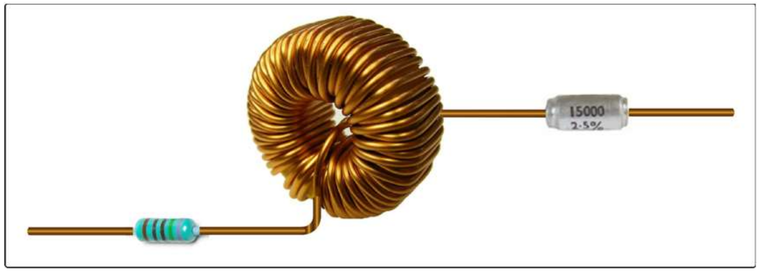
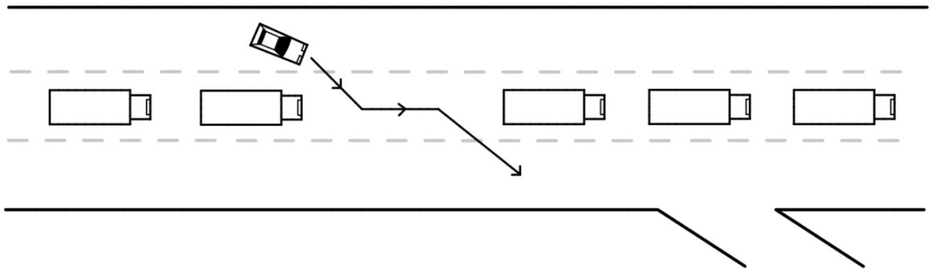
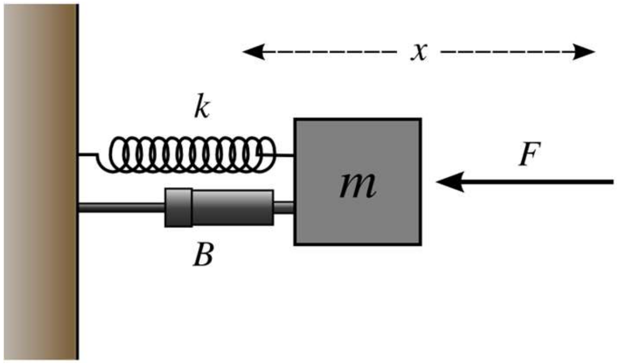


Linear Systems II

CMPT 882

Jan. 14





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- Issues
  - Controller saturation
  - Full state information required

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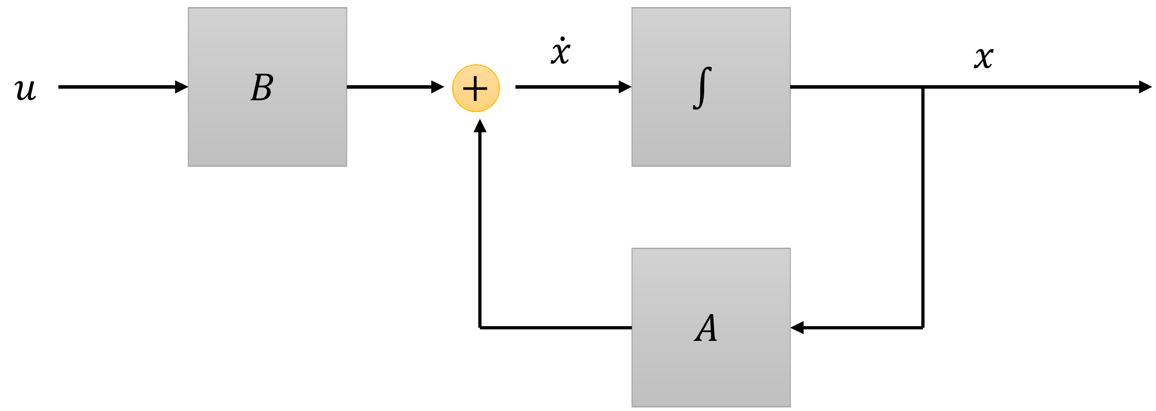

