Convex Optimization: Part I

CMPT 882

Jan. 30

Outline

- Optimization program
  - Examples and classes
- Convex optimization
  - Convex functions
  - Optimality conditions
- Numerical solutions
- cvx software

Optimization Program: Terminology

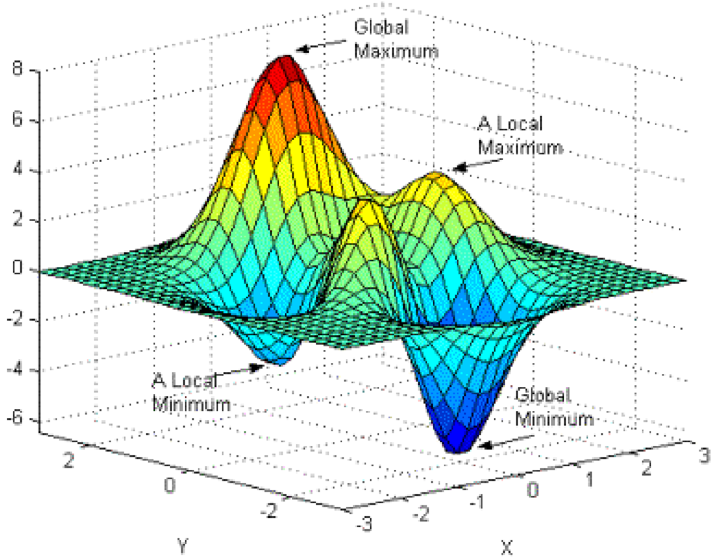
$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, i = 1, \ldots, n \\ & h_j(x) = 0, j = 1, \ldots, m \end{array}$$

Objective function

Inequality constraints

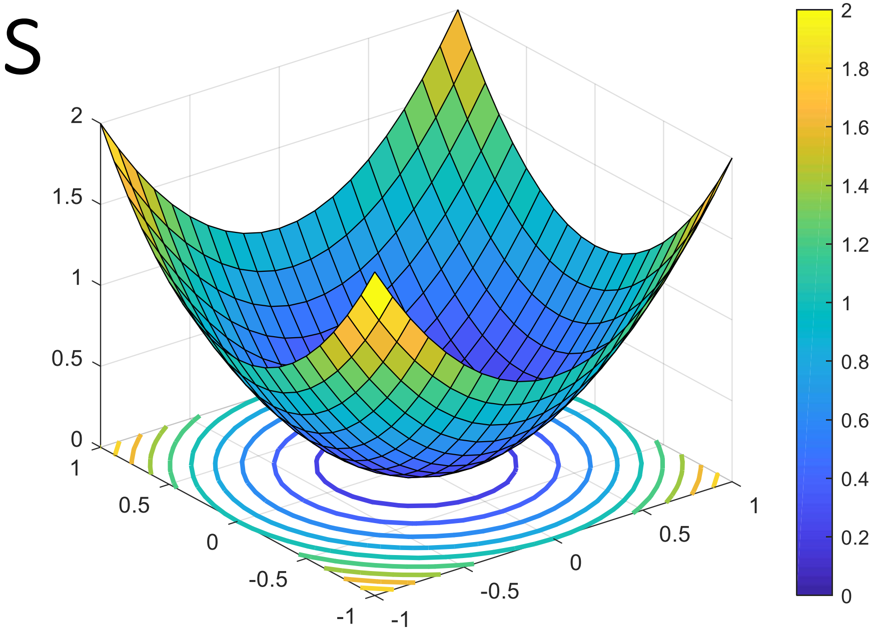
Equality constraints

- In this class, assume f, g_i, h_j are twice differentiable
- Look for an **optimal solution**, the vector x^*
  - **Locally optimal**: x^* is a local minimum of $f(x)$
  - **Globally optimal**: x^* is a global minimum of $f(x)$



Optimization Program: Examples

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, i = 1, \dots, n \\ & h_j(x) = 0, j = 1, \dots, m \end{aligned}$$

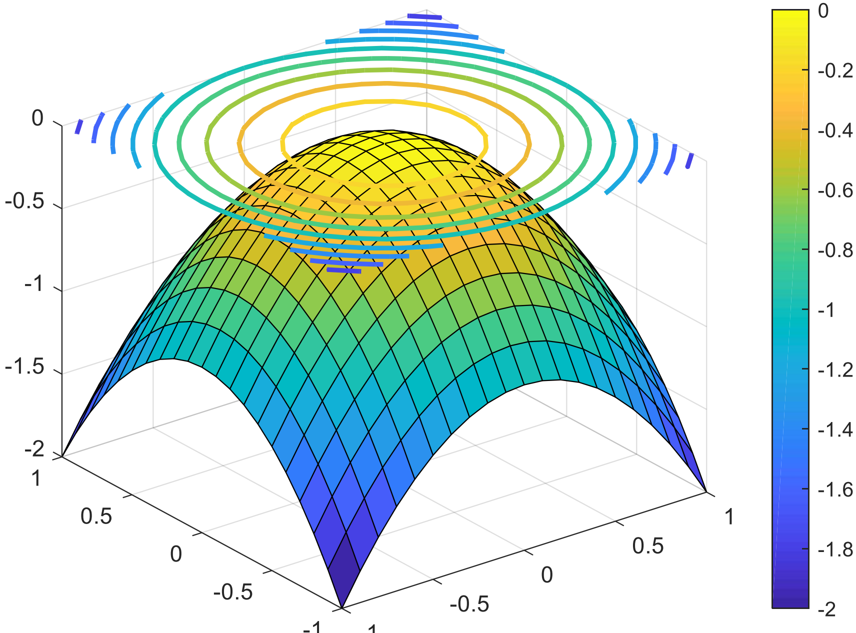


- Applications: Portfolio management

maximize Expected profit

subject to Maximum budget
Maximum acceptable risk

$\min f(x) = \max\{-f(x)\}$



Optimization Program: Examples

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \dots, n \\ & h_j(x) = 0, j = 1, \dots, m \end{aligned}$$

- Applications: Portfolio management

minimize Overall risk

subject to Maximum budget
Minimum acceptable expected profit

Constraints vs. objectives

- Sometimes constraints can be "moved" to the objective as a "penalty"

Optimization Program: Examples

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \ldots, n \\ & h_j(x) = 0, j = 1, \ldots, m \end{aligned}$$

- Applications: Building heating, ventilation, and air conditioning

minimize Energy consumption

subject to Acceptable temperature range by location
Acceptable noise level
Internal and external heat transfer

Optimization Program: Examples

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \ldots, n \\ & h_j(x) = 0, j = 1, \ldots, m \end{aligned}$$

