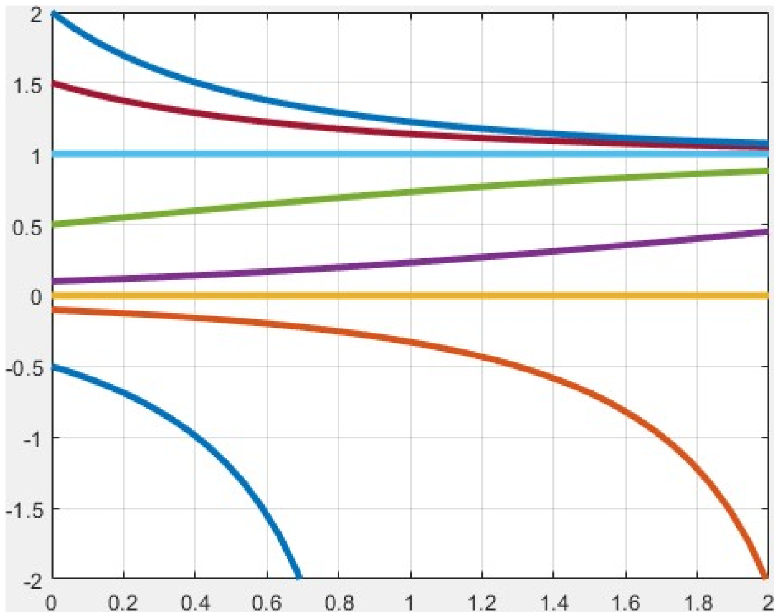

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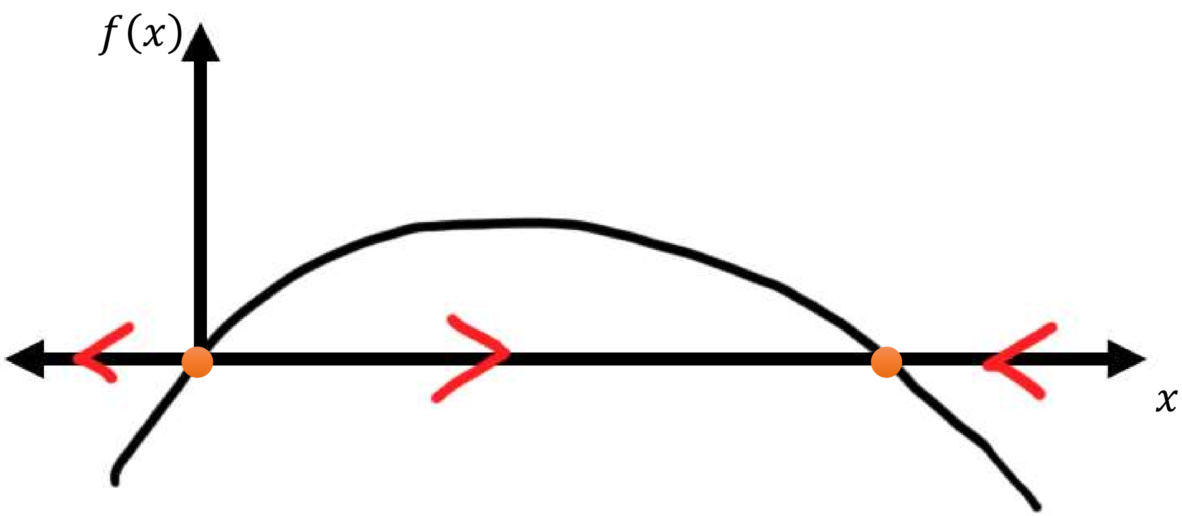


