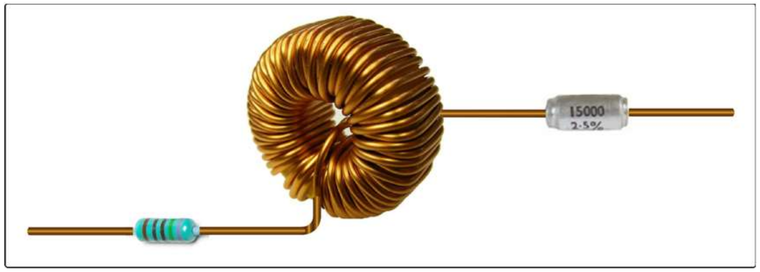
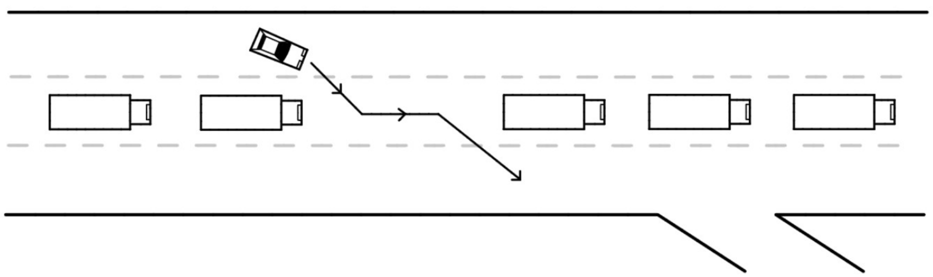
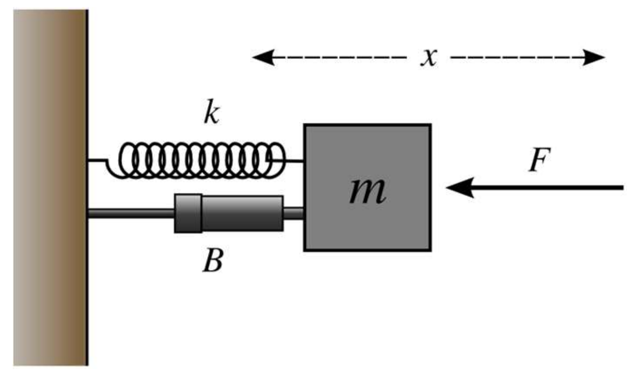


Linear Systems I

CMPT 882

Jan. 11





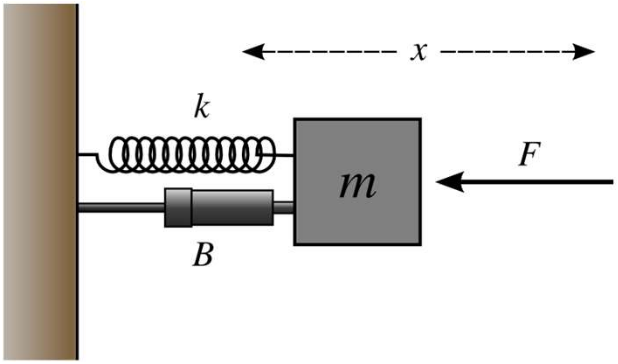
Linear Systems

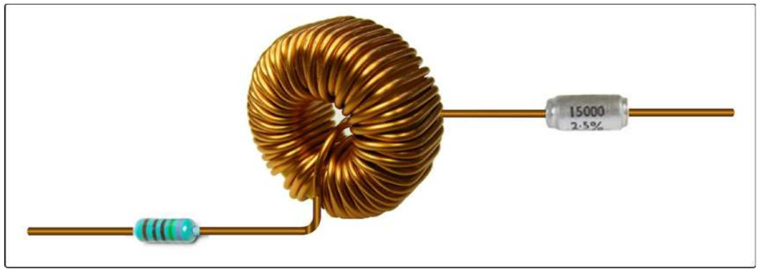
- Differential equations generally do not have closed-form solutions
  - Numerical methods can be used to obtain approximate solutions
  - Other analysis techniques offer insight into the solutions

