Project Proposal

- Due Feb. 18: two paragraphs

- Project options
  - Thoroughly understand and critically evaluate 3 to 5 papers in a course topic
  - Reproduce the results of 1 to 2 papers in a course topic, and suggest or make improvements
  - Mini Research project related to a course topic
  - Other: please consult with instructor

- Course topics:
  - Dynamical systems
  - Nonlinear optimization
  - Optimal control (we are here)
  - Machine learning in robotics (eg. computer vision in robotics, reinforcement learning)
  - Localization and mapping

Optimal Control Part III

CMPT 882

Feb. 12

Outline

- Open-loop control: Numerical solutions
  - Single shooting
  - Multiple shooting
  - Collocation

Last Time: Single Shooting

$$\begin{aligned}
\underset{u(\cdot)}{\text{minimize}} \quad & l\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} c(x(t), u(t), t)dt \\
\text{subject to} \quad & \dot{x} = f(x, u) \\
& g\big(x(t), u(t)\big) \geq 0, \qquad t \in \big[t_0, t_f\big] \\
\text{where} \quad & x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, x(t_0) = x_0
\end{aligned}$$

- Discretized problem:

$$\begin{aligned}
\underset{q}{\text{minimize}} \quad & l(x(t_N), t_N) + \sum_{i=0}^{N-1} c(x(t_i), q_i, t_i)(t_{i+1} - t_i) \\
\text{subject to} \quad & \forall i \in \{0,1, \dots, N-1\}, \\
& \quad x(t_{i+1}) = x(t_i) + f(x(t_i), q_i)(t_{i+1} - t_i) \\
& \quad g(x(t_i), q_i) \geq 0
\end{aligned}$$

Last Time: Single Shooting

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- Variations: Different numerical schemes
    - For ODE constraint
    - For cost function
- Main disadvantage
    - Integration error
    - Errors in “earlier” controls can greatly affect final state
    - Initial guess matters a lot

Multiple Shooting

$$
\begin{aligned}
&\underset{q}{\text{minimize}} && l(\boldsymbol{x}(\boldsymbol{t_N}), t_N) + \sum_{i=0}^{N-1} c(\boldsymbol{x}(\boldsymbol{t_i}), q_i, t_i)(t_{i+1} - t_i) \\
&\text{subject to} && \forall i \in \{0,1,\dots,N-1\}, \\
& && \quad \boldsymbol{x}(\boldsymbol{t_{i+1}}) = \boldsymbol{x}(\boldsymbol{t_i}) + f(\boldsymbol{x}(\boldsymbol{t_i}), q_i)(t_{i+1} - t_i) \\
& && \quad g(\boldsymbol{x}(\boldsymbol{t_i}), q_i) \geq 0
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\underset{\boldsymbol{s},q}{\text{minimize}} && h(\boldsymbol{s_N}, t_N) + \sum_{i=0}^{N-1} c(\boldsymbol{s_i}, q_i, t_i)(t_{i+1} - t_i) \\
&\text{subject to} && \forall i \in \{0,1,\dots,N-1\}, \\
& && \quad \boldsymbol{s_{i+1}} = \boldsymbol{s_i} + f(\boldsymbol{s_i}, q_i)(t_{i+1} - t_i) \\
& && \quad g(\boldsymbol{s_i}, q_i) \geq 0
\end{aligned}
$$

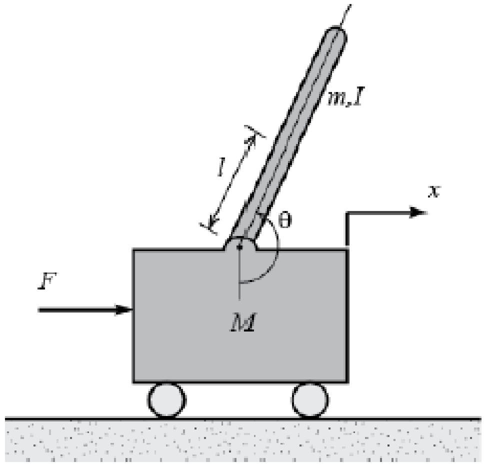
Multiple Shooting

- Discretized problem:

