Bayes’ Filter

CMPT 882

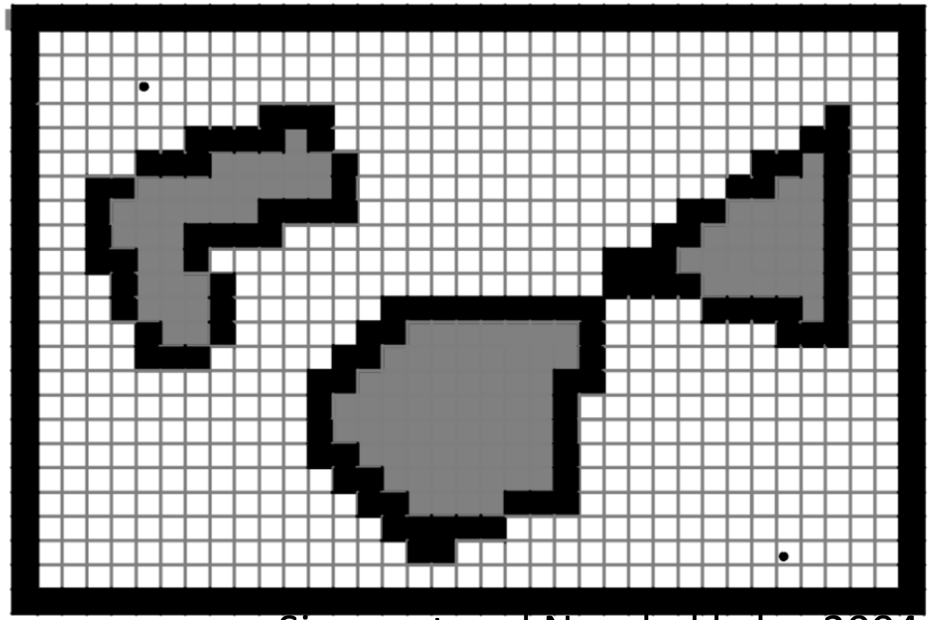
Mar. 25

Outline

- Localization Problem Setup

- Bayes’ Filter

Localization: Problem Setup



Siegwart and Nourbakhshs, 2004

- Assume a map is given: $m = \{m_1, m_2, \ldots, m_N\}$
  - Location based: each m_i represents a specific location and whether it's occupied (eg. Occupancy grid)

  - Feature based: each m_i contains the location and properties of a feature (eg. Topological map)

Localization: Problem Setup

- Assume a map is given: $m = \{m_1, m_2, \ldots, m_N\}$
  - Location based: each m_i represents a specific location and whether it's occupied (eg. Occupancy grid)
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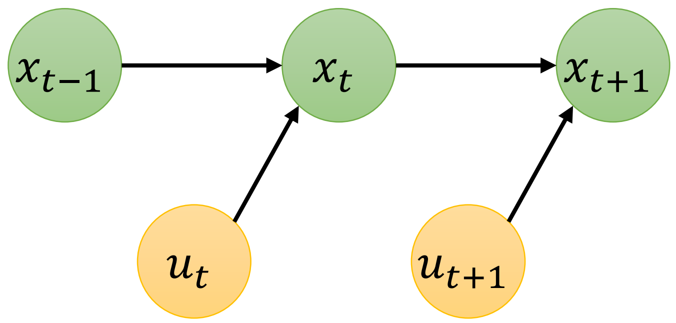
- Robot maintains and updates its belief about where it is with respect to the map
  - Position belief is updated based on sensor data
  - Position belief is a probability distribution