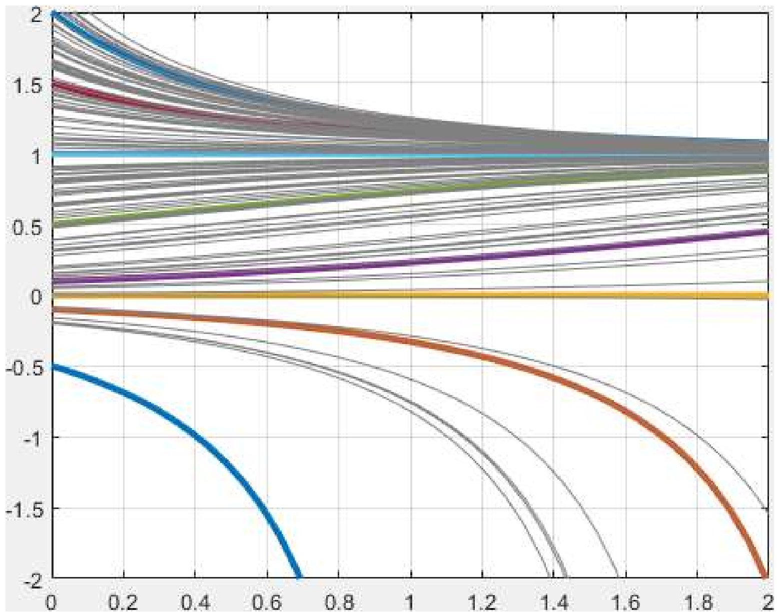

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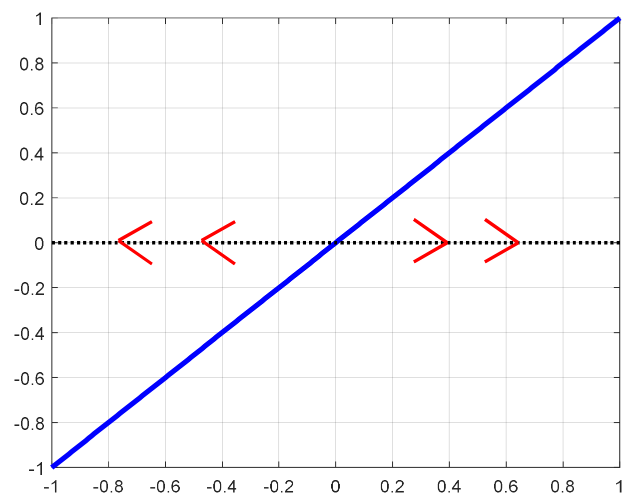
- Look at eigenvalues of linearization around equilibrium point

- Examples:
  - $\dot{x} = ax$



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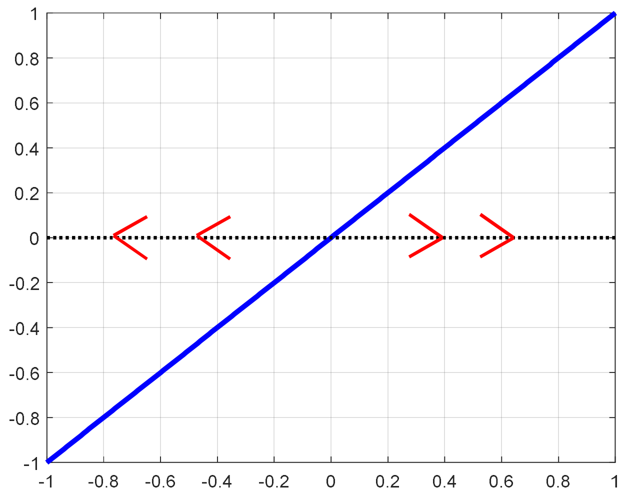
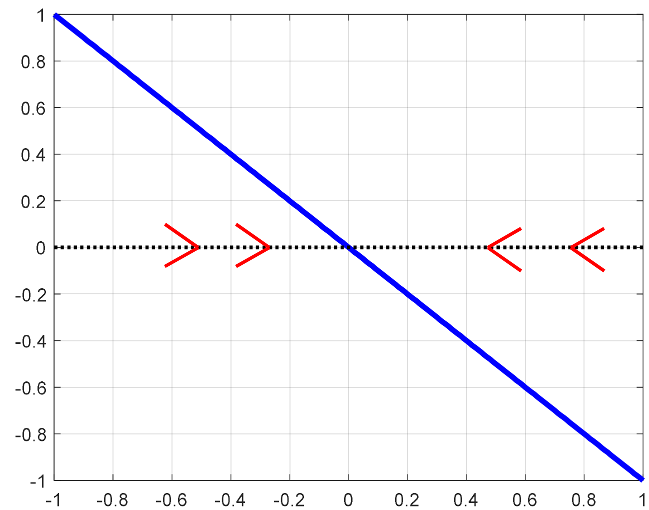



$a > 0$: unstable

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$$\frac{\partial f}{\partial x} = +3x^2$$