$$\begin{aligned} \underset{s,q}{\text{minimize}} \quad & h(s_N, t_N) + \sum_{i=0}^{N-1} c(s_i, q_i, t_i)(t_{i+1} - t_i) \\ \text{subject to} \quad & \forall i \in \{0,1,\dots,N-1\}, \\ & \quad s_{i+1} = s_i + f(s_i, q_i)(t_{i+1} - t_i) \\ & \quad g(s_i, q_i) \geq 0 \end{aligned}$$

- Same variations as single shooting available (numerical schemes)
- State is now a decision variable
  - State constraints do not necessarily need to be satisfied throughout optimization process
  - Improves numerical stability
  - Reduces integration error

Inverted Pole on Cart



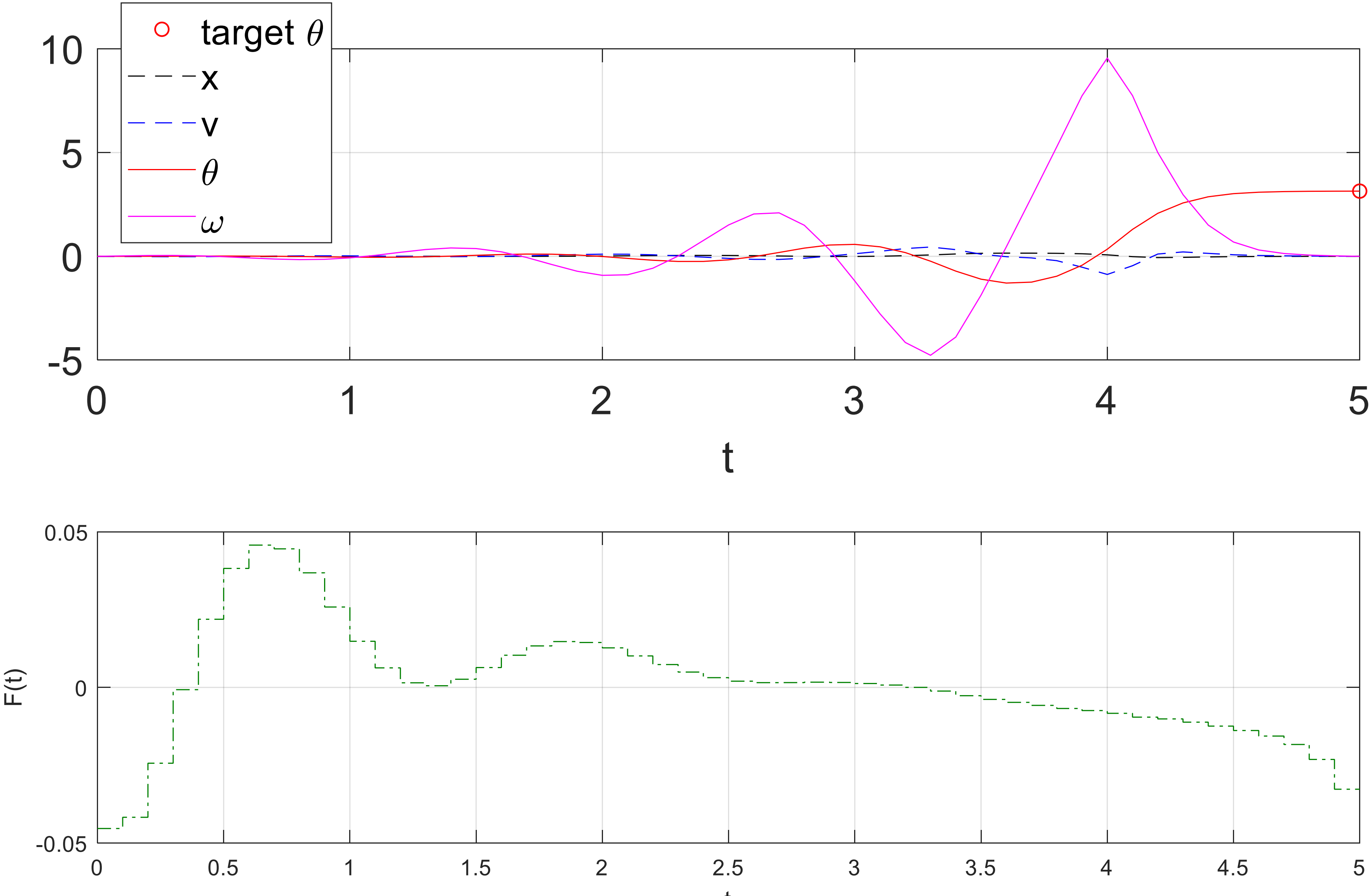
- State: (x, v, θ, ω)
  - Position, speed, angle of pole, angular speed of pole
- Equations of motion:

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$$

$$(I+ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$$

  - Parameters: M, m, l, I, b, g – mass of cart and pole, length and moment of inertial of pole, friction coefficient, acceleration due to gravity
  - Control: F – force of pushing
- Constraints:
  - Start from initial state $(0,0,0,0)$, reach final state $(0,0,\pi,0)$ at time T
  - Maximum force limit
- Cost: Control effort: $\int_0^T F^2(t)dt$

Inverted Pole on Cart



Direct Collocation

$$\begin{aligned}\underset{s,q}{\text{minimize}} \quad & h(s_N, t_N) + \sum_{i=0}^{N-1} c(s_i, q_i, t_i)(t_{i+1} - t_i) \\ \text{subject to} \quad & \forall i \in \{0,1,\dots,N-1\}, \\ & \cancel{s_{i+1} = s_i + f(s_i, q_i)(t_{i+1} - t_i)} \\ & g(s_i, q_i) \geq 0\end{aligned}$$

- No numerical integration
- Directly approximates $x(t)$ and $u(t)$
  - **Piecewise: eg. Hermite-Simpson method**
  - Global: eg. Pseudospectral methods
- Impose dynamics constraints at discrete time points (“collocation points”)

Hermite-Simpson Collocation

x_i and u_i are decision variables

- Discretize time: $$t_0 < t_1 < \cdots < t_N \coloneqq t_f, \qquad h \coloneqq t_{i+1} - t_i$$ $$x_i \coloneqq x(t_i), \qquad u_i = u(t_i)$$
- (Assume scalar x for now, and) write $$x(t) = b_{i,0} + b_{i,1}(t - t_i) + b_{i,2}(t - t_i)^2 + b_{i,3}(t - t_i)^3, \qquad t \in [t_i, t_{i+1}]$$ $$\dot{x}(t) = b_{i,1} + 2b_{i,2}(t - t_i) + 3b_{i,3}(t - t_i)^2, \qquad t \in [t_i, t_{i+1}]$$

1. Some algebra: At t_i and t_{i+1}:

$$\begin{bmatrix} x(t_i) \\ \dot{x}(t_i) \\ x(t_{i+1}) \\ \dot{x}(t_{i+1}) \end{bmatrix} = \begin{bmatrix} x_i \\ f(x_i, u_i) \\ x_{i+1} \\ f(x_{i+1}, u_{i+1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & h & h^2 & h^3 \\ 0 & 1 & 2h & 3h^2 \end{bmatrix} \begin{bmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \\ b_{i,3} \end{bmatrix}$$

- Obtain coefficients in terms of decision variables by taking inverse

$$\begin{bmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \\ b_{i,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/h^2 & -2/h & 3/h^2 & -1/h \\ 2/h^3 & 1/h^2 & -2/h^3 & 1/h^2 \end{bmatrix} \begin{bmatrix} x_i \\ f(x_i, u_i) \\ x_{i+1} \\ f(x_{i+1}, u_{i+1}) \end{bmatrix}$$