Robot-Environment Interaction: Definitions

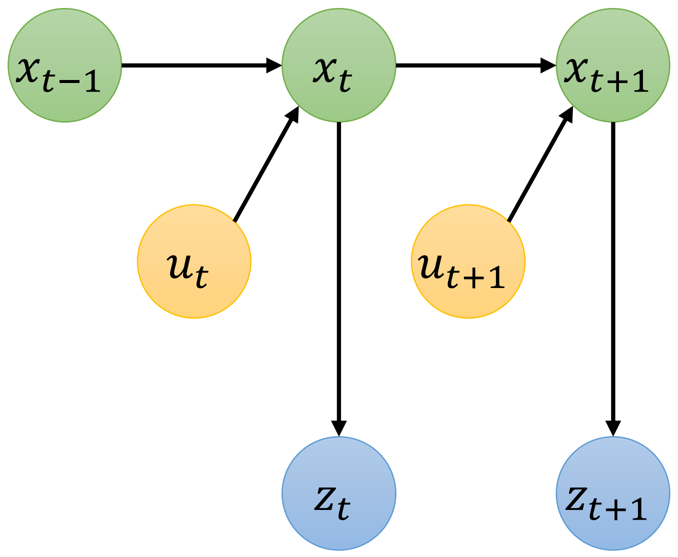
- State x_t: includes the environment (eg. objects, features)
  - Assume the state x_t is complete / the Markov property
- Control data u_t
  - Usually decreases robot's knowledge
- Probabilistic model of state evolution
  - $p(x_t|x_{t-1}, u_t)$



Robot-Environment Interaction: Definitions

- Measurement data z_t
  - Increases robot's knowledge

- Measurement equation:
  - $p(z_t|x_t)$



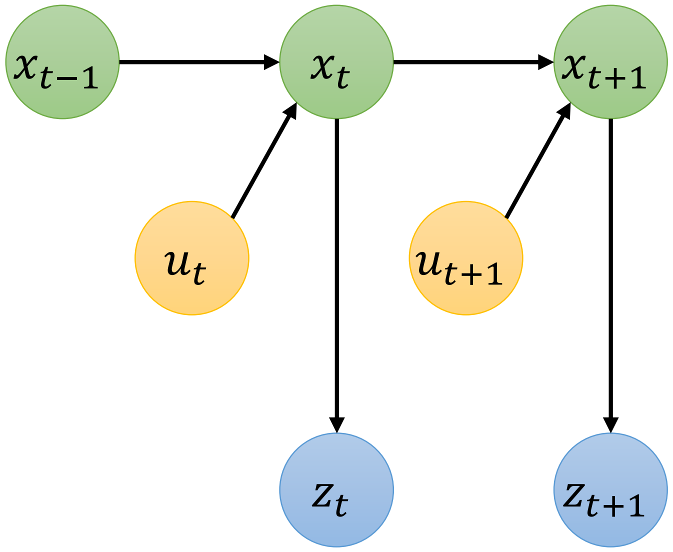
Prediction and Belief Distributions

- Prediction distribution:
  - Robot's prediction of the state before making an observation

$$\overline{\mathrm{bel}}(x_t) := p(x_t | z_{1:t-1}, u_{1:t})$$

- Belief distribution:
  - Robot's internal knowledge about the state

$$\mathrm{bel}(x_t) := p(x_t | z_{1:t}, u_{1:t})$$

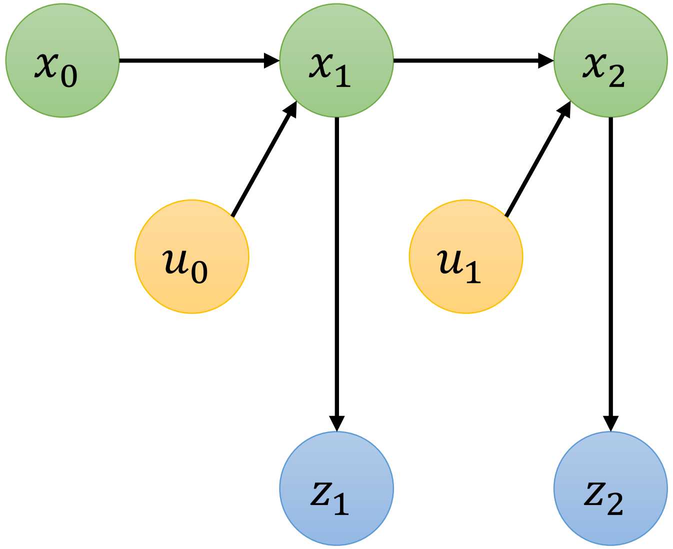


Bayes Filter (Continuous)