- Applications: Robotic trajectory planning

$$\begin{aligned} \text{minimize} \quad & \text{Fuel consumption} \\ \text{subject to} \quad & \text{Goal reaching} \\ & \text{System dynamics} \\ & \text{Collision avoidance} \end{aligned}$$

Optimization Program: Examples

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \dots, n \\ & h_j(x) = 0, j = 1, \dots, m \end{aligned}$$

- Applications: Robotic trajectory planning

minimize Distance to goal

subject to Fuel limitations
System dynamics
Collision avoidance

Optimization Program: Examples

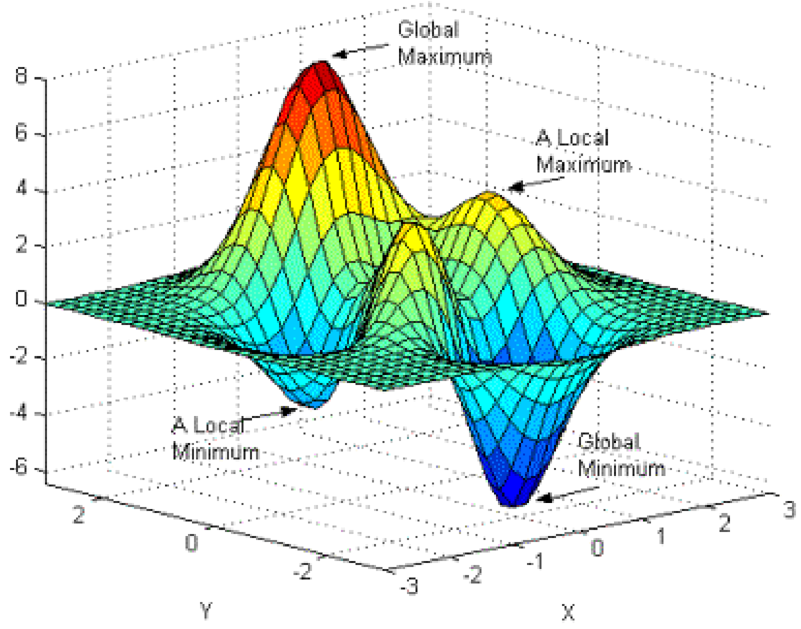
$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, i = 1, \dots, n \\ & h_j(x) = 0, j = 1, \dots, m \end{aligned}$$

- Applications: Machine learning

maximize Performance (eg. Accuracy of object recognition)

subject to Problem constraints

Optimization Program

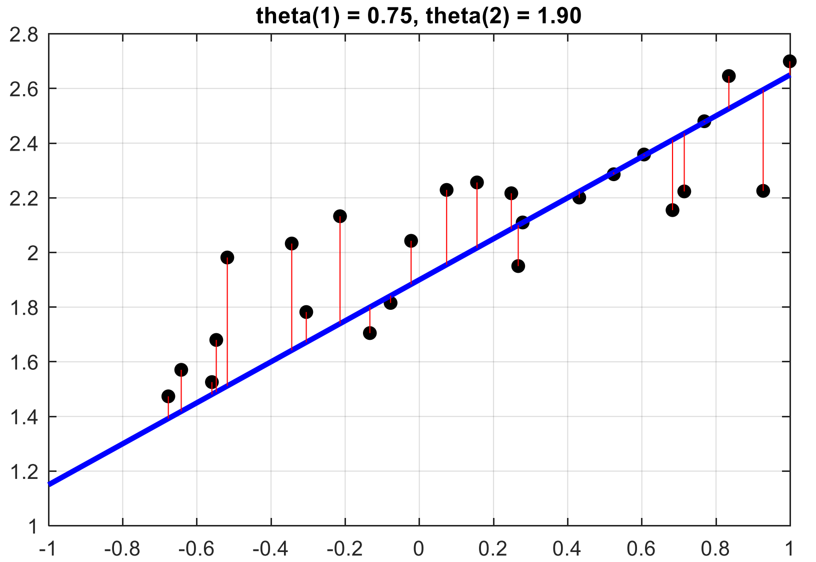


$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \dots, n \\ & h_j(x) = 0, j = 1, \dots, m \end{aligned}$$

- Very difficult to solve in general
  - Trade-offs to consider: computation time, solution optimality

- Easy cases:
  - Find global optimum for **linear program**: f, g_i, h_j are linear
  - Find global optimum for **convex program**: f, g_i are convex, h_j is linear
  - Find local optimum for **nonlinear program**: f, g_i, h_j are differentiable

Example: Least Squares



$$\underset{\theta}{\text{minimize}} \|X\theta - Y\|_2^2$$

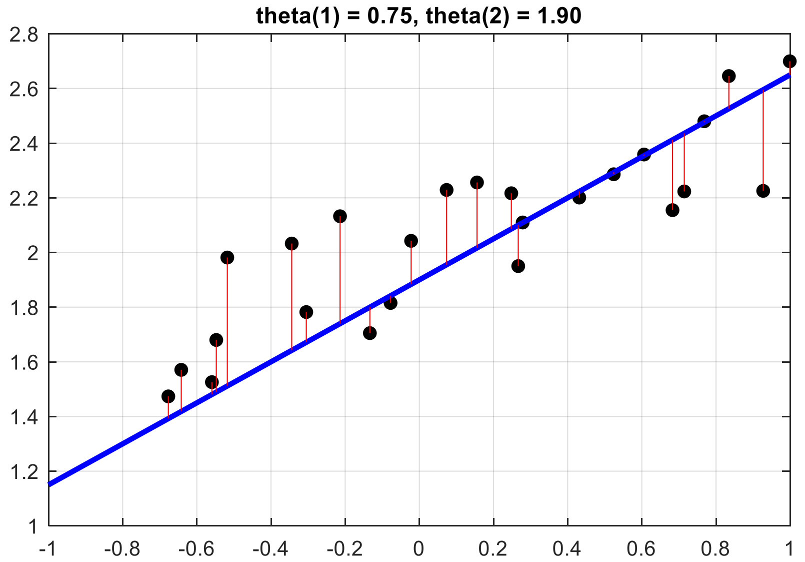
- Scalar example:
  - Data: $\{x_i, y_i\}_{i=1}^n, x_i, y_i \in \mathbb{R}$
  - Model: $y = mx + b, m, b \in \mathbb{R}$
  - Sum of error of model: $\sum_{i=1}^n (y_i - mx_i - b)^2$
  - No constraints: allow *any* m, b
- Error in matrix form: $e_i = y_i - \begin{bmatrix} x_i & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$
  - Stacking the data points: $E_i = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{Y} - \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\theta}$