Dynamics Constraint

2. Choice of collocation points:

$$t_{i,c} = \frac{t_i + t_{i+1}}{2}$$

$$u_{i,c} := \frac{u_{i+1} + u_i}{2}$$

- Plug in $t_{i,c}$:

$$x_{i,c} = b_{i,0} + b_{i,1}\left(t_{i,c} - t_i\right) + b_{i,2}\left(t_{i,c} - t_i\right)^2 + b_{i,3}\left(t_{i,c} - t_i\right)^3$$

$$\dot{x}_{i,c} = b_{i,1} + 2b_{i,2}\left(t_{i,c} - t_i\right) + 3b_{i,3}\left(t_{i,c} - t_i\right)^2$$

3. Dynamics constraint at collocation points:

$$\dot{x}_{i,c} - f\left(x_{i,c}, u_{i,c}\right) = 0$$

- $x_{i,c}, \dot{x}_{i,c}$ depend on $b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}$
- $b_{i,0}, b_{i,1}, b_{i,2}, b_{i,3}$ depend on $x_i, x_{i+1}, u_i, u_{i+1}$
- $u_{i,c}$ depends on u_i, u_{i+1}

$$\boxed{\begin{bmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \\ b_{i,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/h^2 & -2/h & 3/h^2 & -1/h \\ 2/h^3 & 1/h^2 & -2/h^3 & 1/h^2 \end{bmatrix} \begin{bmatrix} x_i \\ f(x_i, u_i) \\ x_{i+1} \\ f(x_{i+1}, u_{i+1}) \end{bmatrix}}$$

Hermite-Simpson collocation

- Optimization problem, with simple integration

$$\underset{\{x_i\}_{i=1}^{N},\{u_i\}_{i=1}^{N-1}}{\text{minimize}} \quad h(x_N, t_N) + \sum_{i=0}^{N-1} c(x_i, u_i, t_i)(t_{i+1} - t_i)$$

subject to $\quad \forall i \in \{0,1,\dots,N-1\},$

$$\dot{x}_{i,c} - f\left(x_{i,c}, u_{i,c}\right) = 0$$

$$g(x_i, u_i) \geq 0$$

$$x_{i,c} = b_{i,0} + b_{i,1}\left(t_{i,c} - t_i\right) + b_{i,2}\left(t_{i,c} - t_i\right)^2 + b_{i,3}\left(t_{i,c} - t_i\right)^3$$

$$\dot{x}_{i,c} = b_{i,1} + 2b_{i,2}\left(t_{i,c} - t_i\right) + 3b_{i,3}\left(t_{i,c} - t_i\right)^2$$

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$$u_{i,c} = \frac{u_{i+1} + u_i}{2}$$

Hermite-Simpson collocation

- Optimization problem, with simple integration

$$\underset{\{x_i\}_{i=1}^{N},\{u_i\}_{i=1}^{N-1}}{\text{minimize}} \quad h(x_N, t_N) + \sum_{i=0}^{N-1} c(x_i, u_i, t_i)(t_{i+1} - t_i)$$

$$\text{subject to} \quad \forall i \in \{0,1,\dots,N-1\},$$

$$\boldsymbol{\dot{x}_{i,c} - f\left(x_{i,c}, u_{i,c}\right) = 0}$$

$$g(x_i, u_i) \geq 0$$

- Key difference from shooting methods
  - Dynamics constraint: no numerical integration

