

Continuous Time LQR and Robotic Safety via Reachability

CMPT 882

Feb. 27

References

- Dynamic Programming:
 - Bertsekas, Dynamic Programming and Optimal Control. Athena Scientific, 2017, 1886529434.
- Reachability Analysis:
 - Chen & Tomlin. “Hamilton-Jacobi Reachability: Some Recent Theoretical Advances and Applications in Unmanned Airspace Management”. *Annual Review*.

Hamilton-Jacobi Equation

- Hamilton-Jacobi partial differential equation

$$\frac{\partial V}{\partial t} + \min_u \left[C(x, u) + \frac{\partial V}{\partial x} \cdot f(x, u) \right] = 0, \quad V(x, t_f) = l(x)$$

- Minimization over u is typically easy
 - Most systems are control affine: $f(x, u)$ has the form $f(x) + g(x)u$
 - Control constraints are typically “box” constraints, e.g. $|u_i| \leq 1$
- PDE is solved on a grid
 - $x \in \mathbb{R}^n$ means $V(t, x)$ is computed on an $(n + 1)$ -dimensional grid
- $V(x, t)$ is often not differentiable (or continuous)
 - Viscosity solutions
 - Lax Friedrichs numerical method



Example: Continuous LQR

$$J(x(t), t) = \int_t^T C(x(s), u(s)) ds + l(x(T))$$

- Linear system: $\dot{x} = Ax + Bu$
- Cost involving quadratic expressions:

$$J(x, t) = \frac{1}{2} \int_t^T (x(t)^\top Q x(t) + u(t)^\top R u(t)) dt + \frac{1}{2} x(T)^\top L x(T)$$

- L, Q, R are symmetric positive semidefinite
- T is given
- $x(t)$ and $u(t)$ are unconstrained
- The Hamilton-Jacobi equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2} \min_u \left[\underbrace{x(t)^\top Q x(t) + u(t)^\top R u(t)}_{\text{Pre-Hamiltonian}} + \frac{\partial V}{\partial x} \cdot (Ax(t) + Bu(t)) \right] = 0$$

Pre-Hamiltonian: $H\left(x, u, \frac{\partial V}{\partial x}\right)$

Continuous Time LQR

- Pre-Hamiltonian: $H\left(x, u, \frac{\partial V}{\partial x}\right) = x(t)^\top Qx(t) + u(t)^\top Ru(t) + \frac{\partial V}{\partial x} \cdot (Ax(t) + Bu(t))$

- Take Jacobian to optimize H : $\frac{\partial H}{\partial u} = Ru(s) + B \cdot \frac{\partial V}{\partial x}$

- Observe that $\frac{\partial^2}{\partial u^2} = R \succcurlyeq 0$, so first order condition is sufficient

- Setting $\frac{\partial H}{\partial u}$ to zero, we get $u^*(t) = -R^{-1}B^\top \frac{\partial V}{\partial x}$

- Plugging this back into H , we get the Hamiltonian:

$$H^*\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2}x(t)^\top Qx(t) + \frac{1}{2}\left(R^{-1}B^\top \frac{\partial V}{\partial x}\right)^\top RR^{-1}B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top \left(Ax - BR^{-1}B^\top \frac{\partial V}{\partial x}\right)$$

$$H^*\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2}x(t)^\top Qx(t) + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top BR^{-1}B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top \left(Ax - BR^{-1}B^\top \frac{\partial V}{\partial x}\right)$$

$$H^*\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2}x(t)^\top Qx(t) - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top BR^{-1}B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top Ax$$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}x(t)^\top Qx(t) - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top BR^{-1}B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top Ax = 0, \quad V(x(T), T) = \frac{1}{2}x(T)^\top Lx(T)$$

- Strategy for obtaining solution: guess something that works
 - $V(x, t) = \frac{1}{2}x^\top K(t)x, K(t) \geq 0$

$$\frac{\partial V}{\partial t} = \frac{1}{2}x^\top \dot{K}(t)x, \quad \frac{\partial V}{\partial x} = K(t)x$$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Qx(t) - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top BR^{-1}B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top Ax = 0, \quad V(x(t_f), t_f) = \frac{1}{2}x(t_f)^\top Lx(t_f)$$

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Continuous Time LQR

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Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Qx(t) - \frac{1}{2}x^\top K(t)^\top BR^{-1}B^\top K(t)x + \underbrace{x^\top K(t)^\top Ax + \frac{1}{2}x^\top A^\top K(t)x}_{\text{Scalar, therefore symmetric}} = 0$$