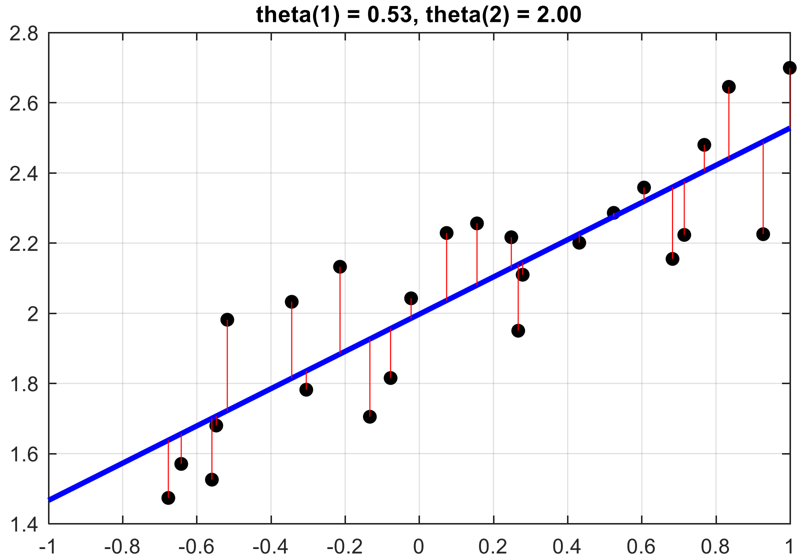
Example: Least Squares

$$\underset{\theta}{\text{minimize}} \|X\theta - Y\|_2^2$$

- Analytic solution available!
  - Objective: $f(\theta) = \|X\theta - Y\|_2^2$, set derivative to zero
  - $f(\theta) = (X\theta - Y)^\top (X\theta - Y)$
  - $f(\theta) = \theta^\top X^\top X\theta - 2Y^\top X\theta + Y^\top Y$

$$\frac{\partial f}{\partial \theta} = 2X^\top X\theta - 2X^\top Y$$
$$0 = 2X^\top X\theta - 2X^\top Y$$
$$X^\top Y = X^\top X\theta$$
$$x^* = (X^\top X)^{-1} X^\top Y$$