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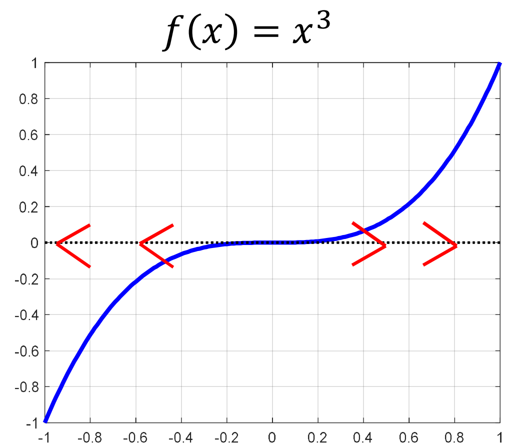

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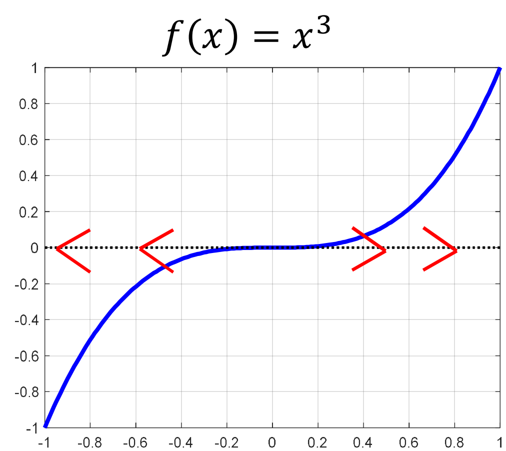


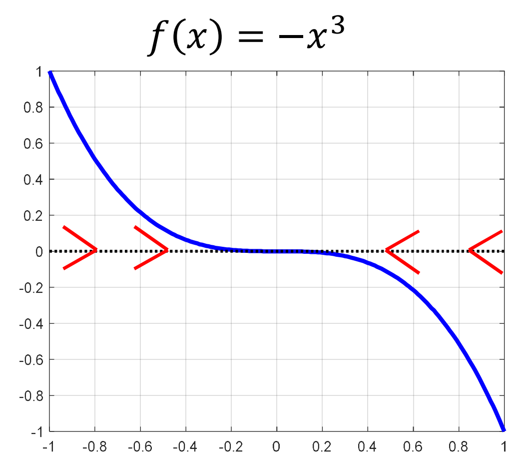

# Duffing's Equation

- Damped ($\delta > 0$) and no forcing:

$$\dot{x} = y$$
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$$\frac{\partial f}{\partial(x, y)} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -1 \end{bmatrix}$$

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Eigenvalues:

$$s^2 + s + 2 = 0$$

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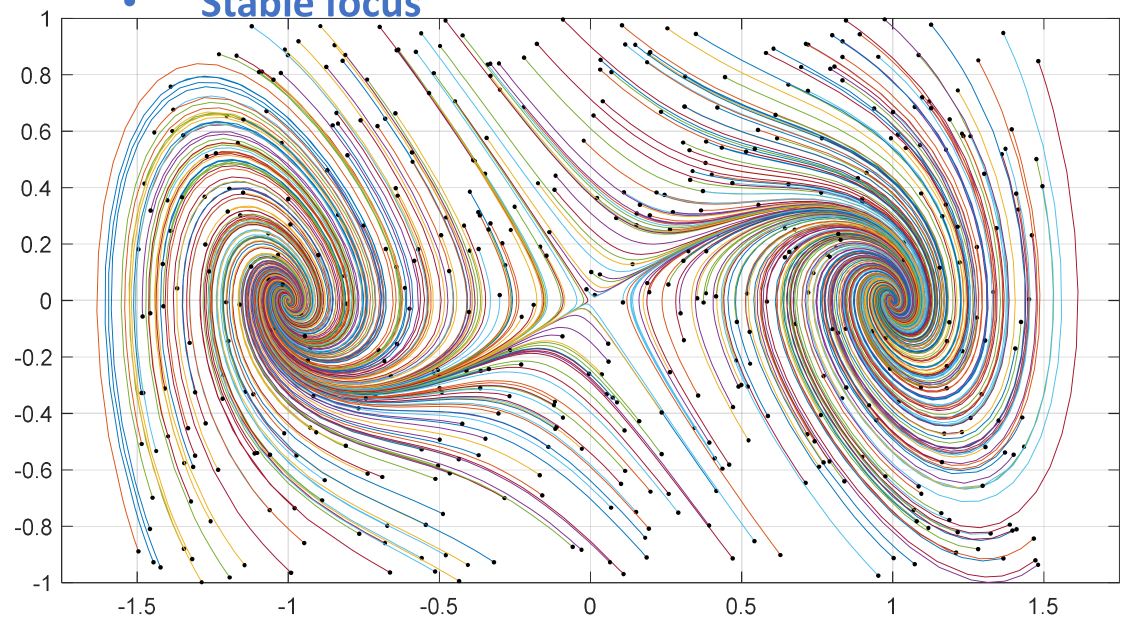