Linear Systems

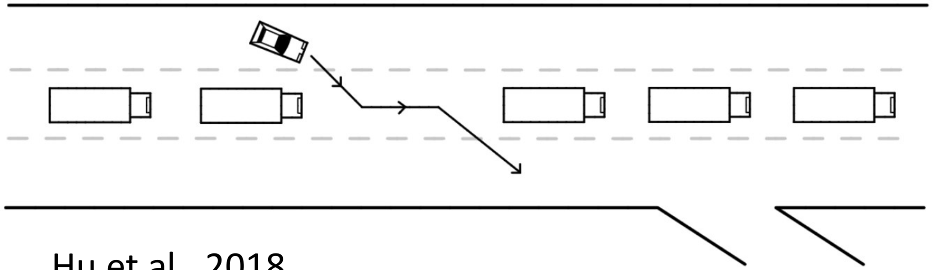
- Differential equations generally do not have closed-form solutions
  - Numerical methods can be used to obtain approximate solutions
  - Other analysis techniques offer insight into the solutions

- Linear time-invariant (LTI) systems: $\dot{x} = Ax + Bu$
  - Damped mass spring systems
  - Circuits involving resistors, capacitors, inductors
  - Approximations of nonlinear systems





Linear Systems



Hu et al., 2018

(If flying near hover, and slowly)
Bouffard, 2012

Road Map

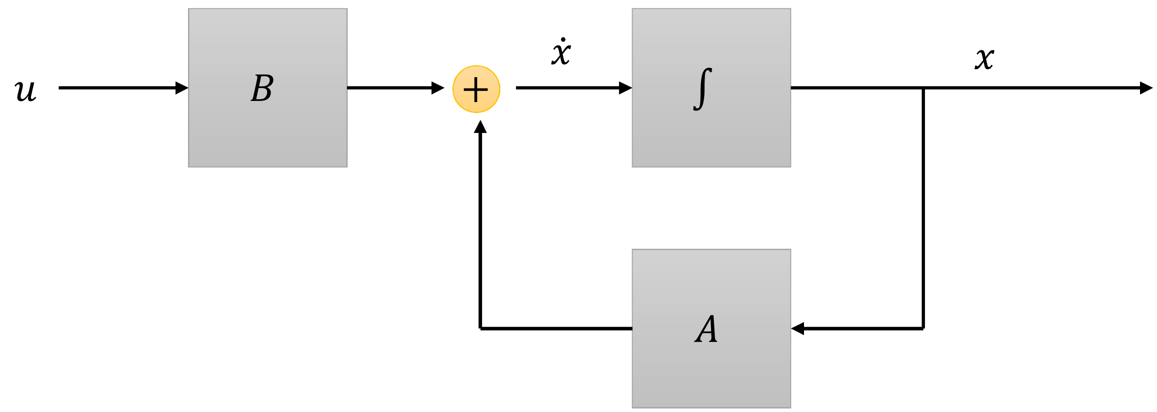
- Basic properties and closed form solution
- Stability
- Linearization
- Controllability and observability

Road Map

- Linear Systems (This and next lecture)
  - Basic properties and closed form solution
  - Stability
  - Linearization
  - Controllability and observability
- Nonlinear systems (Two lectures)
- Optimization and optimal control (New unit, ~8 lectures)

LTI Systems

- Linear time-invariant (LTI) systems: $\dot{x} = Ax + Bu$



Linear System

- Existence and Uniqueness of Solutions of $\dot{x} = f(x, u)$
  - $\exists L > 0, \forall u, x_1, x_2, \|f(x_1, u) - f(x_2, u)\| \leq L\|x_1 - x_2\|$

- Existence and Uniqueness of Solutions of $\dot{x} = Ax + Bu$
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  - $\Leftrightarrow \exists L > 0, \forall u, x_1, x_2, \|Ax_1 - Ax_2\| \leq L\|x_1 - x_2\|$
  - But $\|Ax_1 - Ax_2\| = \|A(x_1 - x_2)\| \leq \|A\|_i\|x_1 - x_2\|$

- Recall:
  - $\|A\|_{p,i} = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$
  - $\|A\|_{\infty,i} = \max_i \sum_{j=1}^{n} |a_{ij}|$ (maximum row sum)

LTI systems: Closed form solution

- $\dot{x} = Ax + Bu, \;\; x(0) = x_0$
  - $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$$

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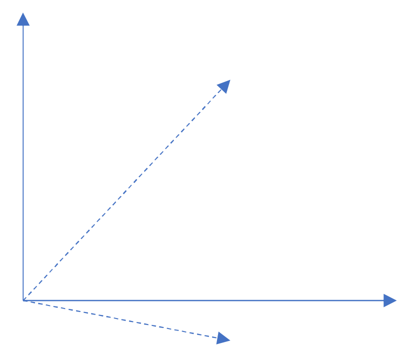