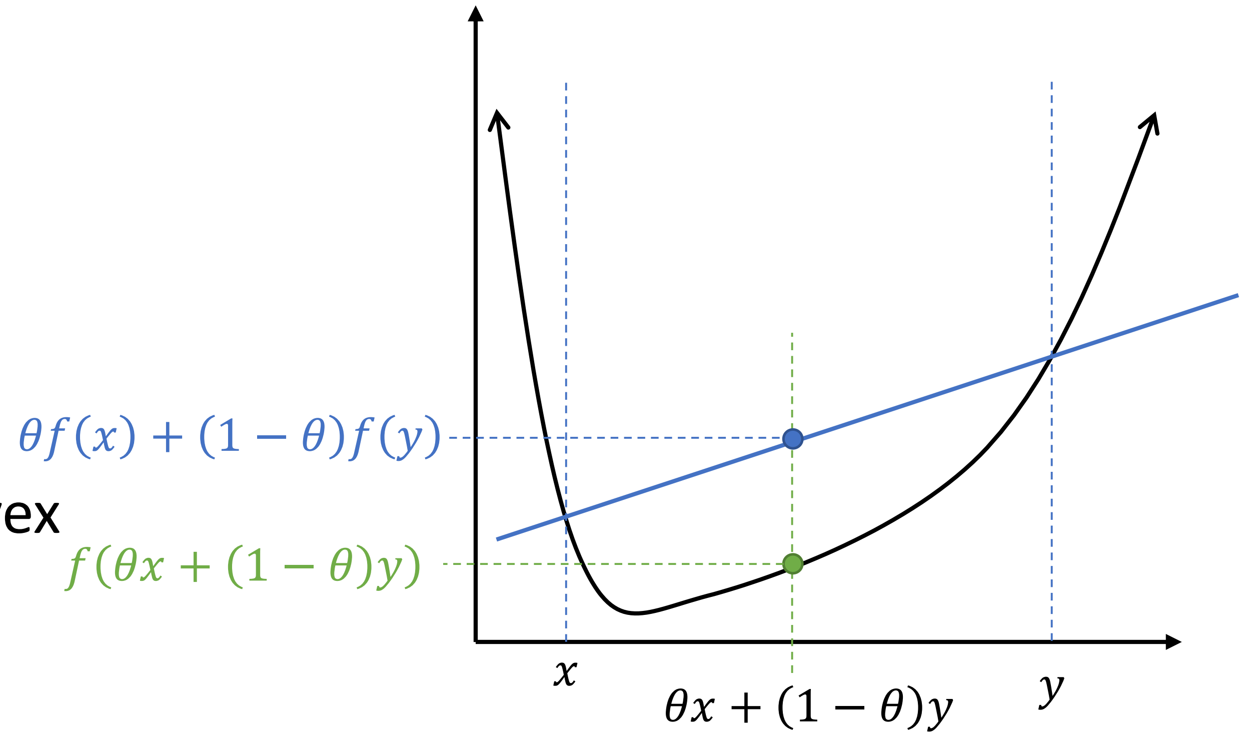
Convex Programs

minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, \ldots, n,$

where $g_i(x)$ are convex

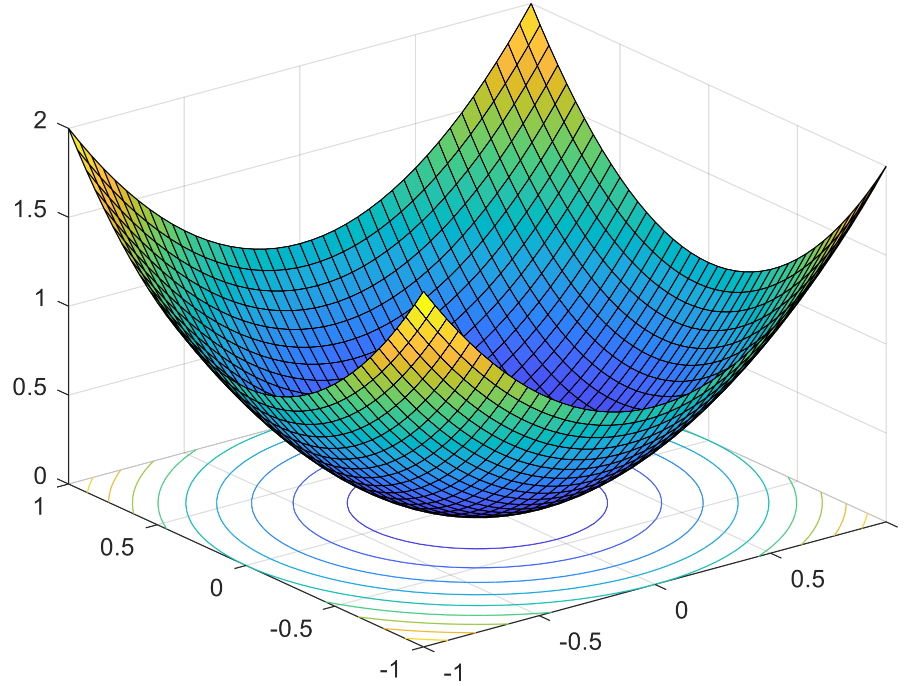
$h_j^\top x = 0, j = 1, \ldots, m$



- **Convex function**

  $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$ for all $x, y \in \mathbb{R}^n$,
  for all $\theta \in [0,1]$

Convex Programs



$$\text{minimize} \quad f(x)$$

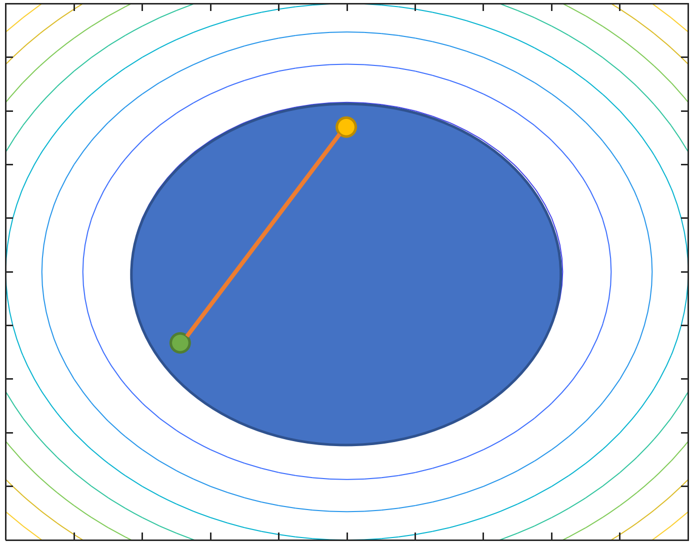
$$\text{subject to} \quad g_i(x) \leq 0, i = 1, \ldots, n,$$

where $g_i(x)$ are convex

$$h_j^\top x = 0, j = 1, \ldots, m$$

- **Convex function**
  $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$ for all $x, y \in \mathbb{R}^n$,
  for all $\theta \in [0,1]$

- Sublevel sets of convex functions, $\{x: f(x) \leq C\}$, are convex
  - **Convex shape** $\mathcal{C}$:
    $x_1, x_2 \in \mathcal{C}, \theta \in [0,1] \Rightarrow \theta x_1 + (1-\theta) x_2 \in \mathcal{C}$
  - *Super*level sets of convex functions are *not* convex!



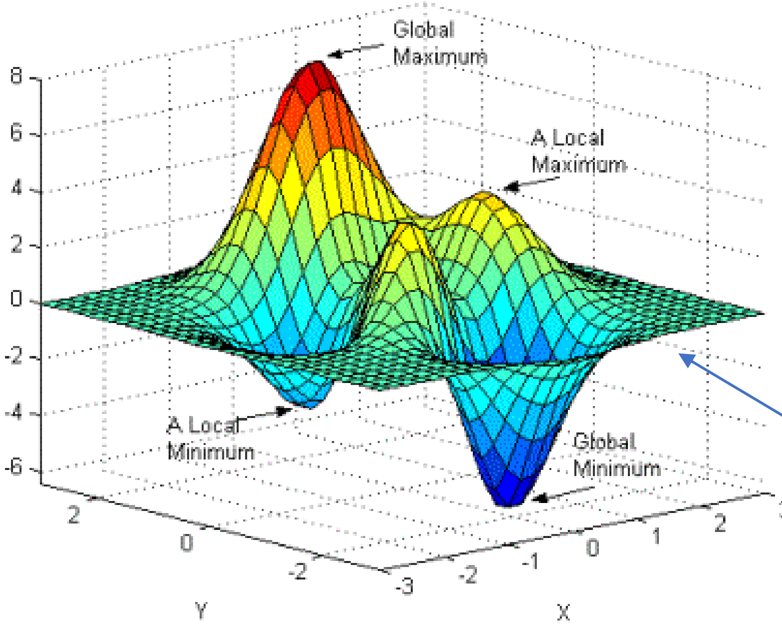
Convex Programs

minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, \dots, n,$

where $g_i(x)$ are convex

$h_j^{\top} x = 0, j = 1, \dots, m$



Not convex

- **Convex function**
  $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$ for all $x, y \in \mathbb{R}^n$,
  for all $\theta \in [0,1]$

- Sublevel sets of convex functions, $\{x: f(x) \leq C\}$, are convex
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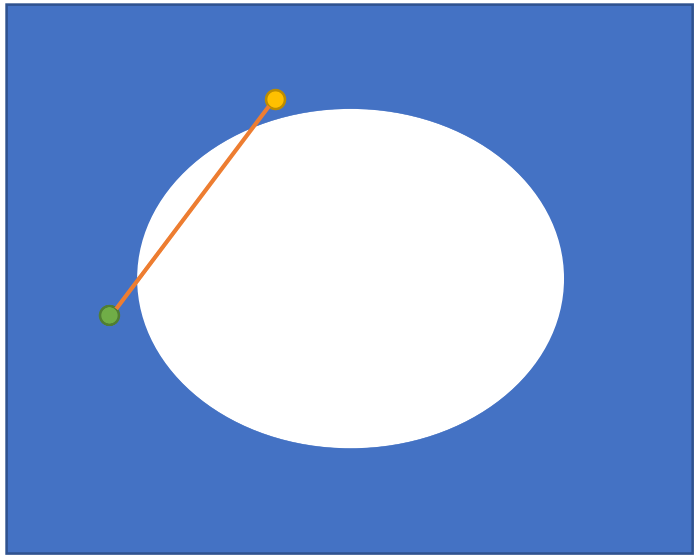


Convex Programs

minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, \dots, n,$

where $g_i(x)$ are convex

$h_j^\top x = 0, j = 1, \dots, m$

minimize A convex objective function

subject to Convex inequality constraints

Linear equality constraints

Detailed observations:

- Linear functions are convex
- Any equality constraints must be linear
  - $h(x) = 0 \Leftrightarrow h(x) \geq 0$ AND $h(x) \leq 0$