# Phase Portraits

- Phase portraits: Graphs of $y(t)$ vs. $x(t)$ for 2D systems

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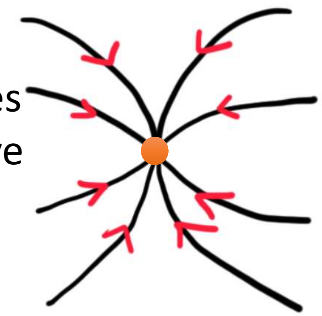
**Stable node**
- Both eigenvalues real and negative



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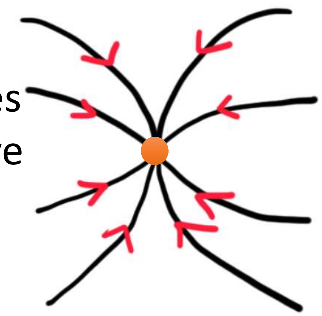

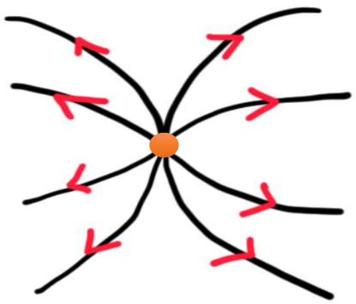

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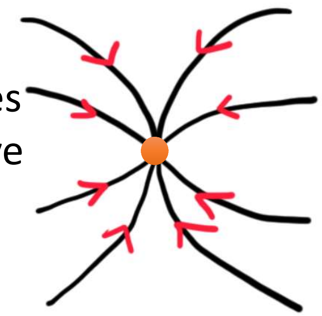

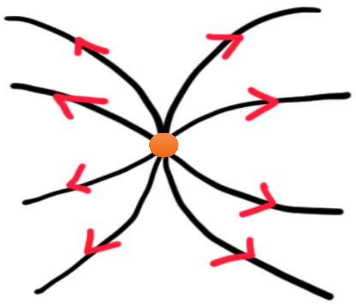

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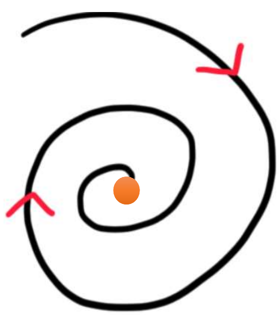

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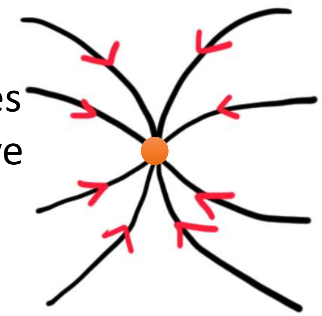

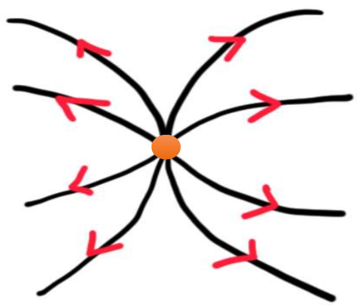

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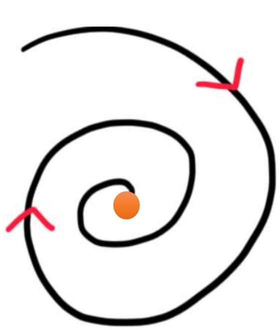