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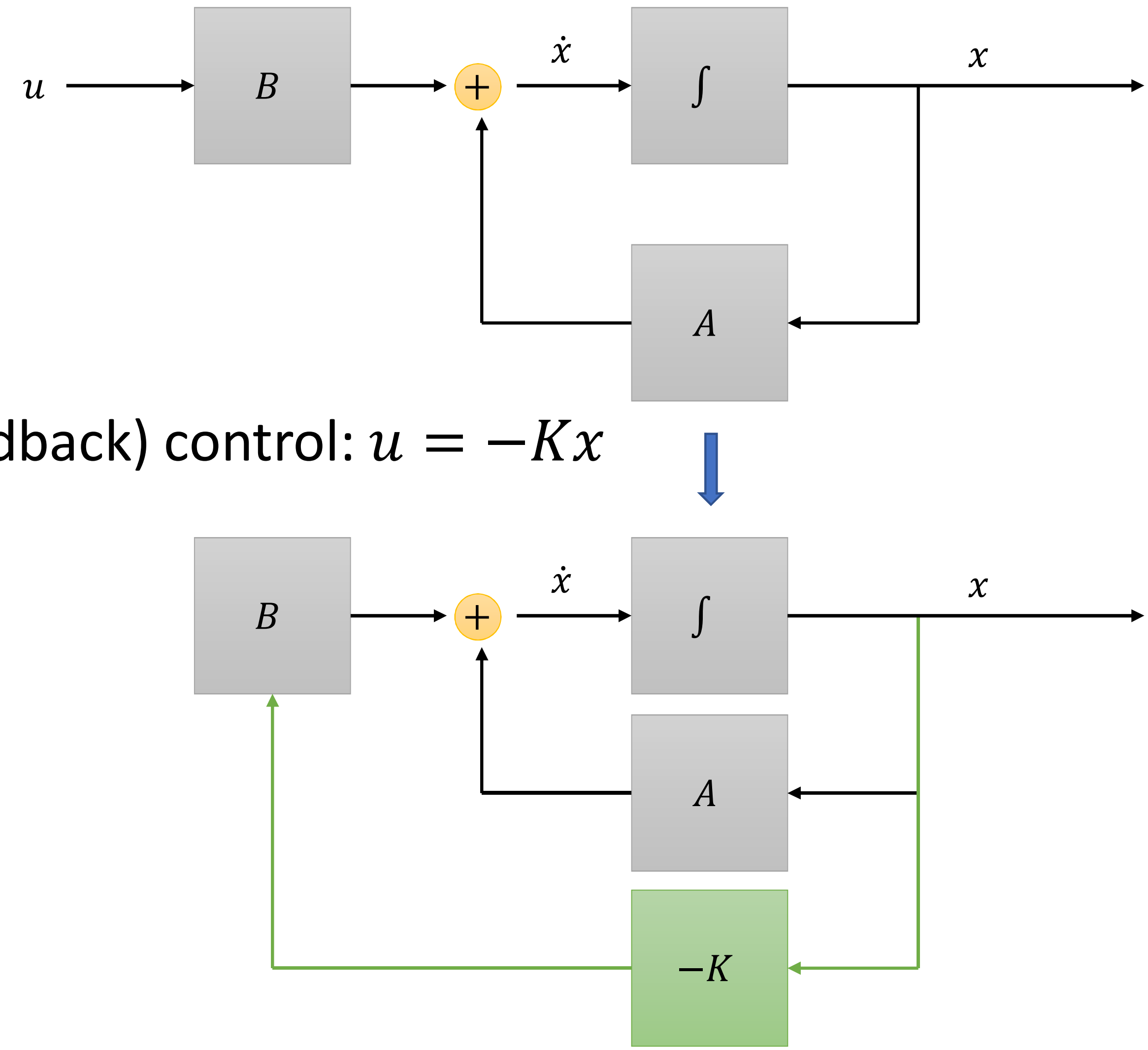