- Collect like terms

$$\frac{1}{2}x^\top \left(\dot{K}(t) + Q - K(t)^\top BR^{-1}B^\top K(t) + K(t)^\top A + A^\top K(t) \right) x = 0$$

- This equation must hold for all $x(t)$, so

$$\dot{K}(t) + Q - K(t)^\top BR^{-1}B^\top K(t) + K(t)^\top A + A^\top K(t) = 0$$

- Boundary condition: $V(x(T), T) = \frac{1}{2}x(T)^\top Lx(T)$

- Therefore $K(T) = L$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Qx(t) - \frac{1}{2}x^\top K(t)^\top BR^{-1}B^\top K(t)x + x^\top K(t)^\top Ax = 0, \quad V(x(t_f), t_f) = \frac{1}{2}x(T)^\top Lx(T)$$

- PDE becomes ODE:

$$\dot{K}(t) + Q - K(t)^\top BR^{-1}B^\top K(t) + K(t)^\top A + A^\top K(t) = 0, \quad K(T) = L$$

- “Riccati equation”
- Integrate backwards in time
- Optimal control is linear state feedback!

$$u^*(t, x) = -R^{-1}B^\top \frac{\partial V}{\partial x} = -R^{-1}B^\top K(t)x$$

Comments

- No control constraint

- What if there is control constraint?

- Easy: Let controllers saturate
- Difficult but proper: Explicitly treat it in the minimization of J

Suppose $|u(t)| \leq 1$ is a constraint

- Use $u(t) = -R^{-1}B^T K(t)x$ if $|-R^{-1}B^T K(t)x| \leq 1$
- Use $u(t) = 1$ if $-R^{-1}B^T K(t)x > 1$
- Use $u(t) = -1$ if $-R^{-1}B^T K(t)x < -1$

- MATLAB commands

- Discrete time: `d1qr`; continuous time: `lqr`

- In general, need to solve

$$\frac{\partial V}{\partial t} + \min_u \left[C(x, u) + \frac{\partial V}{\partial x} \cdot f(x, u) \right] = 0, \quad V(x, T) = l(x)$$

- $V(x, t)$ is $(n + 1)$ -dimensional, if $x \in \mathbb{R}^n$

- Optimal state feedback control: $u^*(t, x) = \arg \min_u \left[C(x, u) + \frac{\partial V}{\partial x} \cdot f(x, u) \right]$

Time Horizon

- Finite time horizon problems
 - Time-varying value function: optimal cost from some time and state
 - $J_k^*(x)$ for discrete time, $V(t, x)$ for continuous time
 - Time-varying control policy: achieves optimal cost
 - $u_k^*(x)$ for discrete time, $u^*(t, x)$ for continuous time
- Infinite time horizon problems
 - Let final time be 0, and apply DP backwards until convergence
 - Convergence not guaranteed; if $J_k^*(x) / V(t, x)$ converges, then we have a time-invariant value function and control policy
 - $J_\infty^*(x), u_\infty^*(x) / V_\infty(x), u_\infty^*(x)$

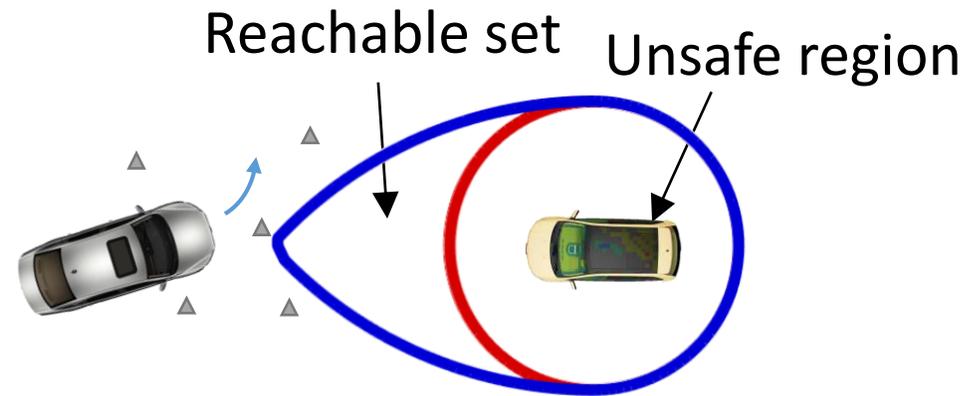
Robotic Safety

- Verification methods



- Considers all possible system behaviours, given assumptions
- Can be written as an optimal control problem