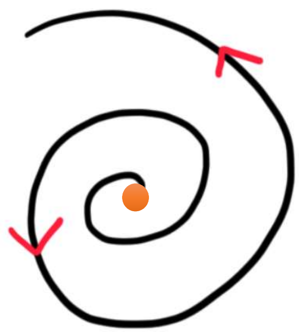

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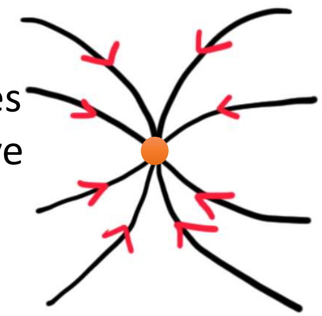

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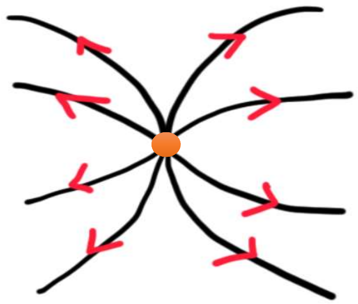
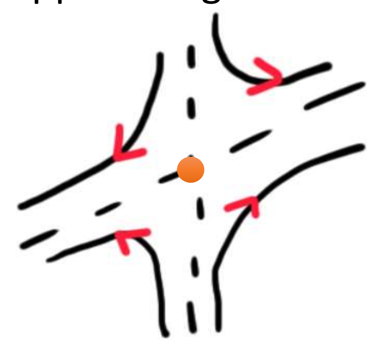


**Saddle**
- Real eigenvalues with opposite signs



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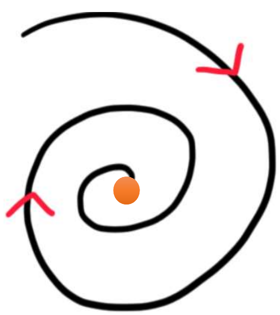

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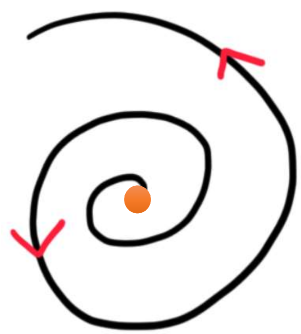
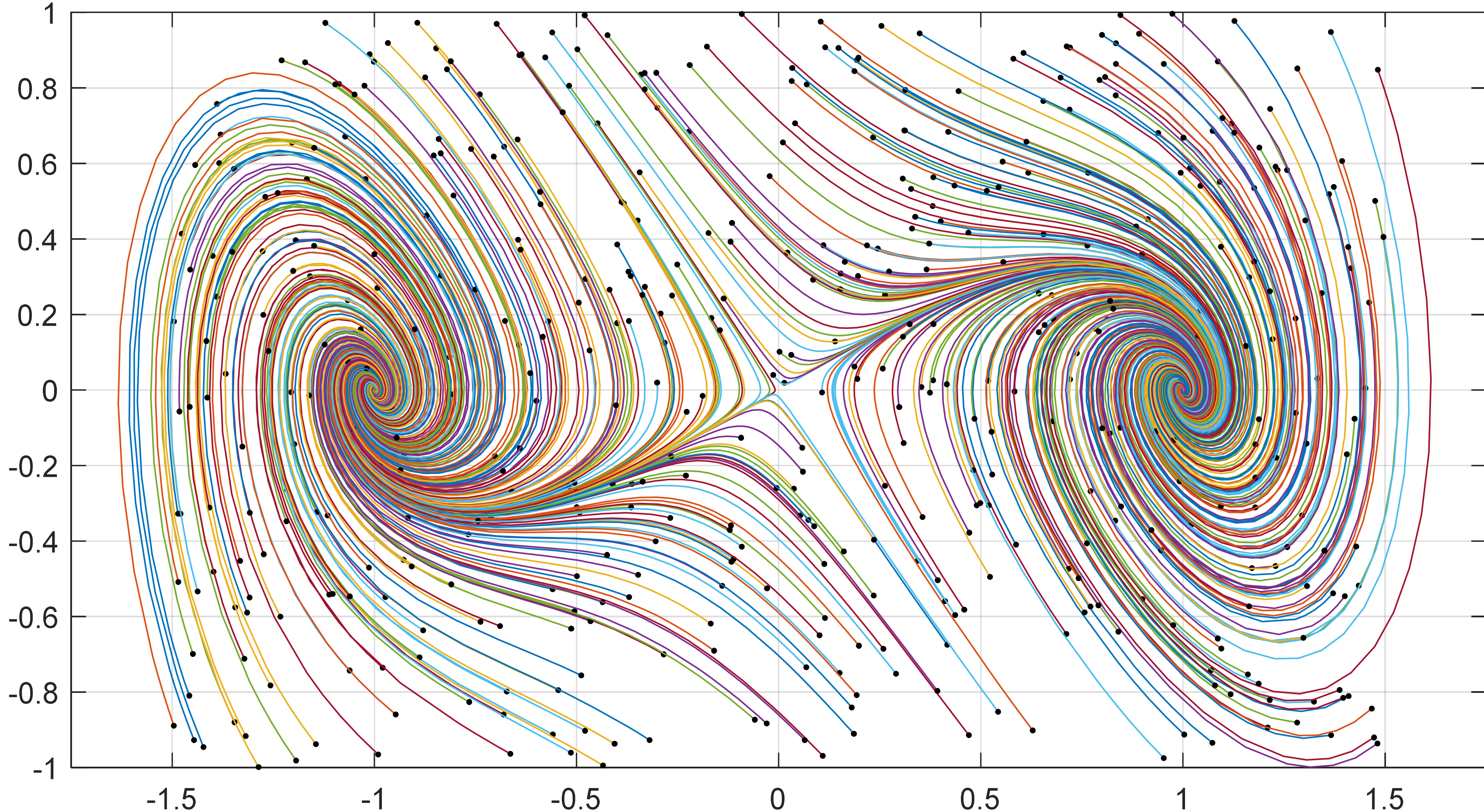
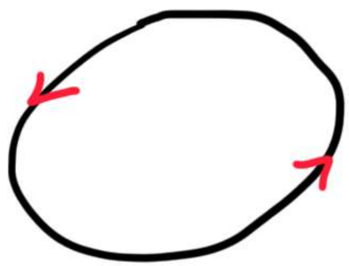