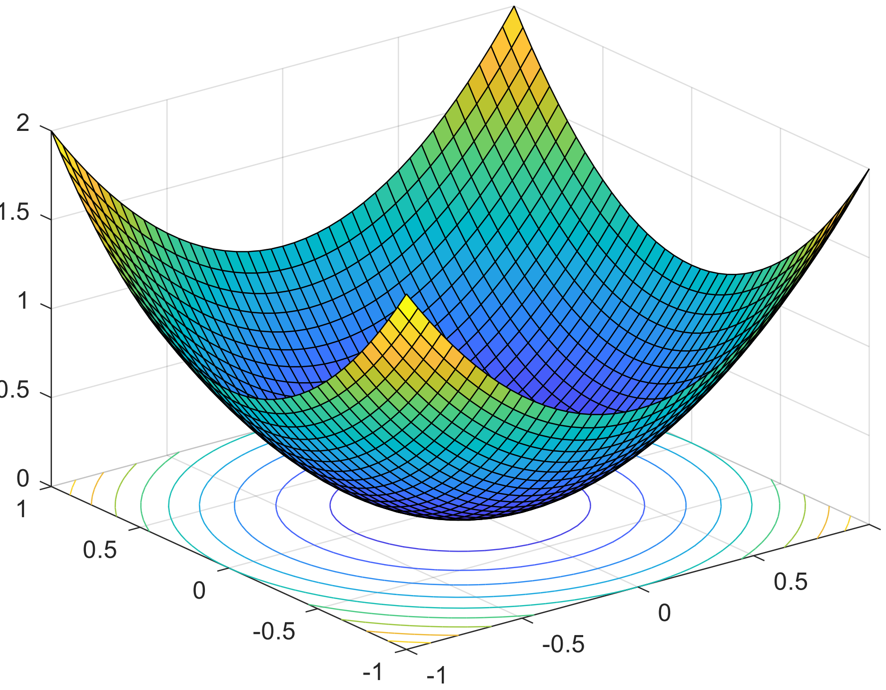
Convex Programs

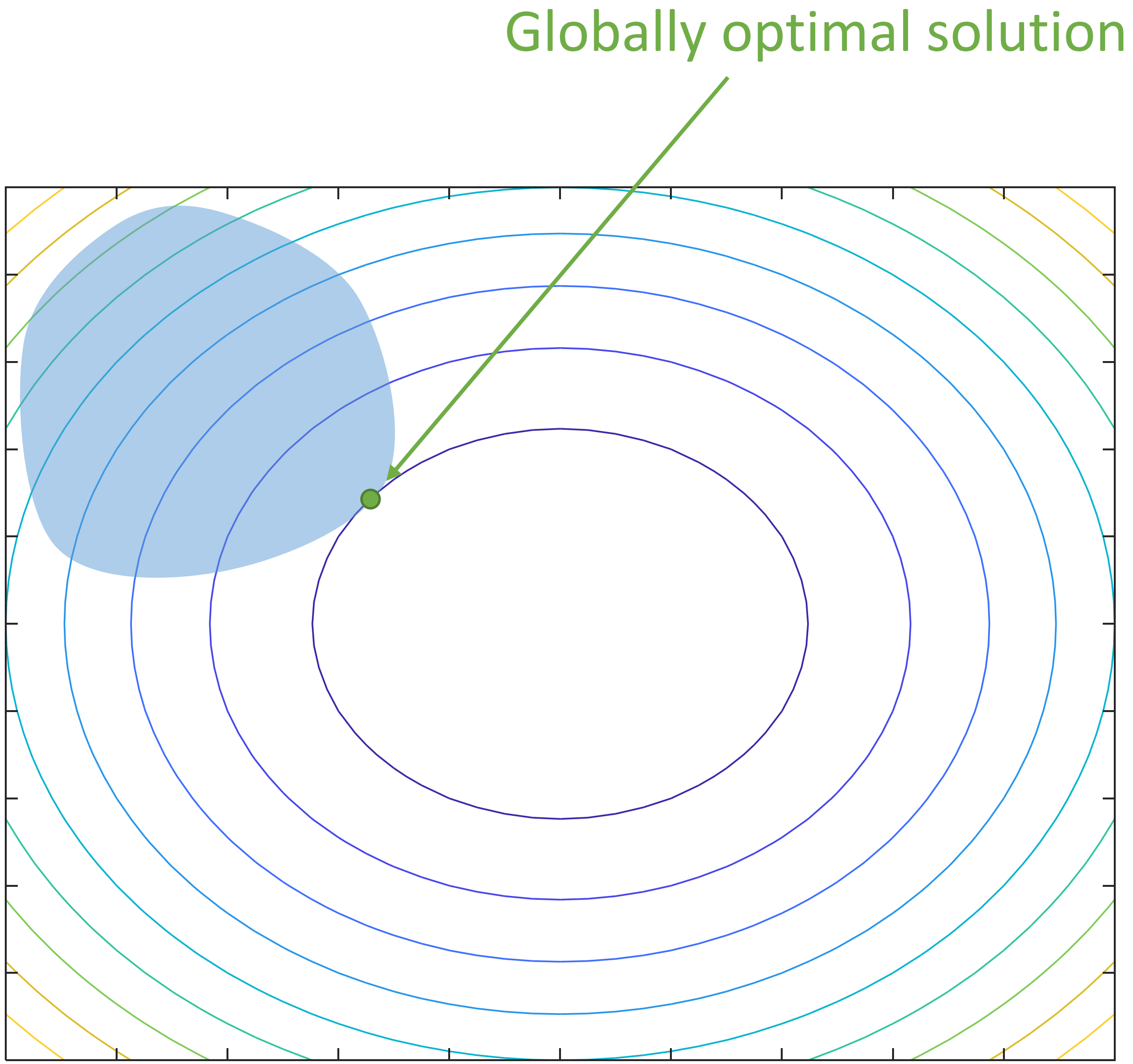
$$\text{minimize} \quad f(x)$$

$$\text{subject to} \quad g_i(x) \le 0, i = 1, \dots, n,$$

where $g_i(x)$ are convex

$$h_j^\top x = 0, j = 1, \dots, m$$





Convex Programs

$$\text{minimize} \quad f(x)$$

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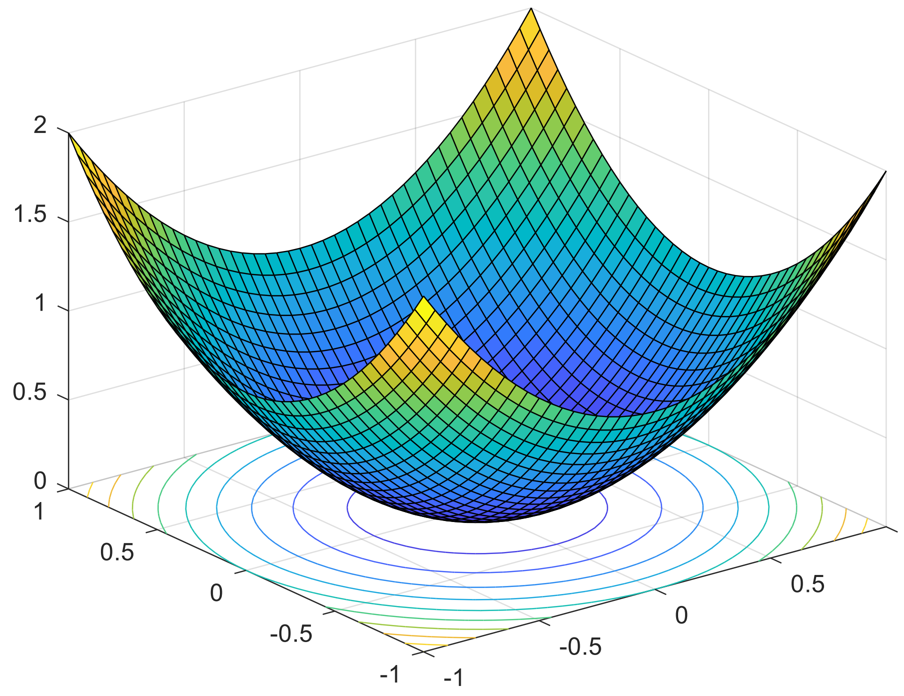
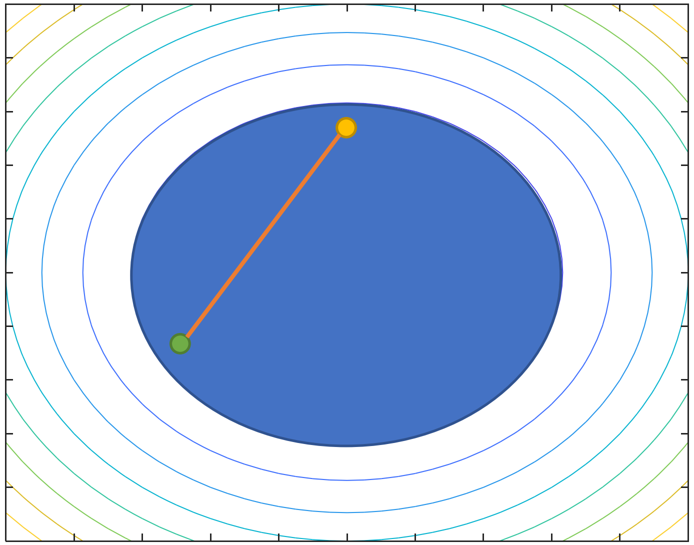