Reachability Analysis: Avoidance



Assumptions:

- Model of robot
- Unsafe region: Obstacle



Control policy

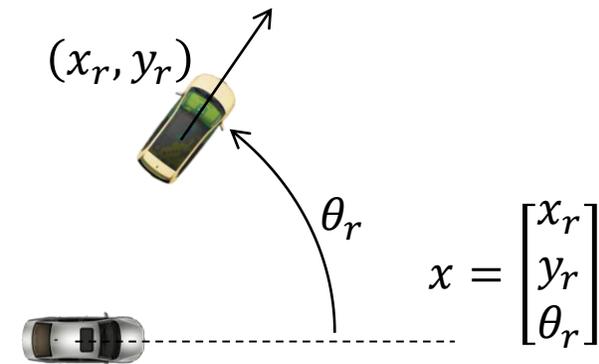
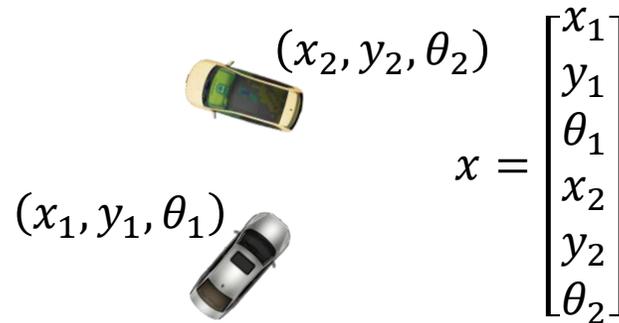
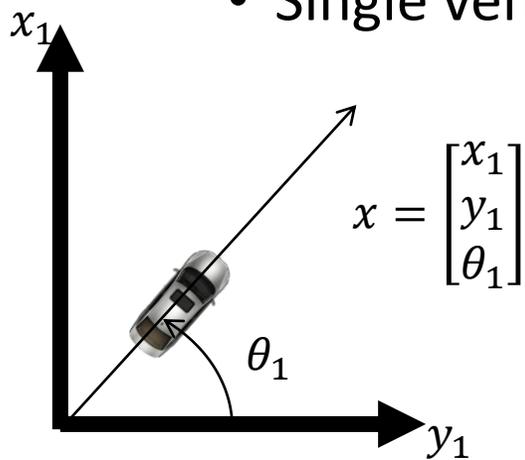
Backward reachable set
(States leading to danger)

Assumptions

- System dynamics: $\dot{x} = f(x, u, d), t \leq 0$ (by convention, final time is 0)

- State x

- Single vehicle, multiple vehicle, relative coordinates

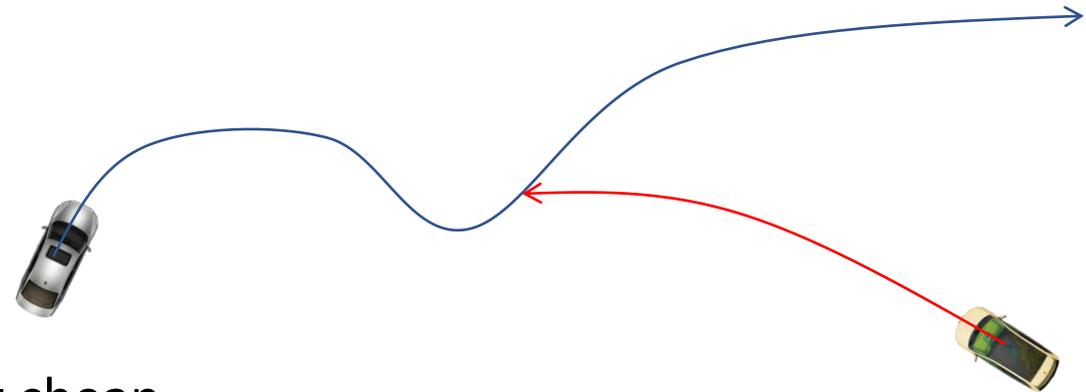


- Disturbance d : uncontrolled factors that affect the system, such as wind
 - Can be used to model other agents, when state includes them
 - Assume worst case