1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.5
-1
-0.5
0
0.5
1
1.5

# Closed orbits

- Closed orbit: trace of the trajectory of a periodic solution



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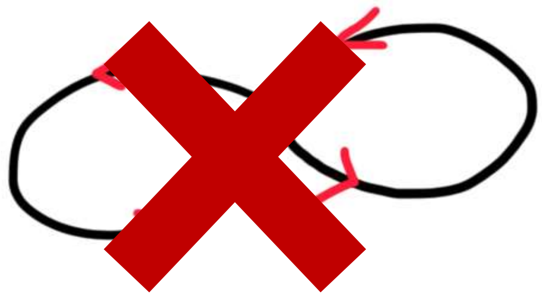

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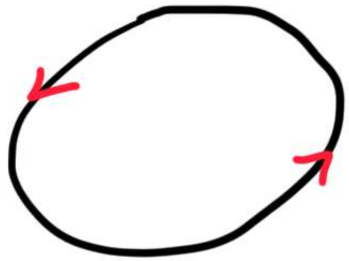

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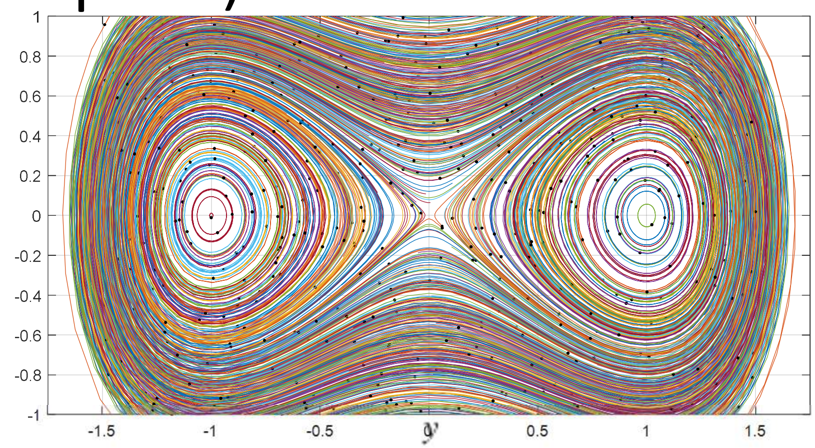