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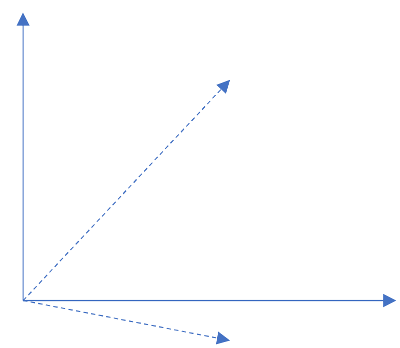

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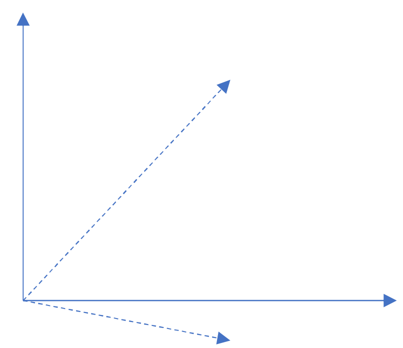

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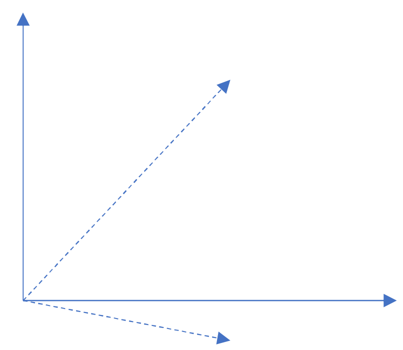

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  - Solution in terms of z: $z(t) = e^{Jt}z_0$
  - Diagonal J: $z(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}$

LTI systems: Closed form solution

- General J:
- $f(J) = \begin{bmatrix} f(\lambda_1) & & & & & \\ & f(\lambda_1) & f'(\lambda_1) & & & \\ & & f(\lambda_1) & & & \\ & & & f(\lambda_2) & f'(\lambda_2) & \frac{1}{2}f''(\lambda_2) \\ & & & & f(\lambda_2) & f'(\lambda_2) \\ & & & & & f(\lambda_2) \end{bmatrix}$

LTI systems: Closed form solution

$$f(J) = e^{Jt}$$
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Matrix Exponential properties

- If $\dot{x} = Ax,\ x(0) = x_0$, then $x(t) = e^{At}x_0$

e^{At} "propagates" a state forward by a duration of t, according to the system dynamics A

Matrix Exponential properties

$t = 0$

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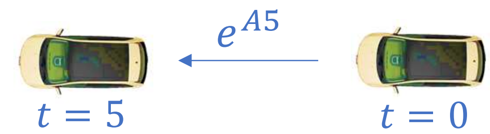

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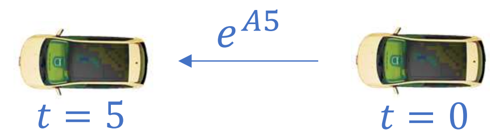

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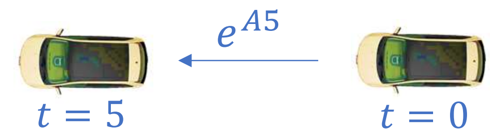
- **State transition matrix**

Matrix Exponential properties



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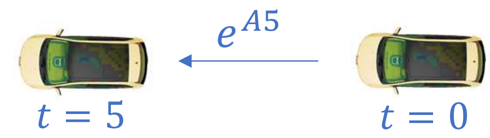

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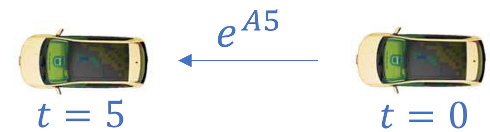

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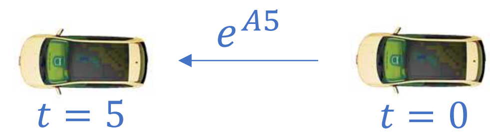

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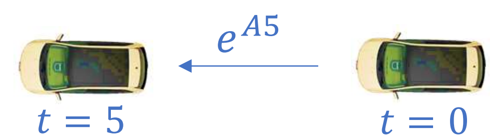

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- $(e^{At})^{-1} = e^{-At}$
  - So $e^{At}e^{-At} = I$
- $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$
  - From definition: $e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$

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Solution to LTI System: Proof

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- Differentiate:

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  - $\dot{x} = \frac{d}{dt}(e^{At}x_0) + \frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)$
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$$\begin{aligned}\frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) &= \frac{d}{dt}\left(\int_0^t e^{At}e^{-A\tau}Bu(\tau)d\tau\right)\\ &= \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau\right)\\ &= Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-A}\ Bu(t)\\ &= A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)\end{aligned}$$