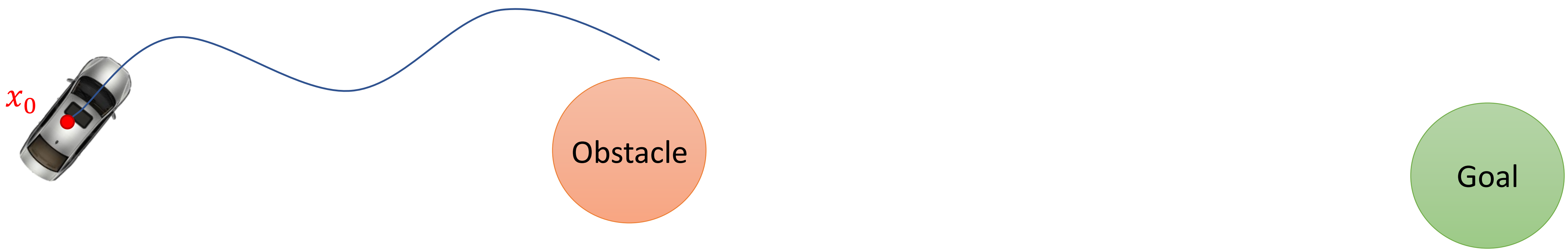
Pseudospectral Methods

- Represent entire state trajectory as sum of weighted basis functions
  - Chebyshev polynomials, Legendre polynomials, etc.
- Pros:
  - Fewer decision variables
  - Numerically more accurate
- Cons:
  - Dense optimization problems

Receding Horizon Control

$$\begin{aligned} \underset{q}{\text{minimize}} \quad & l(x(t_N), t_N) + \sum_{i=0}^{N-1} c(x(t_i), q_i, t_i)(t_{i+1} - t_i) \\ \text{subject to} \quad & \forall i \in \{0,1,\ldots,N-1\}, \\ & \quad x(t_{i+1}) = x(t_i) + f(x(t_i), q_i)(t_{i+1} - t_i) \\ & \quad g(x(t_i), q_i) \geq 0 \end{aligned}$$

- Start from x_0, initial state; solve optimization
  - q provides control from time steps 0 to $N-1$ ← not necessary a long time horizon
  - Apply control only at time step 0



Receding Horizon Control

$$\underset{q}{\text{minimize}} \quad l(x(t_N), t_N) + \sum_{i=0}^{N-1} c(x(t_i), q_i, t_i)(t_{i+1} - t_i)$$
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$$g(x(t_i), q_i) \geq 0$$

- Start from x_0, initial state; solve optimization
  - q provides control from time steps 0 to $N-1$ ← not necessary a long time horizon
  - Apply control only at time step 0
- Now, the state is at $x(t_{i+1})$; re-solve the optimization
  - Obtain control from time steps $i+1$ to $i+N$

Receding Horizon Control

$$\begin{array}{ll} \underset{q}{\text{minimize}} & l(x(t_N), t_N) + \sum_{i=0}^{N-1} c(x(t_i), q_i, t_i)(t_{i+1} - t_i) \\ \text{subject to} & \forall i \in \{0,1,\ldots,N-1\}, \\ & \quad x(t_{i+1}) = x(t_i) + f(x(t_i), q_i)(t_{i+1} - t_i) \\ & \quad g(x(t_i), q_i) \geq 0 \end{array}$$

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  - Apply control at time step $i+1$

Receding Horizon Control

$$\begin{aligned}
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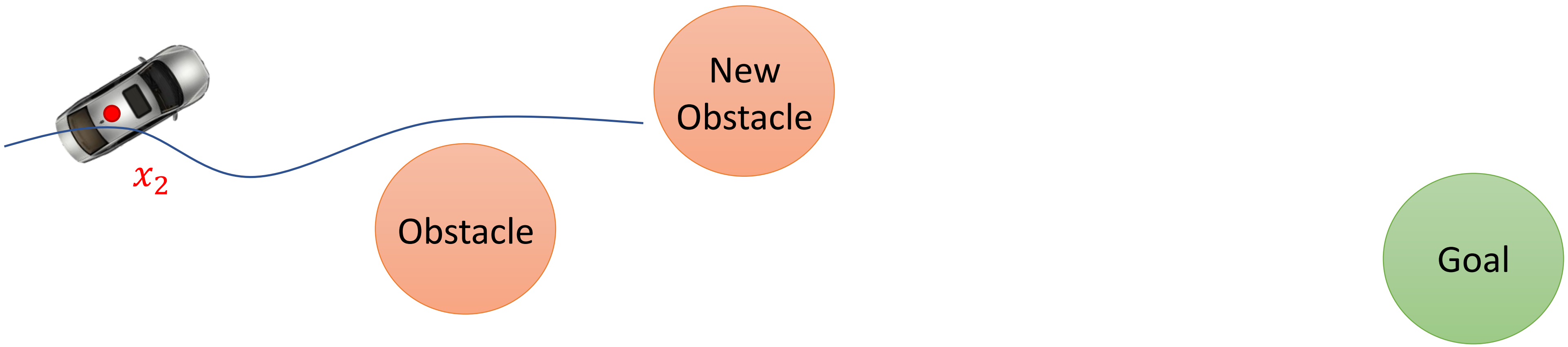
  - Apply control only at time step 0
- Now, the state is at $x(t_{i+1})$; re-solve the optimization
  - Obtain control from time steps $i+1$ to $i+N$
  - Apply control at time step $i+1$
  - Repeat