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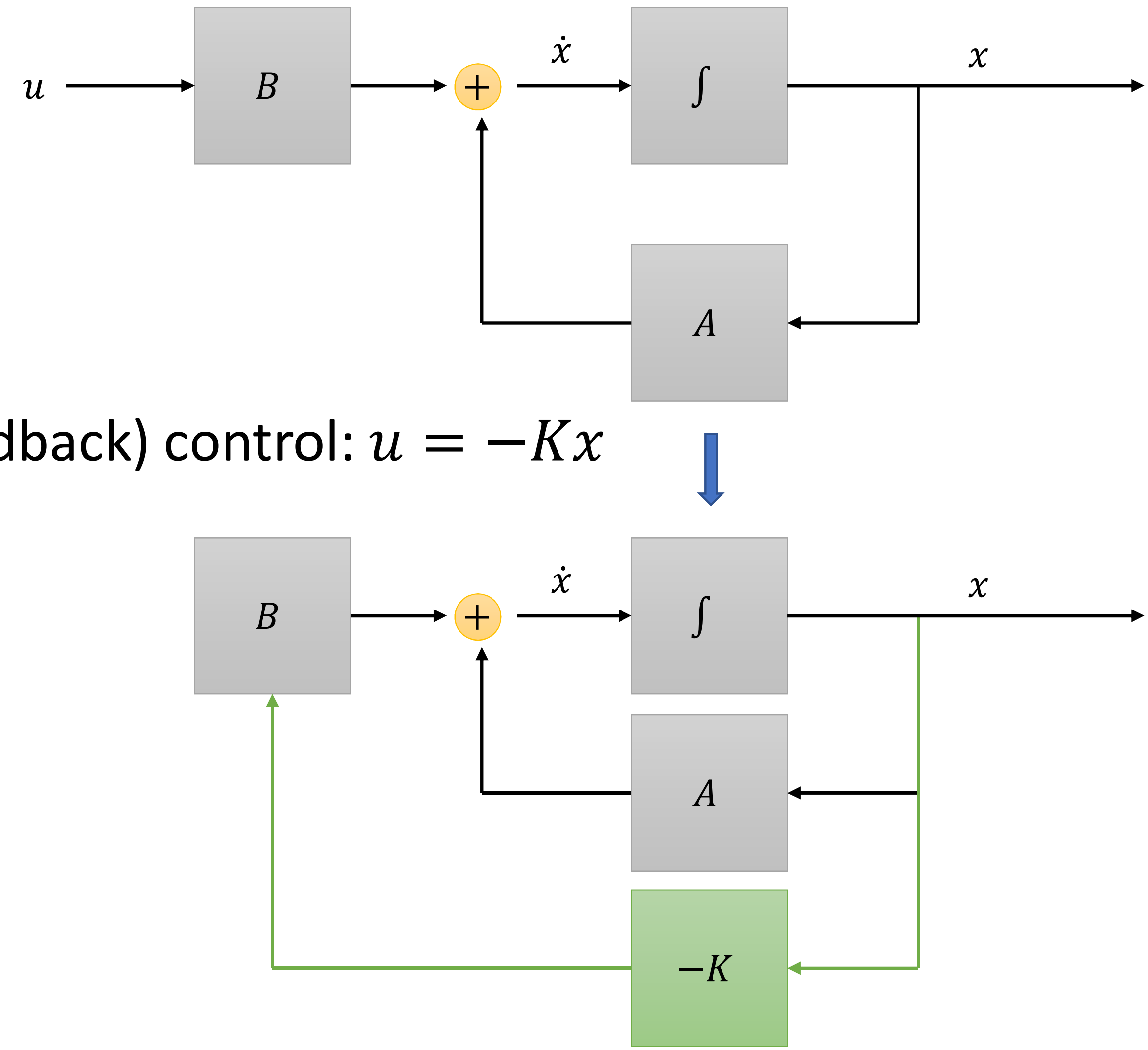