Information Pattern

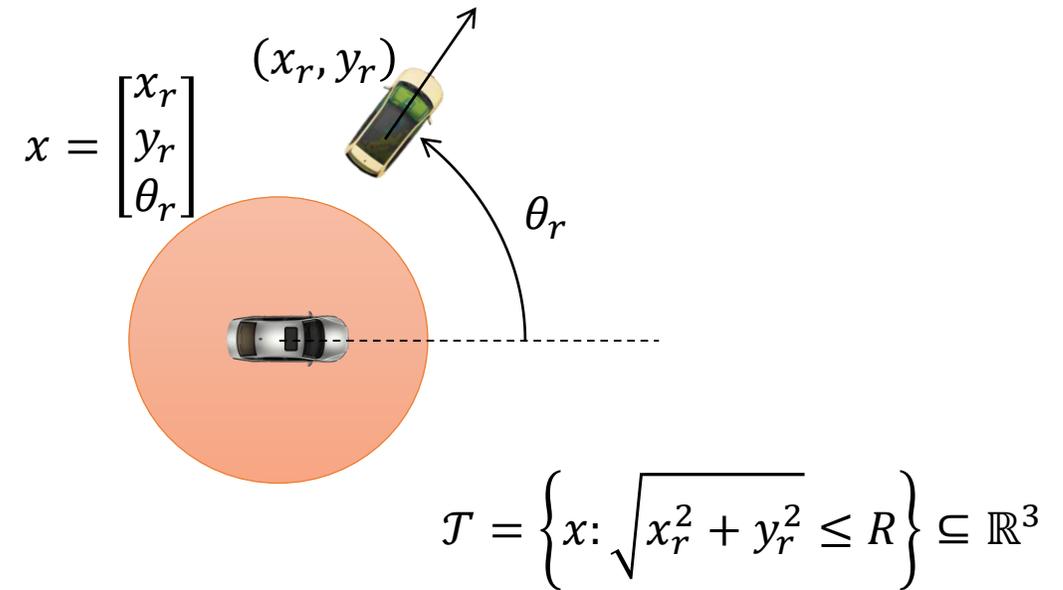
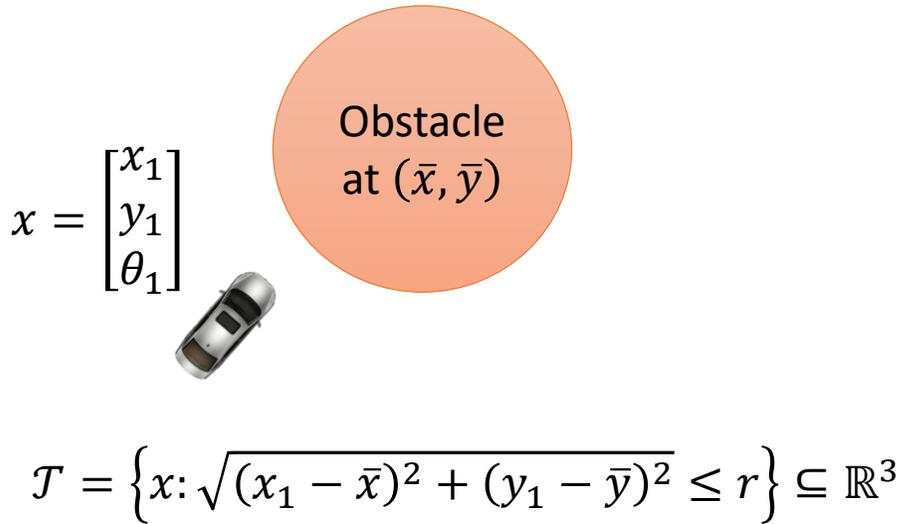


- Control: chosen by “ego” robot
- Disturbances: chosen by other robot (or weather gods)
 - Assume worst case
- “Open-loop” strategies
 - Ego robot declares entire plan
 - Other robot responds optimally (worst-case)
 - Conservative, unrealistic, but computationally cheap
- “Non-anticipative” strategies
 - Other robot acts based on state and control trajectory up current time
 - Notation: $d(\cdot) = \Gamma[u](\cdot)$
 - Disturbance still has the advantage: it gets to react to the control!