Solution to LTI System: Proof

- If $\dot{x} = Ax + Bu,\ \ x(0) = x_0$, then $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- Initial conditions:
  - $x(0) = e^{A(0)}x_0 + \int_0^0 e^{A(t-\tau)}Bu(\tau)d\tau = x_0$

$$\frac{d}{dt}\int_a^t g(\tau)d\tau = g(t)$$

- Differentiate:
  - $\dot{x} = \frac{d}{dt}(e^{At}x_0) + \frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)$
  - $\dot{x} = Ae^{At}x_0 + A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)$
  - $\dot{x} = Ax(t) + Bu(t)$

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) &= \frac{d}{dt}\left(\int_0^t e^{At}e^{-A}\ Bu(\tau)d\tau\right) \\
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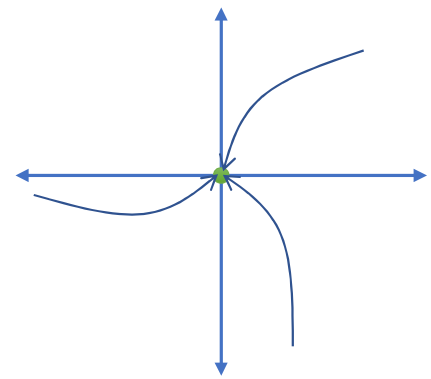
\end{aligned}$$

LTI System: Stability of $\dot{x} = Ax$



- **Equilibrium point** of $\dot{x} = f(x)$ is where $f(x) = 0$
  - For $\dot{x} = Ax$, in general $\mathbf{0}_n$ is an equilibrium point: $x_e = \mathbf{0}_n$
  - Also, $x_e \in N(A)$

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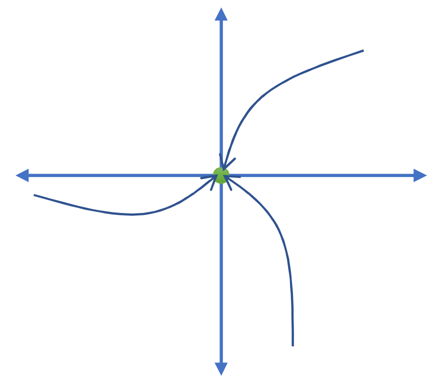

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- **Stable**: $x(t)$ is bounded for all $t \geq 0$, for all initial conditions x_0
- **Asymptotically stable**: $x(t) \to x_e$ as $t \to \infty$
- **Exponentially stable**: $\exists M, \alpha > 0$ such that $\|x(t)\| \leq Me^{-\alpha t}\|x_0\|$

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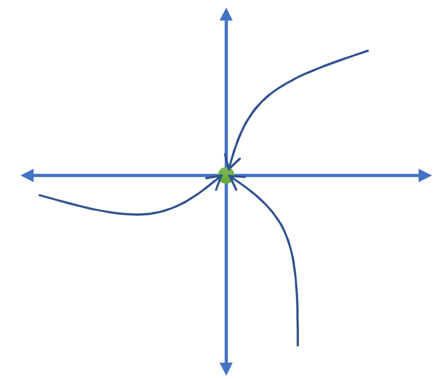

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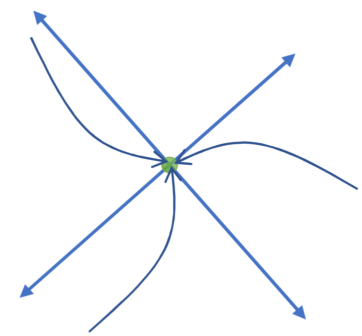

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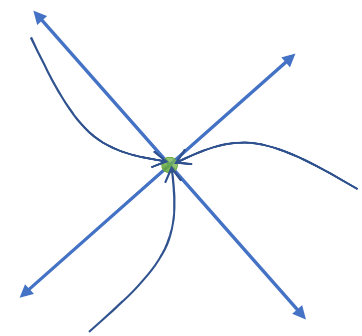

- If $\max \mathrm{Re}(\lambda_k) = 0$, $z(t)$ stays bounded only if $\bar{\lambda}_k$ has Jordan block of size 1

Eigenvalue with largest real part

LTI System: Stability

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- When $\lambda_i = 0$…

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- When $\lambda_i = 0$…
- Not stable!