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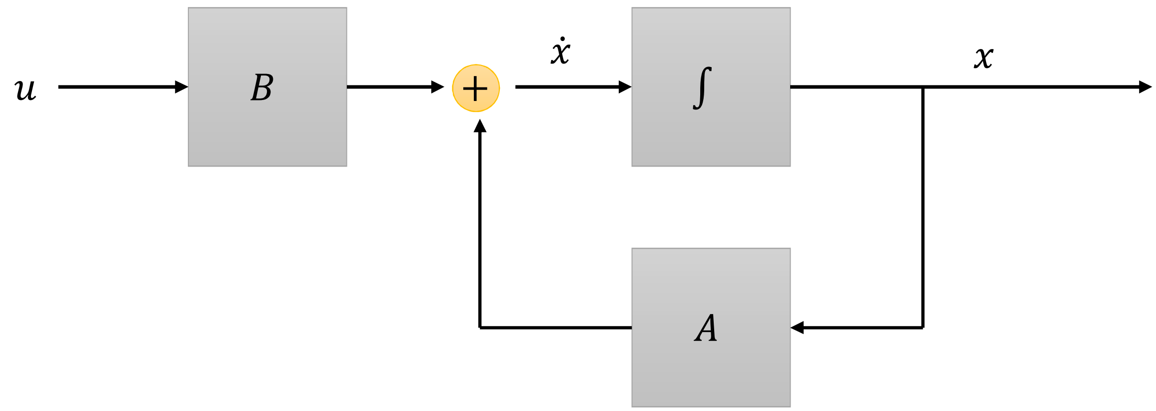

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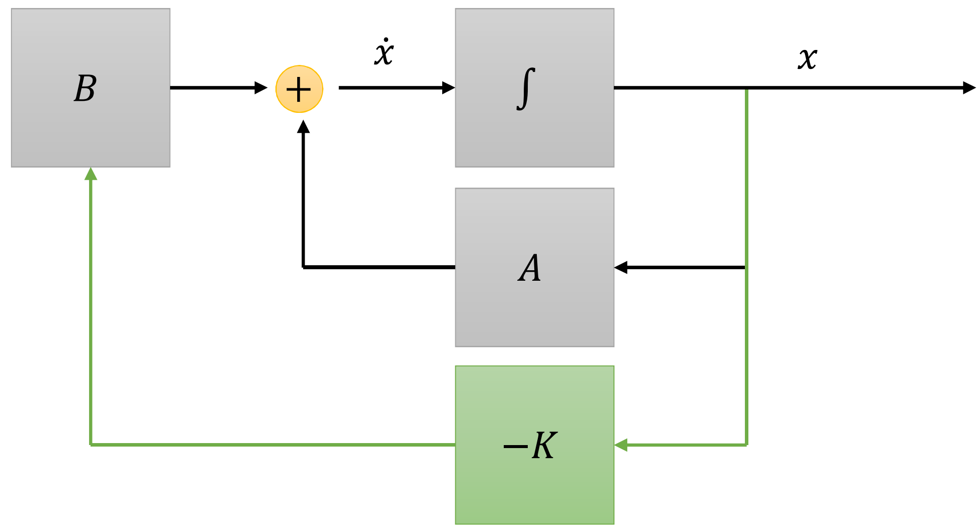

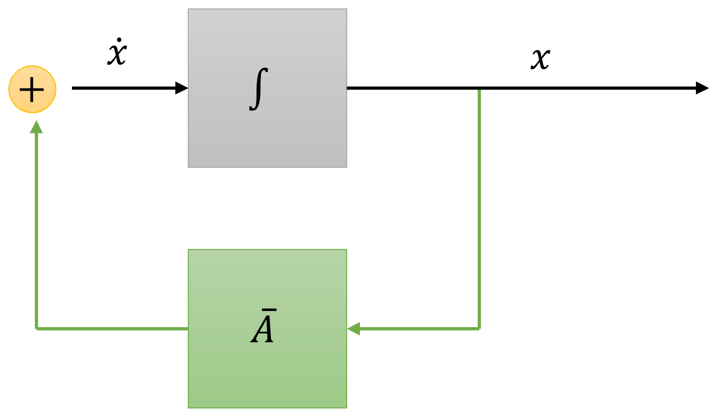

Controllability

- A system is controllable on $[t_0, t_1]$ if for all pairs of states x_0, x_1, there exists a control function $u_{[t_0,t_1]}(\cdot)$ which steers the system from x_0 at t_0 to x_1 at t_1