Receding Horizon Control

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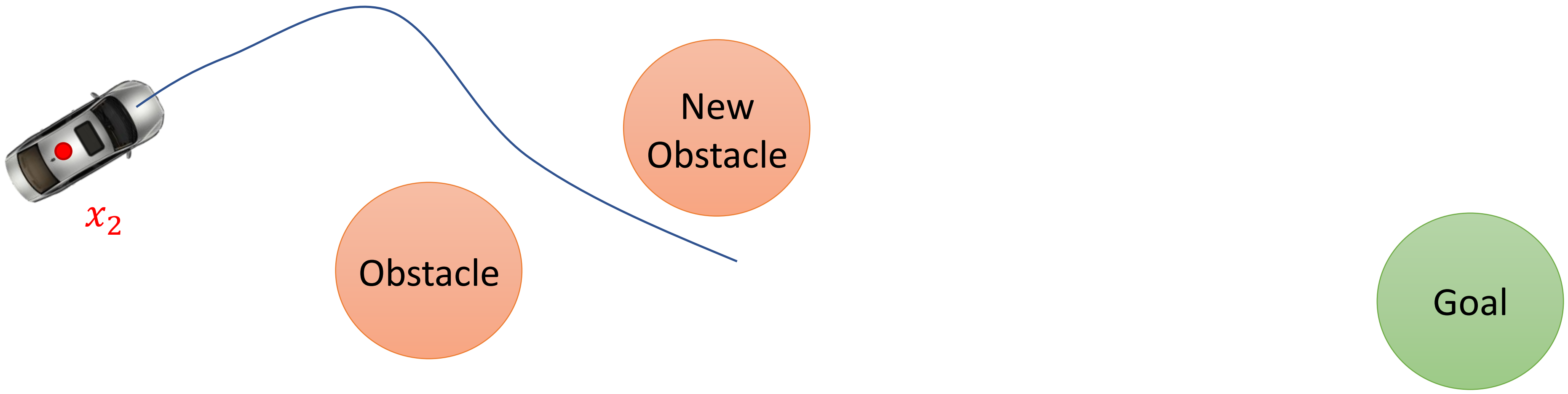

Receding Horizon Control

- Main requirement
  - Computation must be fast enough compared to re-planning frequency
  - Re-planning frequency varies greatly depending on application
    - Agile mobile vehicles: milliseconds to a second
    - Building temperature control: minutes to hours

- Theoretical considerations
  - Recursive feasibility: feasible first optimization problem $\Rightarrow$ feasible kth optimization problem
  - Performance guarantees: eg. goal satisfaction

- Special popular case
  - Model-predictive control: uses a model of the system

Optimal Control

- Open-loop solutions
  - Differential flatness
  - Shooting methods
  - Collocation
- Receding horizon control:
  - Apply first part of the open-loop solution
  - Resolve open-loop optimization
- Relevant software packages
  - Optimization: cvx, Gurobi, SeDuMi, Mosek, Cplex, Matlab (`fmincon`)
  - Shooting/collocation: casadi, ACADO, Matlab `bvp4c` (and similar)
  - Receding horizon control: ACADO, Matlab (MPC toolbox)