Linearization

- Local behaviour of nonlinear system $\dot{x} = f(x, u)$ at operating point $(x, u) = (\bar{x}, \bar{u})$
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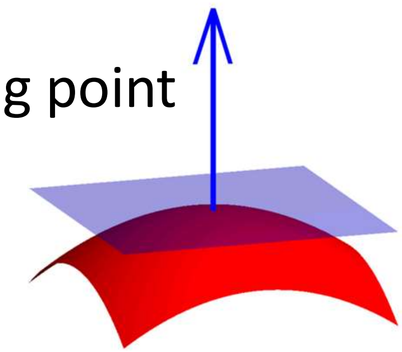
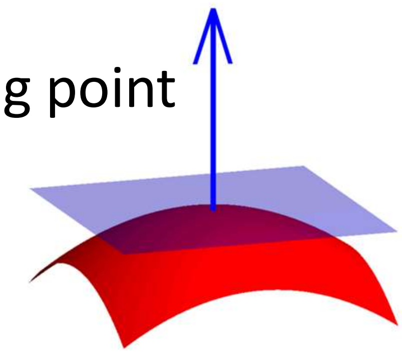


Linearization



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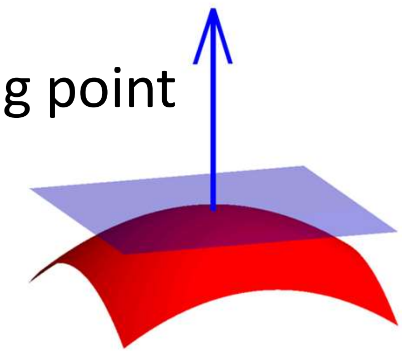
Linearization



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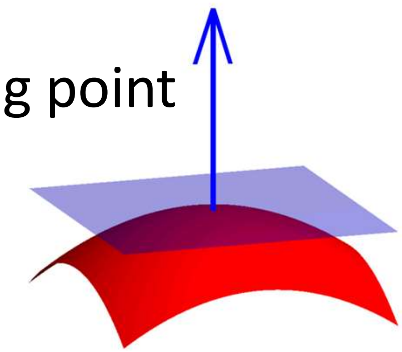
Linearization



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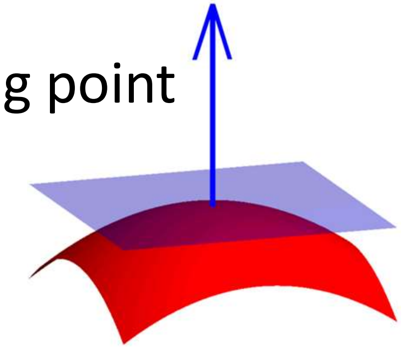

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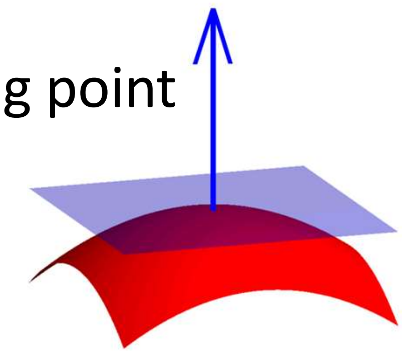
Linearization



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Linearization

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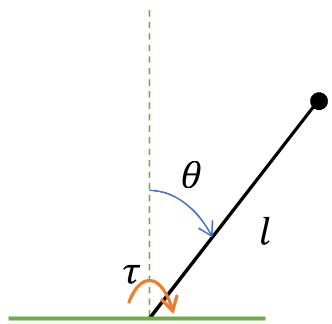
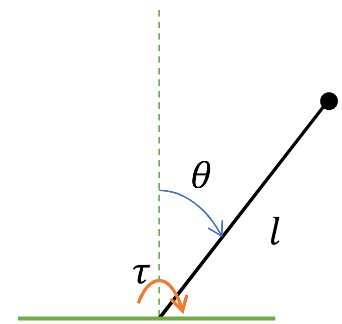
Linearization



- Inverted pendulum
  - Newton's laws: $\ddot{\theta} = \frac{\tau}{ml^2} + \frac{g}{l}\sin\theta$

Linearization



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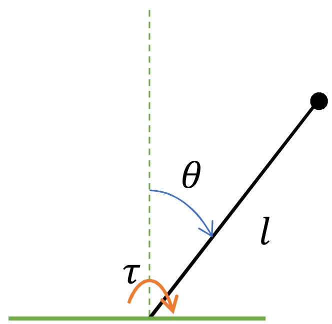
Linearization