- Robot and environment have state x_0
  - Initialize $\mathrm{bel}(x_0)$ (eg. to be uniform or dirac distribution)

- From x_0, choose a control u_0 → robot moves to x_1
  1. Predict the next state by computing $\overline{\mathrm{bel}}(x_1)$ using dynamics
  2. Make an observation z_1, and use it to compute $\mathrm{bel}(x_1)$

- Repeat for $x_2, x_3, \ldots$



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- Repeat for $x_2, x_3, \ldots$

- Bayes' filter algorithm:

Input: $\mathrm{bel}(x_{t-1}), u_t, z_t$

Output: $\mathrm{bel}(x_t)$

For every x_t,

Perform prediction:

$$\overline{\mathrm{bel}}(x_t) = \int p(x_t | u_t, x_{t-1}) \mathrm{bel}(x_{t-1}) dx_{t-1}$$

Perform measurement update:

$$\mathrm{bel}(x_t) = \eta p(z_t | x_t) \overline{\mathrm{bel}}(x_t)$$

Return $\mathrm{bel}(x_t)$

Bayes Filter (Continuous)

$$\overline{\mathrm{bel}}(x_t) = p(x_t|z_{1:t-1}, u_{1:t})$$
$$= \int p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1}|z_{1:t-1}, u_{1:t}) dx_{t-1}$$

Theorem of total probability

$$p(y) = \int p(x, y) dx = \int p(y|x) p(x) dx$$

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$$= \int p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t})p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$$
$$= \int p(x_t|x_{t-1}, u_t)p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$$

Markov assumption

- Bayes' filter algorithm:

Input: $\text{bel}(x_{t-1}), u_t, z_t$
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For every x_t,

Perform prediction:
$$\overline{\text{bel}}(x_t) = \int p(x_t|u_t, x_{t-1})\text{bel}(x_{t-1})dx_{t-1}$$

Perform measurement update:
$$\text{bel}(x_t) = \eta p(z_t|x_t)\overline{\text{bel}}(x_t)$$

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$$= \int p(x_t|x_{t-1}, u_t)p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$$
$$= \int p(x_t|x_{t-1}, u_t)p(x_{t-1}|z_{1:t-1}, u_{1:t-1})dx_{t-1}$$

u_t does not affect probability of x_{t-1}

- Bayes' filter algorithm:

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Output: $\text{bel}(x_t)$
For every x_t,

Perform prediction:
$$\overline{\text{bel}}(x_t) = \int p(x_t|u_t, x_{t-1})\text{bel}(x_{t-1})dx_{t-1}$$

Perform measurement update:
$$\text{bel}(x_t) = \eta p(z_t|x_t)\overline{\text{bel}}(x_t)$$

Return $\text{bel}(x_t)$

Bayes Filter (Continuous)

$$
\begin{aligned}
\overline{\mathrm{bel}}(x_t) &= p(x_t | z_{1:t-1}, u_{1:t}) \\
&= \int p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} | z_{1:t-1}, u_{1:t}) dx_{t-1} \\
&= \int p(x_t | x_{t-1}, u_t) p(x_{t-1} | z_{1:t-1}, u_{1:t}) dx_{t-1} \\
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&= \int p(x_t | x_{t-1}, u_t) \mathrm{bel}(x_{t-1}) dx_{t-1}
\end{aligned}
$$

- Bayes' filter algorithm:

Input: $\mathrm{bel}(x_{t-1}), u_t, z_t$
Output: $\mathrm{bel}(x_t)$
For every x_t,

Perform prediction:
$\overline{\mathrm{bel}}(x_t) = \int p(x_t | u_t, x_{t-1}) \mathrm{bel}(x_{t-1}) dx_{t-1}$