where $g_i(x)$ are convex

$$h_j^\top x = 0, j = 1, \ldots, m$$



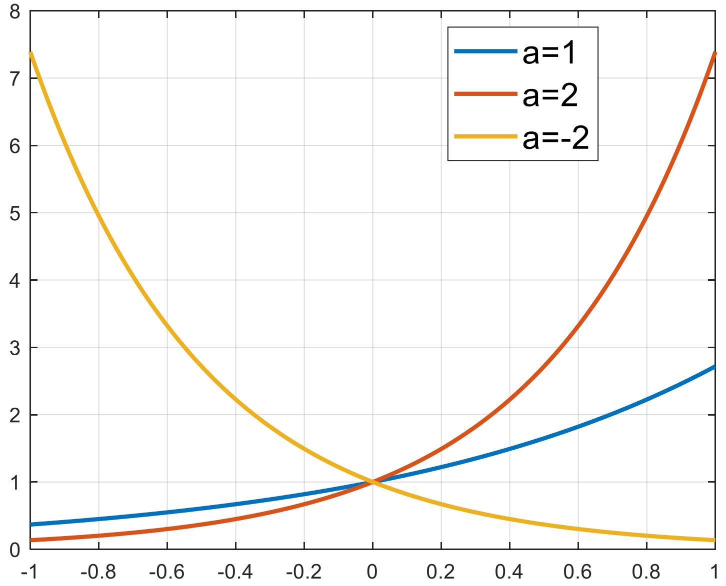
- Local optimum is global!
- Relatively easy to solve using simple algorithms
- When you see an optimization problem, first hope it's convex (although this is almost never true)
  - If an optimization problem is not convex, usually one can only hope for local optimum
- It is useful to recognize convex functions