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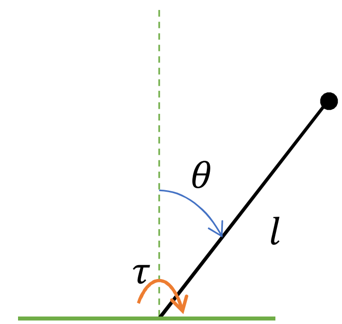
Linearization



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- Linearize around $\theta = x_1 = 0, \dot{\theta} = x_2 = 0, u = 0$

Linearization

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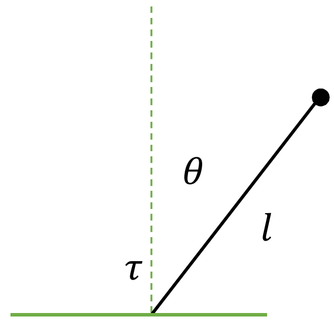

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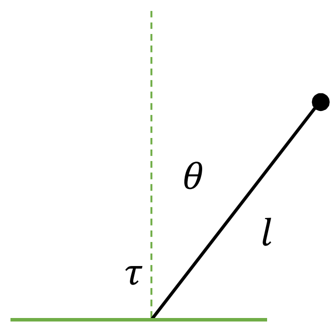

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Linearization

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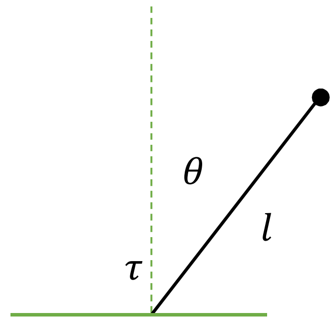

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Linearization

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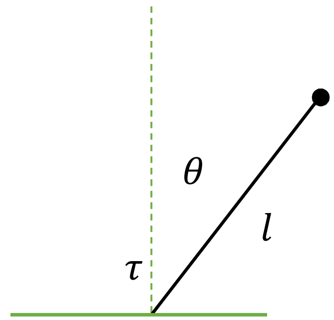

- Linearize around $\theta = x_1 = 0, \dot{\theta} = x_2 = 0, u = 0$
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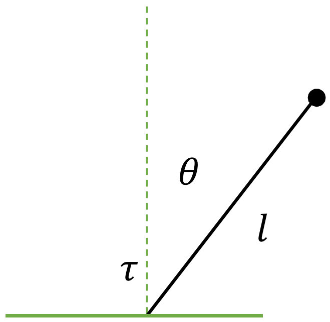

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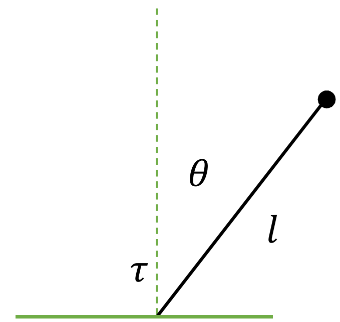

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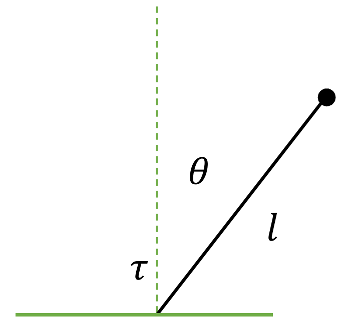

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- So $\dot{x} \approx \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \Rightarrow \qquad \begin{aligned} \dot{x}_1 &\approx x_2 \\ \dot{x}_2 &\approx \frac{g}{l} x_1 + u \end{aligned}$

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- Suppose $\dot{x} = Ax + Bu$, can we design u to make $x = \mathbf{0}_n$ stable?

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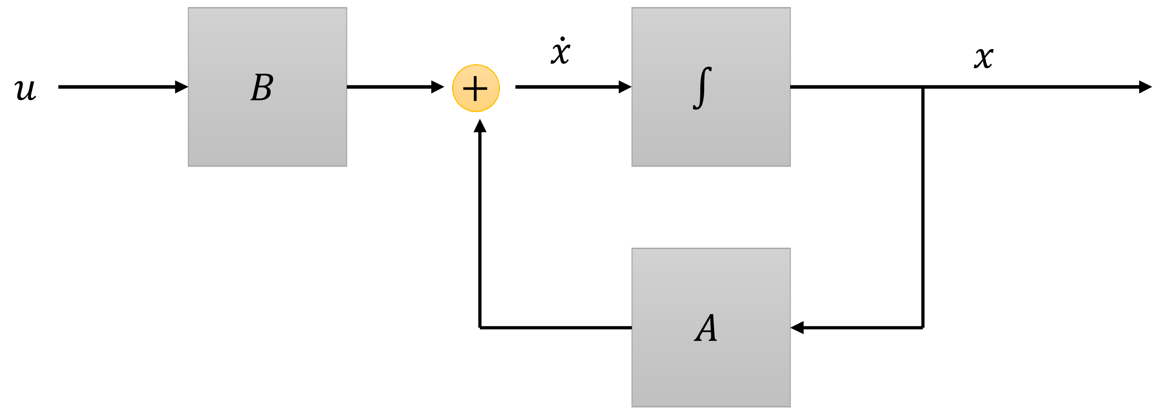
- Issues
  - Controller saturation
  - Full state information required