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- Special case for a controllable system in the form $\dot{x} = Ax = Bu$:
  - For all $x_0 \in \mathbb{R}^n$, there exists $u_{[t_0,t_1]}(\cdot)$ that steers (x_0, t_0) to (θ_n, t_1)
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Eigenvalues of A

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- Suppose we have the system $\dot{x} = Ax + Bu$, where

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  - $x_1^{(n-1)} = -\alpha_0 x_1 - \alpha_1 \dot{x}_1 - \cdots \alpha_{n-1} x_n^{(n-1)} + u$

Controllable Canonical Form

- Observation 2: Eigenvalues, $\det(A - sI) = 0$:

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Characteristic equation
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Characteristic equation
- of the ODE
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- Eigenvalues are solutions to the polynomial with coefficients given by the negative of last row

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Recall $A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

- Observation 3: suppose $u = -Kx, K \in \mathbb{R}^{1\times n}$: $u = -k_0 x_1 - \cdots - k_{n-1} x_n$

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- In matrix form, $\dot{x} = \bar{A}x$, with $\bar{A} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\alpha_0 - k_0 & -\alpha_1 - k_1 & \cdots & -\alpha_{n-1} - k_{n-1} \end{bmatrix}$

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- Observation 3: suppose $u = -Kx, K \in \mathbb{R}^{1\times n}$: $u = -k_0 x_1 - \cdots - k_{n-1} x_n$

- Writing out the last component in $\dot{x} = Ax + Bu$, we have
  - $\dot{x}_n = -\alpha_0 x_1 - \cdots - \alpha_{n-1} x_n - k_0 x_1 - \cdots - k_{n-1} x_n$
  - $\dot{x}_n = -(\alpha_0 + k_0) x_1 - \cdots - (\alpha_{n-1} + k_{n-1}) x_n$

- In matrix form, $\dot{x} = \bar{A}x$, with $\bar{A} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\alpha_0 - k_0 & -\alpha_1 - k_1 & \cdots & -\alpha_{n-1} - k_{n-1} \end{bmatrix}$
  - Eigenvalues are solutions to $s^n + (\alpha_{n-1} + k_{n-1}) s^{n-1} + \cdots + (\alpha_1 + k_1) s + \alpha_0 + k_0 = 0$

Choosing K determines coefficients, and therefore eigenvalues

Controllable Canonical Form Example

- Suppose $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
  - Choose K such that if $u = -Kx$, all eigenvalues of the closed-loop system has are -1

- Characteristic polynomial of closed-loop system:
$$s^3 + (1 + k_2)s^2 + (2 + k_1)s + (-1 + k_0)$$
- Desired characteristic polynomial: $(s + 1)^3 = s^3 + 3s^2 + 3s + 1$

Three eigenvalues at -1

- Therefore, use $k_2 = 2, k_1 = 1, k_0 = 2$

Transformation Into Controllable Canonical Form

- Given a **controllable** system $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n, u \in \mathbb{R}$, let