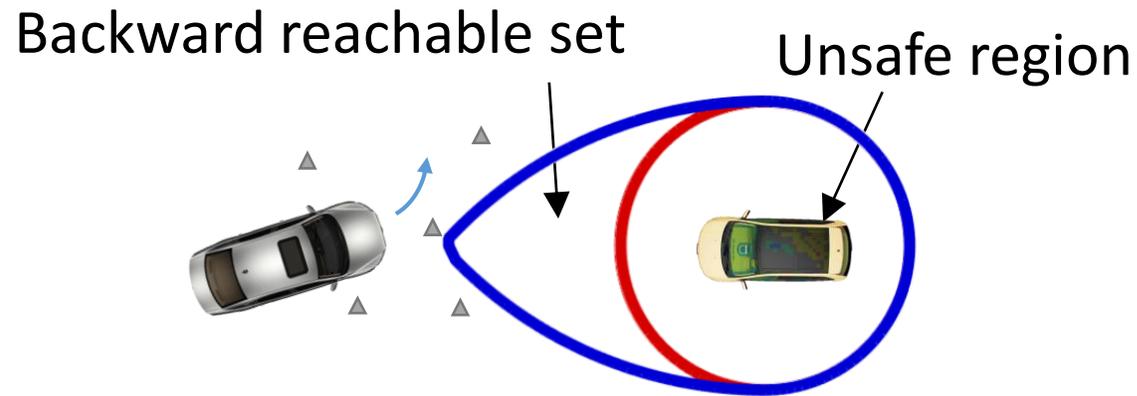


Assumptions

- “Target set”, \mathcal{T}
 - Can specify set of states leading to danger
 - Expressed through set notation



Reachability Analysis: Avoidance



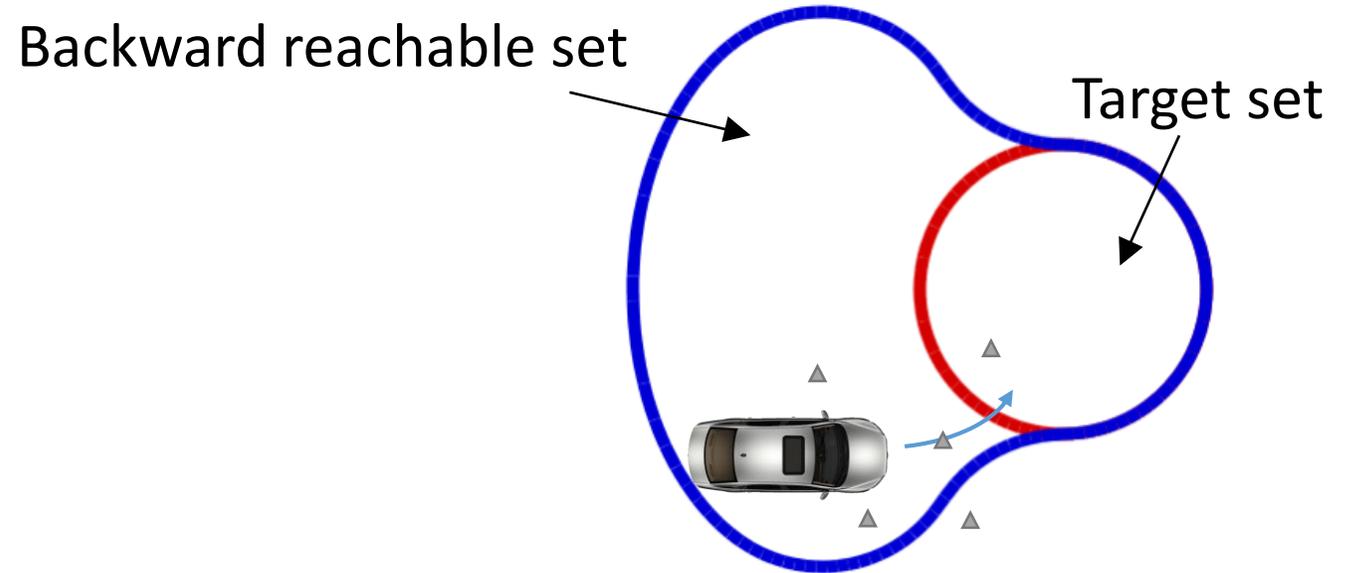
- Model of robot
- Unsafe region



Control policy

Backward reachable set
(States leading to danger)

Reachability Analysis: Goal Reaching



- Model of robot
- Goal region

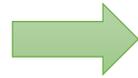


Control policy

Backward reachable set
(States leading to goal)

Reachability Analysis

- Model of robot
- Unsafe region



- $\mathcal{A}(t) = \{\bar{x} : \exists \Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$

Backward reachable set (States leading to danger)