State Feedback



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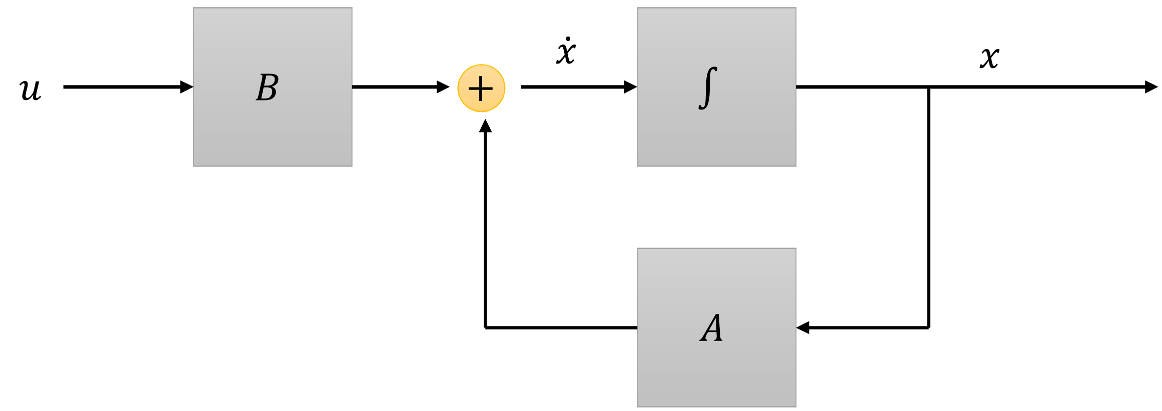

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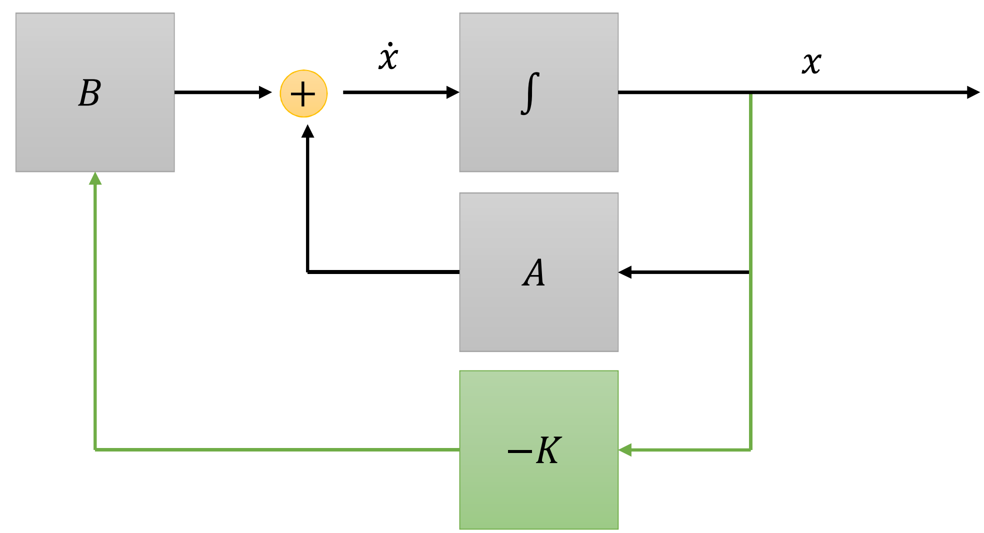