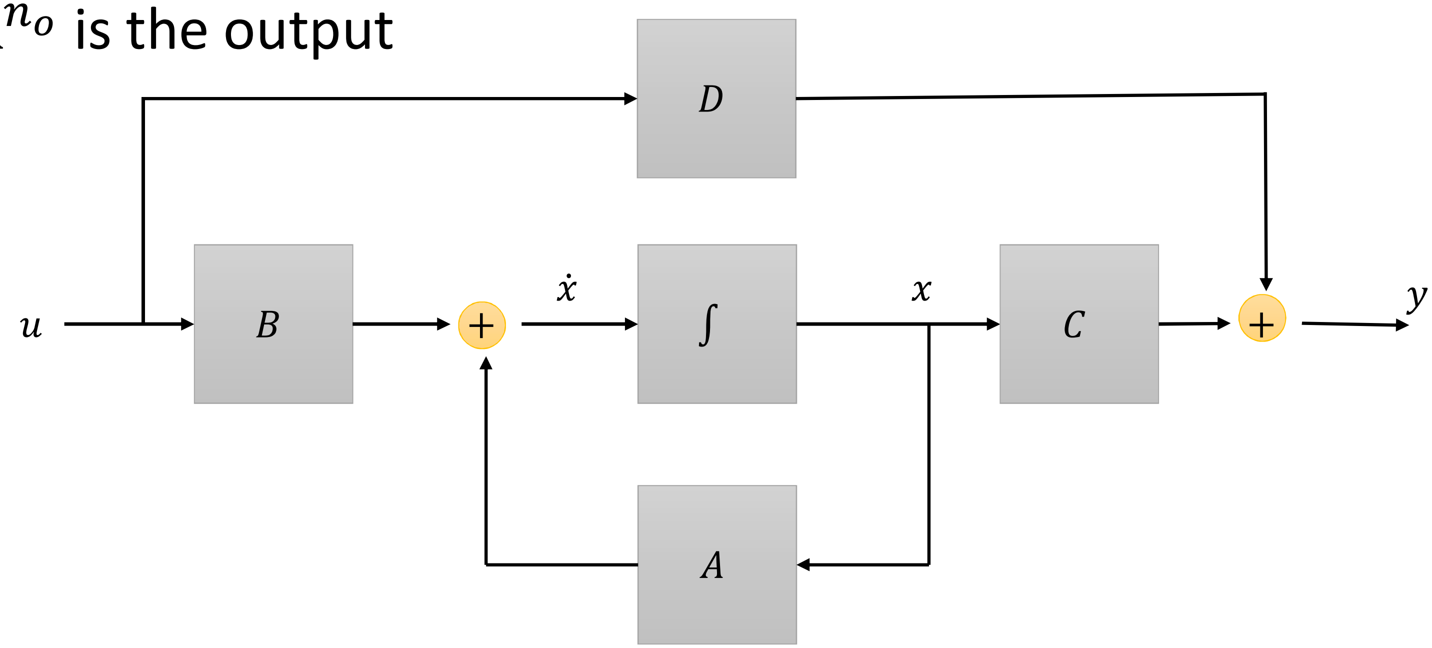
$$T^{-1} = [B \quad AB \quad \cdots \quad A^{n-1}B] \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ \alpha_2 & ⋰ & ⋰ & ⋰ & 0 \\ \vdots & ⋰ & 1 & ⋰ & \vdots \\ \alpha_{n-1} & ⋰ & ⋰ & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

- Then, $\tilde{A} \coloneqq TAT^{-1} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}$

The (A, B, C, D) representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- $y \in \mathbb{R}^{n_o}$ is the output



Observability

- A dynamical system is observable on $[t_0, t_1]$ if for all $u_{[t_0,t_1]}(\cdot)$ and $y_{[t_0,t_1]}(\cdot)$, x_0 at t_0 is uniquely determined

- The following are equivalent
  - The system $\dot{x} = Ax$ with output $y = Cx$ is observable on the time interval $[0, \Delta]$
  - $\mathrm{rank}\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n$
  - $\forall s \in \mathbb{C},\ \mathrm{rank}\begin{pmatrix} sI - A \\ C \end{pmatrix} = n$

Controllability and Observability

- The following are equivalent
  - The system $\dot{x} = Ax + Bu$ is **controllable** on the time interval $[0, \Delta]$
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  - $\forall s \in \mathbb{C},\ \operatorname{rank}([sI - A \quad B]) = n$

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Stabilizability and Detectability

- The following are equivalent
  - The system $\dot{x} = Ax + Bu$ is **stabilizable** on the time interval $[0, \Delta]$
  - $\forall s \in \sigma(A) \cap \mathbb{C}_+$, $\operatorname{rank}([sI - A \quad B]) = n$

- The following are equivalent
  - The system $\dot{x} = Ax$ with output $y = Cx$ is **detectable** on the time interval $[0, \Delta]$
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Eigenvalues of A in the right half plane

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Eigenvalues of $\boldsymbol{A}$ in the right half plane

- The uncontrollable parts of the system are stable

- The unobservable parts of the system are stable

Other Important Topics in Linear Systems

- Singular value decomposition
- Controllable and observable subspaces
- Linear Time-Varying Systems
- Etc.

- F. Callier & C. A. Desoer, Linear System Theory, Springer-Verlag, 1991.
- W. J. Rugh, Linear System Theory, Prentice-Hall, 1996.