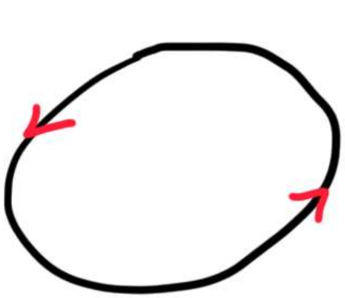

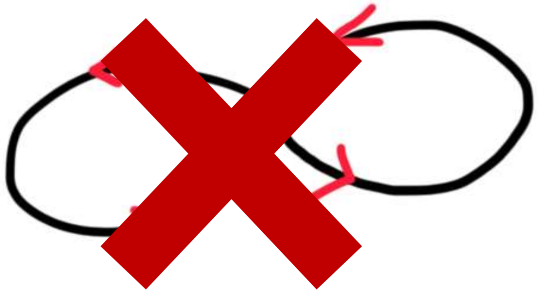

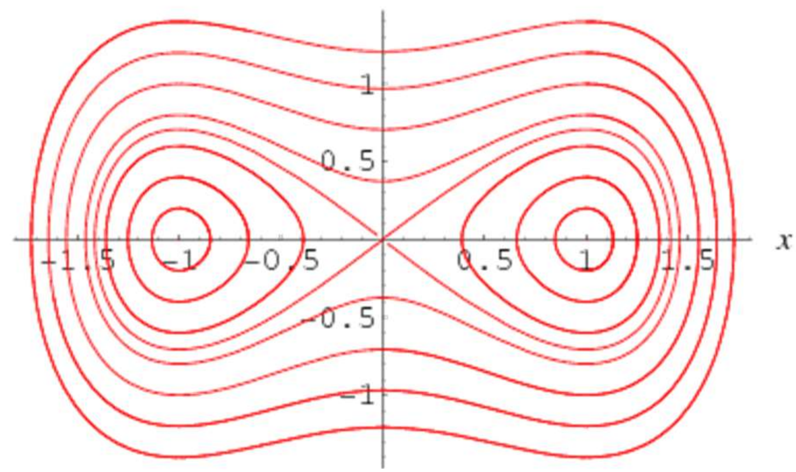

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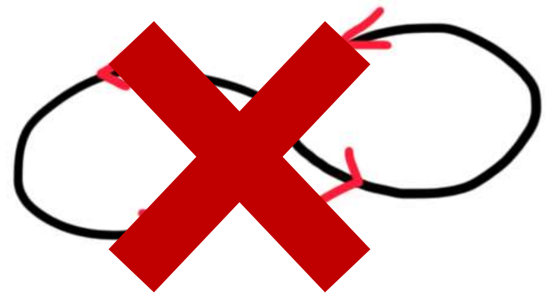

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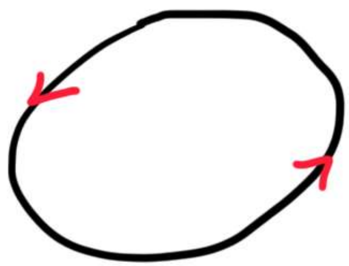
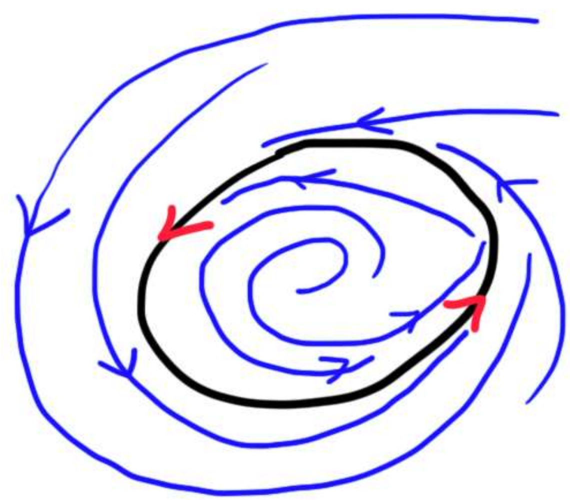


- Limit cycle: a closed orbit $\gamma$ such that there is an initial condition $x_0$ such that $x(t) \to \gamma$ as $t \to \pm\infty$ starting from $x_0$.



# Rayleigh's Model of Violin String

- See assignment 1