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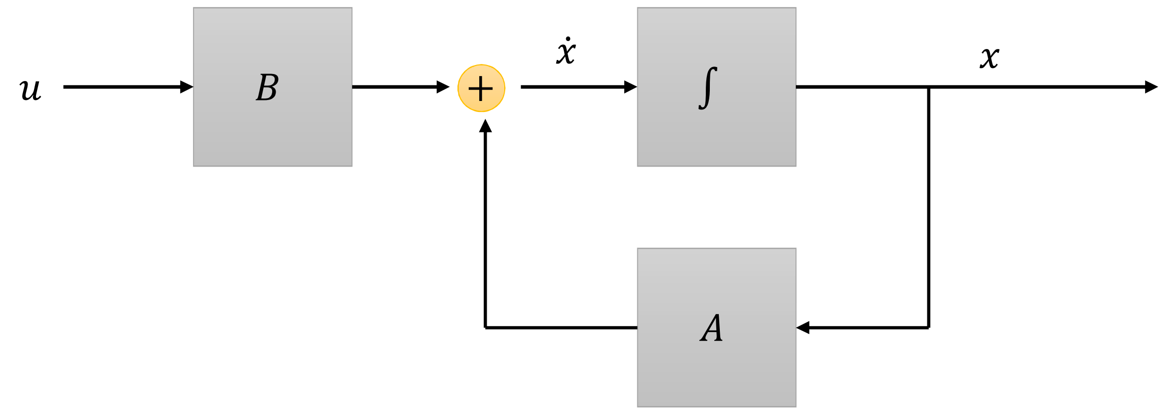

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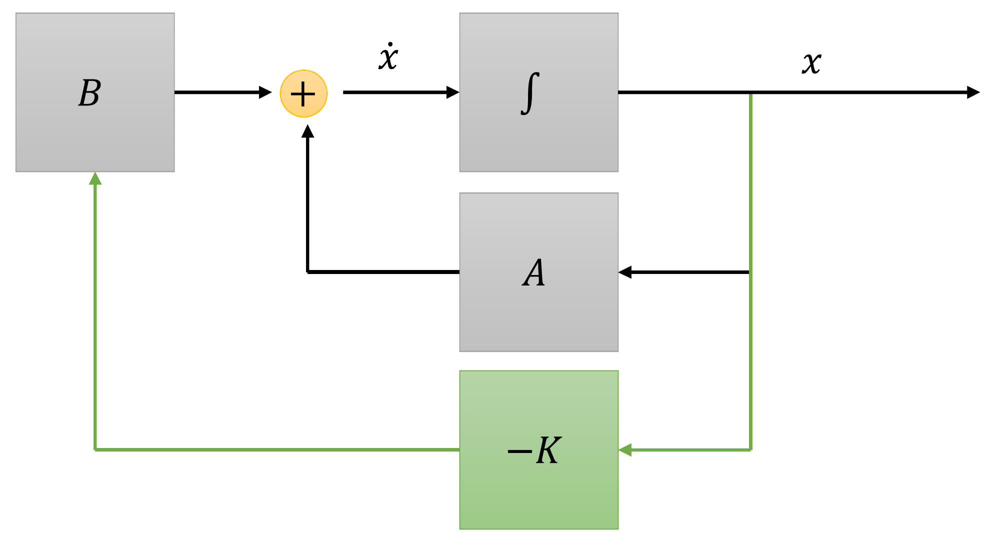

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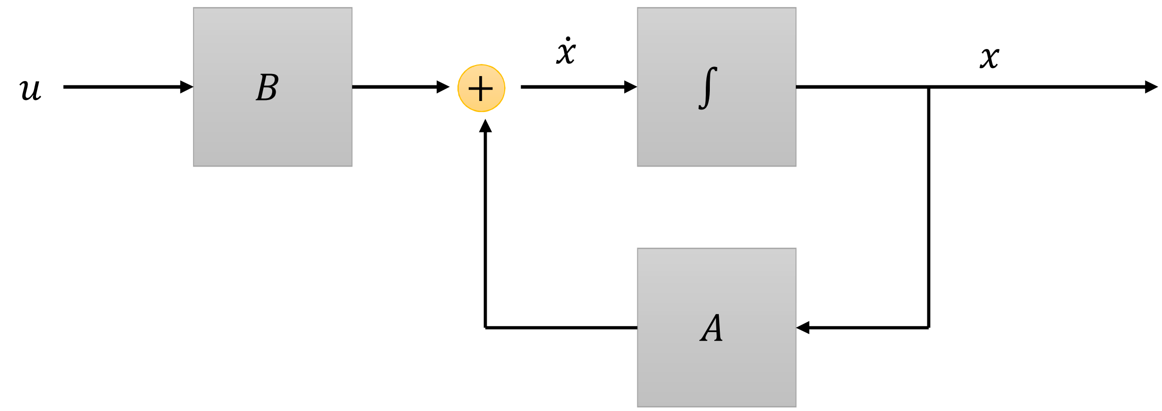

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