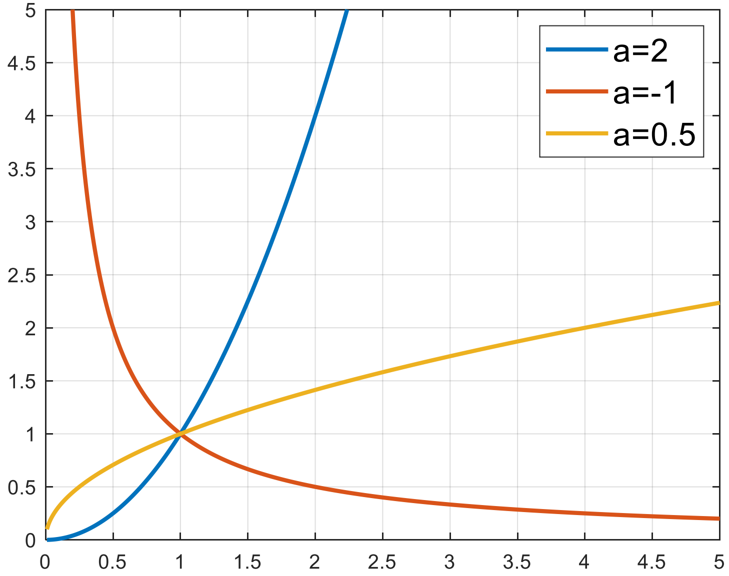
Common Convex Functions on $\mathbb{R}$

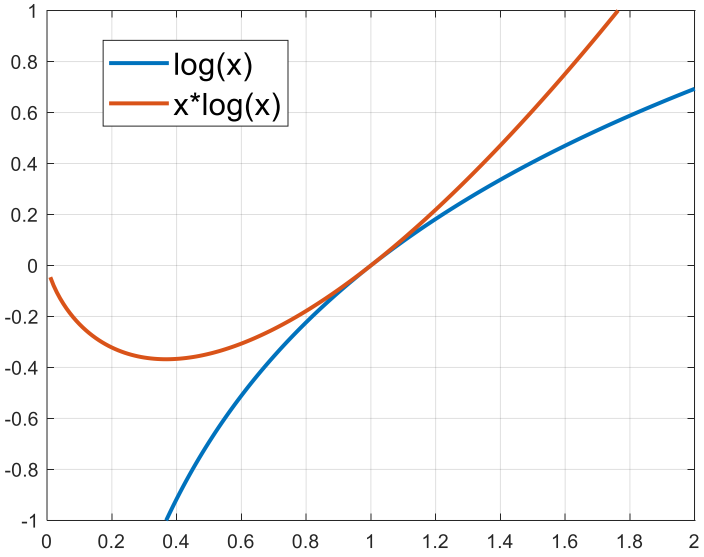
- $f(x) = e^{ax}$ is convex for all $x, a \in \mathbb{R}$
- $f(x) = x^a$ is convex on $x > 0$ if $a \geq 1$ or $a \leq 0$; concave if $0 < a < 1$
- $f(x) = \log x$ is concave
- $f(x) = x \log x$ is convex for $x > 0$ (or $x \geq 0$ if defined to be 0 when $x = 0$)

$f(x) = e^{ax}$



$f(x) = x^a$

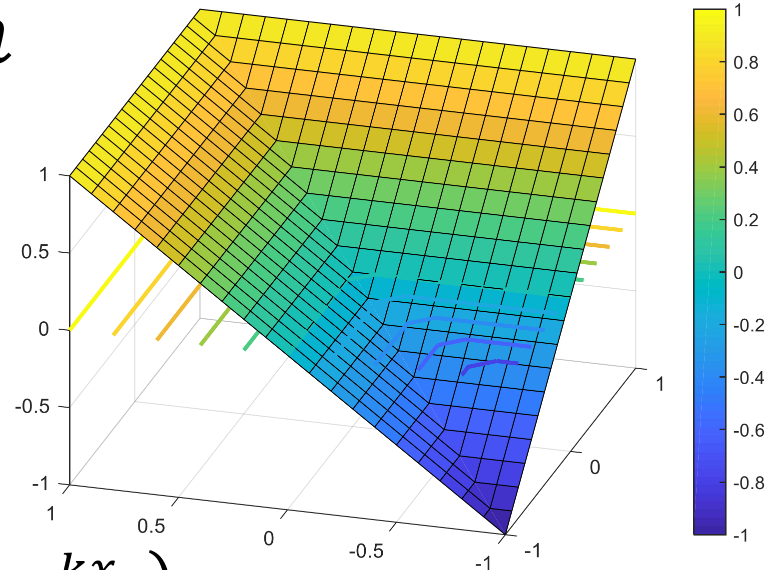




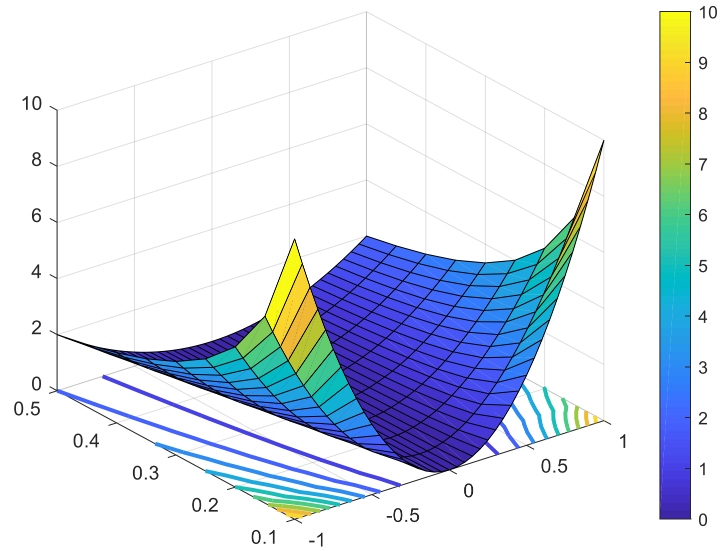
Common Convex Functions on $\mathbb{R}^n$

- $f(x) = Ax + b$ is convex for any A, b
- Every norm on $\mathbb{R}^n$ is convex
- $f(x) = \max(x_1, x_2, \dots, x_n)$ is convex
- $f(x) = \frac{x_1^2}{x_2}$ (for $x_2 > 0$)
- Log-sum-exp softmax: $f(x) = \frac{1}{k}\log\left(e^{kx_1} + e^{kx_2} + \cdots + e^{kx_n}\right)$
- Geometric mean: $f(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}},\ x_i > 0$

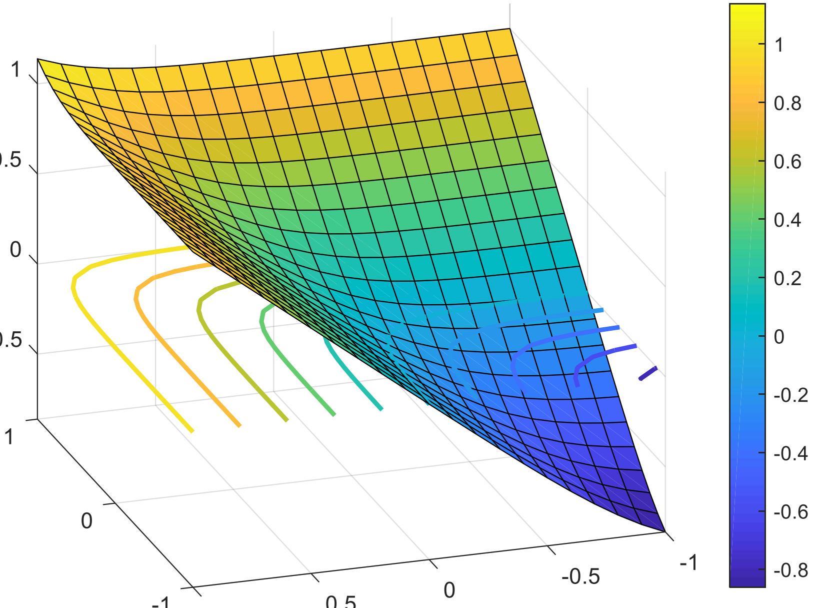
$f(x_1, x_2) = \max(x_1, x_2)$



$f(x) = \frac{x_1^2}{x_2}$



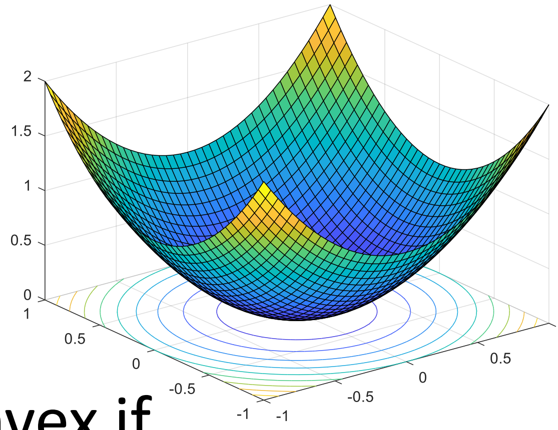
$f(x) = \frac{1}{5}\log(e^{5x_1} + e^{5x_2})$