Perform measurement update:
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\end{aligned}
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\begin{aligned}
\mathrm{bel}(x_t) &= p(x_t|z_{1:t}, u_{1:t}) \\
&= \frac{p(z_t|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t})}{p(z_t|z_{1:t-1}, u_{1:t})}
\end{aligned}
$$

Bayes' rule

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

- Bayes' filter algorithm:

Input: $\mathrm{bel}(x_{t-1}), u_t, z_t$

Output: $\mathrm{bel}(x_t)$

For every x_t,

Perform prediction:

$$\overline{\mathrm{bel}}(x_t) = \int p(x_t|u_t, x_{t-1})\mathrm{bel}(x_{t-1})dx_{t-1}$$

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$$\text{bel}(x_t) = p(x_t|z_{1:t}, u_{1:t})$$
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$$= \eta p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t})$$

Markov property

- Bayes' filter algorithm:

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Output: $\text{bel}(x_t)$
For every x_t,

Perform prediction:
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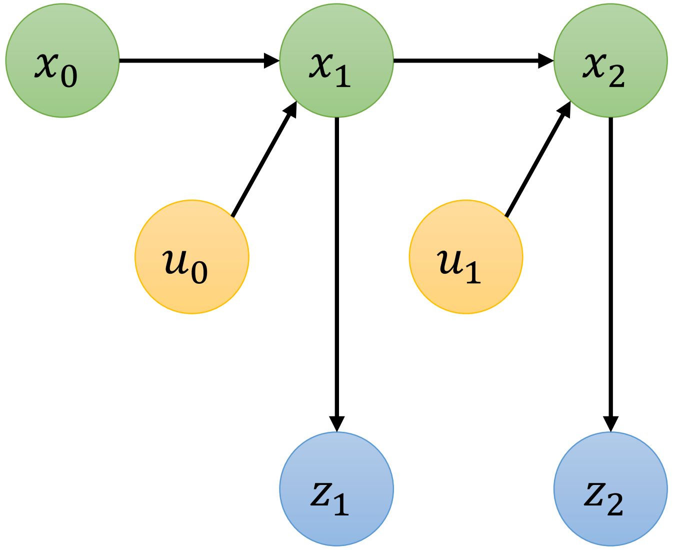
Bayes Filter (Discrete)

- Robot and environment have state x_0
  - Initialize $\mathrm{bel}(x_0)$ (eg. to be uniform or Dirac distribution)

- From x_0, choose a control u_0 → robot moves to x_1
  1. Predict the next state by computing $\overline{\mathrm{bel}}(x_1)$
  2. Make an observation z_1, and use it to compute $\mathrm{bel}(x_1)$

- Repeat for x_2, x_3, ...

- Discrete state space, x_t takes on discrete values index by k
  - $\mathrm{bel}(x_t)$ represented as pmf $\{p_{k,t}\}$



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  2. Make an observation z_1, and use it to compute $\text{bel}(x_1)$
- Repeat for $x_2, x_3, \ldots$

- Discrete state space, x_t takes on discrete values index by k
  - $\text{bel}(x_t)$ represented as pmf $\{p_{k,t}\}$

Input: $\text{bel}(x_{t-1}), u_t, z_t$

Output: $\text{bel}(x_t)$

For every k,

Perform prediction:

$\bar{p}_{k,t} = \sum p(x_t|u_t, x_{t-1})p_{k,t-1}$

Perform measurement update:

$\text{bel}(x_t) = \eta p(z_t|x_t)\bar{p}_{k,t}$

Return $\{p_{k,t}\}$

Bayes Filter

- Continuous state space: Closed-form $\mathrm{bel}(x_t)$ is unlikely. Need discretization and interpolation
- Must iterate through every x_t or every k
  - Number of states is exponential in state space dimension
- Solution: exploit structure or make assumptions
  - Parametric filters: assume a form for distributions
  - Non-parametric filters: represent distributions using samples