Control policy

- $\dot{x} = f(x, u, d)$
- \mathcal{T}

- $u^*(t, x)$

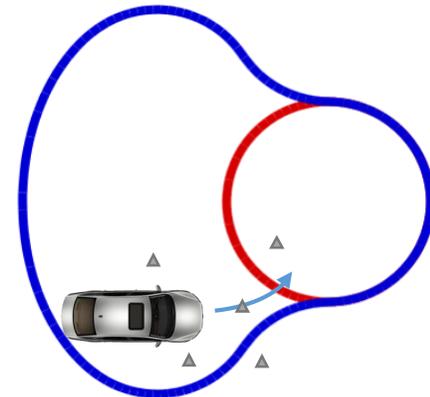
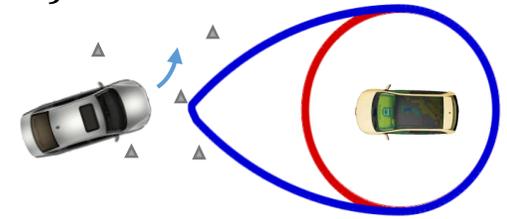
Control policy

- Model of robot
- Goal region



Backward reachable set (States leading to goal)

- $\mathcal{R}(t) = \{\bar{x} : \forall \Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$



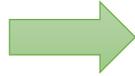
Reachability Analysis

States at time t satisfying the following:

there exists a disturbance such that for all control, system enters target set at $t = 0$

$$\mathcal{A}(t) = \{\bar{x} : \exists \Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$$

- Model of robot
- Unsafe region



Backward reachable set (States leading to danger)

Control policy

- $\dot{x} = f(x, u, d)$
- \mathcal{T}

$$u^*(t, x)$$

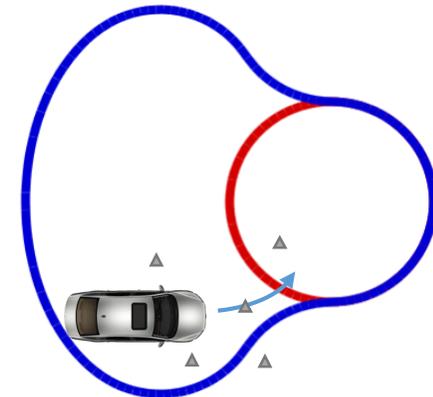
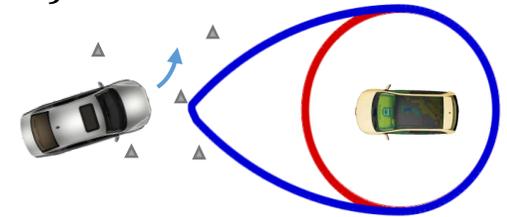
- Model of robot
- Goal region



Control policy

Backward reachable set (States leading to goal)

$$\mathcal{R}(t) = \{\bar{x} : \forall \Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$$



States at time t satisfying the following:

for all disturbances, there exists a control such that system enters target set at $t = 0$

Terminology

- Minimal backward reachable set

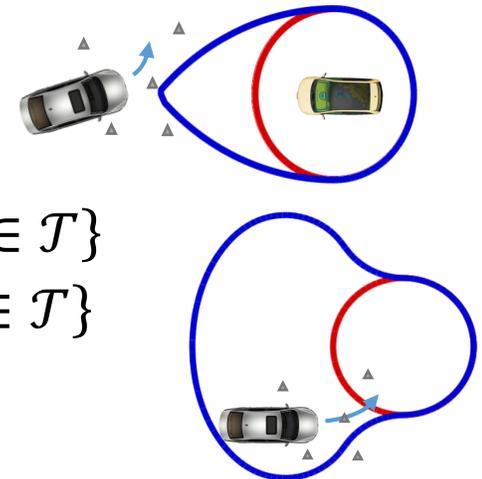
- $\mathcal{A}(t) = \{\bar{x}: \exists \Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$
- Control minimizes size of reachable set

- Maximal backward reachable set

- $\mathcal{R}(t) = \{\bar{x}: \forall \Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$
- Control maximizes size of reachable set

- Minimal and maximal backward reachable tube

- $\bar{\mathcal{A}}(t) = \{\bar{x}: \exists \Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, \exists s \in [t, 0], x(s) \in \mathcal{T}\}$
- $\bar{\mathcal{R}}(t) = \{\bar{x}: \forall \Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x, u, d), x(t) = \bar{x}, \exists s \in [t, 0], x(s) \in \mathcal{T}\}$



Computing Reachable Sets

- Start from continuous time dynamic programming
- Observe that disturbances do not affect the procedure
- Remove running cost
- Pick final cost intelligently