Operations that Preserve Convexity

- Non-negative weighted sum: $\sum_i w_i f_i(x)$ is convex if $f_i(x)$ are convex and $w_i \geq 0$
  - Example: $f(x) = ax^2 + bx^4 + cx^6$, where $a, b, c > 0$
- Composition with affine function: $g(x) = f(Ax + b)$ is convex if $f(x)$ is convex
  - Example: $f(\theta) = \|X\theta - Y\|_2^2$
- Point-wise maximum: $\max\big(f_1(x), f_2(x)\big)$

Operations that Preserve Convexity



- Point-wise minimum of a function: $g(y) \coloneqq \min_z f(y,z)$ is convex if $f(y,z)$ is convex (jointly in (y,z))

- Perspective: $g(x,t) \coloneqq tf\left(\frac{x}{t}\right), t > 0$ is convex if $f(x)$ is convex
  - Example: $\frac{x_1^2}{x_2}$ is convex if $x_2 > 0$, because $f(x_1) = x_1^2$ is convex

- If $g_i: \mathbb{R}^n \to \mathbb{R}$ are convex, and $h: \mathbb{R}^k \to \mathbb{R}$ is convex and non-decreasing in each argument, then $h\big(g_1(x), g_2(x), \dots, g_k(x)\big)$ is convex
  - Example: $\log(e^{x_1} + e^{x_2} + \cdots + e^{x_n})$ is convex, because e^x is convex, and $\log x$ is convex and non-decreasing

How to check if a function is convex

- Use definition: $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$
- Show $f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$ for differentiable functions
- Show $\nabla^2 f(x) \succcurlyeq 0$ for twice differentiable functions
- **Show f is obtained from simple convex functions and operations that preserve convexity**

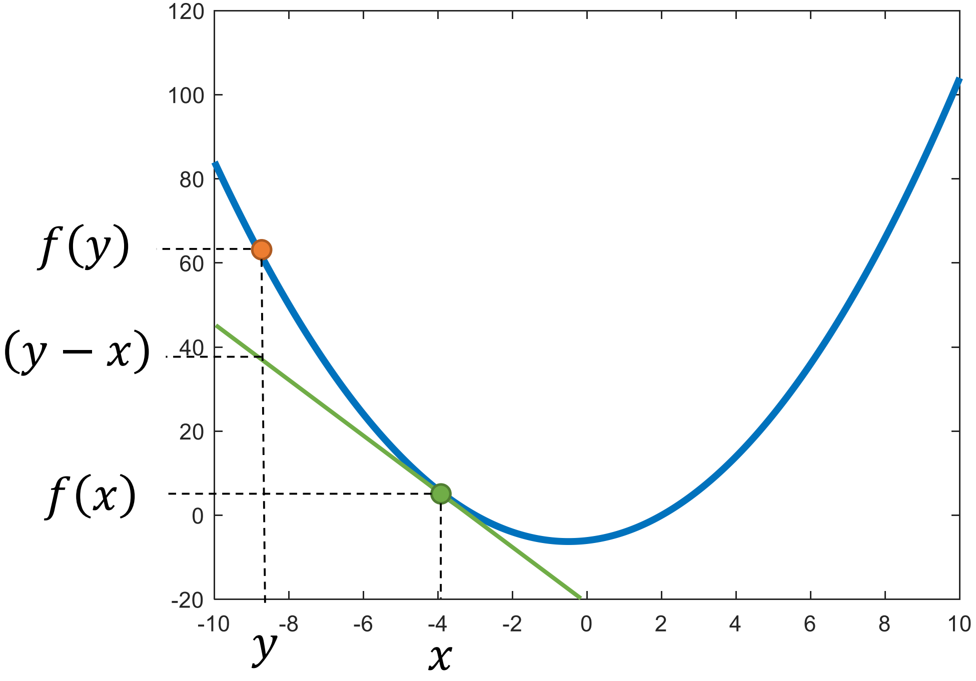
Example 1:

- $f(x) = Ax + b, x \in \mathbb{R}^n$

$$\begin{aligned} f(\theta x + (1-\theta)y) &= A(\theta x + (1-\theta)y) + b \\ &= \theta Ax + (1-\theta)Ay + b \\ &= \theta Ax + (1-\theta)Ay + \theta b + (1-\theta)b \\ &= \theta f(x) + (1-\theta)f(y) \end{aligned}$$

  - Equality!
  - This means f is also concave (i.e. $-f$ is convex)
  - Linear functions are both convex and concave

Example 2:



- $f(x) = x^2 + x - 6$
- Method 1: show $f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$
  - $\nabla f(x) = f'(x) = 2x + 1$

$$
\begin{aligned}
f(y) - f(x) + f'(x)(y - x) &= y^2 + y - 6 - [x^2 + x - 6 + (2x + 1)(y - x)] \\
&= y^2 + y - [x^2 + x + 2xy - 2x^2 + y - x] \\
&= y^2 + y - [-x^2 + 2xy + y] \\
&= y^2 + x^2 - 2xy \\
&= (x - y)^2 \geq 0
\end{aligned}
$$

- Method 2: show $\nabla^2 f(x) \geq 0$

$$\nabla^2 f(x) = f''(x) = 2 \geq 0$$

Example 3:

- $f(x) = \|Ax + b\|_2 + \lambda\|x\|_1$, A is a constant matrix, b is a constant vector, and $\lambda \geq 0$ is a constant scalar.
  - We know $\|x\|_1$ are $\|x\|_2$ are convex
    - All norms are convex
  - So, $\|Ax + b\|_2$ is convex, by the rule of affine composition
    - $g(x) = f(Ax + b)$ is convex if $f(x)$ is convex
  - Finally, $\|Ax + b\|_2 + \lambda\|x\|_1$ is convex, by the rule of non-negative weighted sum
    - $\sum_i w_i f_i(x)$ is convex if $f_i(x)$ are convex and $w_i \geq 0$