Announcements

- Assignment 1 posted, due Sept. 30
- Office hours this week: Today 13:00-14:30
- Office hours after this week: Mondays 14:00-15:30

Nonlinear Systems

CMPT 419/983

11/9/2019

Nonlinear Systems Roadmap

- Introduction
- Analysis
- Control
- Numerical solutions

Nonlinear Systems Roadmap: Today

- Feature of nonlinear systems

- Linearization

- Stability via linearization

- Phase portraits

Nonlinear systems

- $\dot{x} = f(x, u)$
  - **State:** $x(t) \in \mathbb{R}^n, x(t_0) = x_0$
  - **Control:** $u(t) \in \mathcal{U}$

- Differential equations generally do not have closed-form solutions
  - Numerical methods can be used to obtain approximate solutions
  - Other analysis techniques offer insight into the solutions

- Existence and uniqueness of solutions
  - f is a nonlinear function
  - Lipschitz continuous in x
    $$\exists L > 0, \forall u, \|f(x_1, u) - f(x_2, u)\| \leq L\|x_1 - x_2\|$$
  - $u(\cdot)$ is piecewise continuous

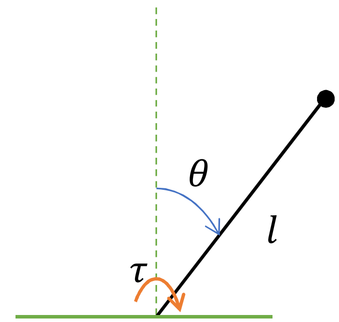
Study of Nonlinear Systems

- In general, no closed form solutions
- Numerical approximations of solutions can be helpful
    - Widely used for simulations to predict system behaviour
- Analysis involves studying
    - equilibrium points
    - stability
    - limit cycles
    - bifurcations

Features of nonlinear systems

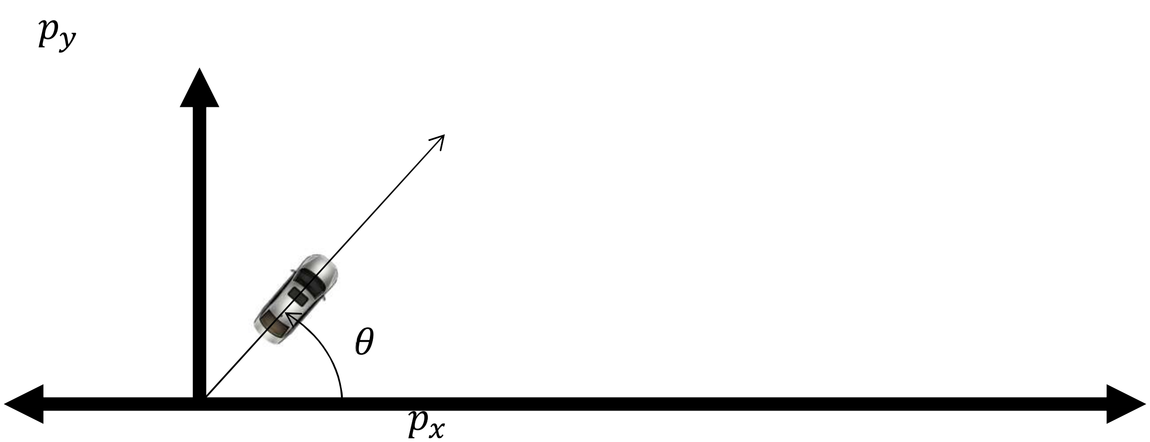
- Almost all real-world robots are modelled by nonlinear systems

Examples of Nonlinear Systems



- Dubins Car

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = u$$



- Inverted pendulum

$$\ddot{\theta} - \frac{g}{l}\sin\theta = 0$$
$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

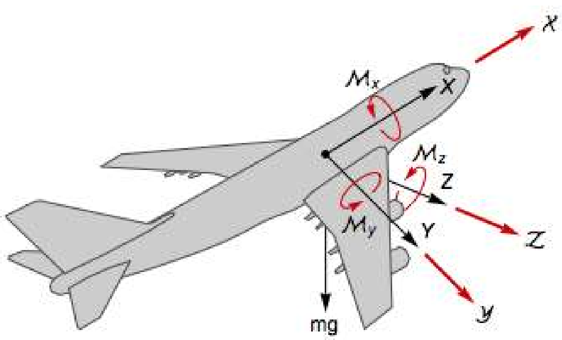
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g}{l}\sin x_1$$

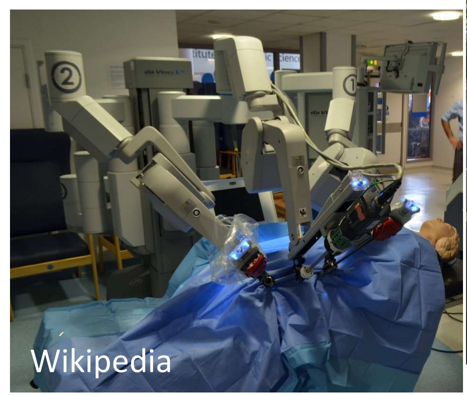
Examples of Nonlinear Systems

- Bicycle

$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{v}_x &= \omega v_y + u_1 \\
\dot{y} &= v_y \\
\dot{v}_y &= -\omega v_x + \frac{2}{m}\left(F_{c,f}\cos u_2 + F_{c,r}\right) \\
\dot{\psi} &= \omega \\
\dot{\omega} &= \frac{2}{I_z}\left(l_f F_{c,f} - l_r F_{c,r}\right) \\
\dot{X} &= v_x \cos\psi - v_y \sin\psi \\
\dot{Y} &= v_x \sin\psi + v_y \cos\psi
\end{aligned}
$$

Examples of Nonlinear Systems





Features of Nonlinear Systems

- Almost all real-world robots are modelled by nonlinear systems

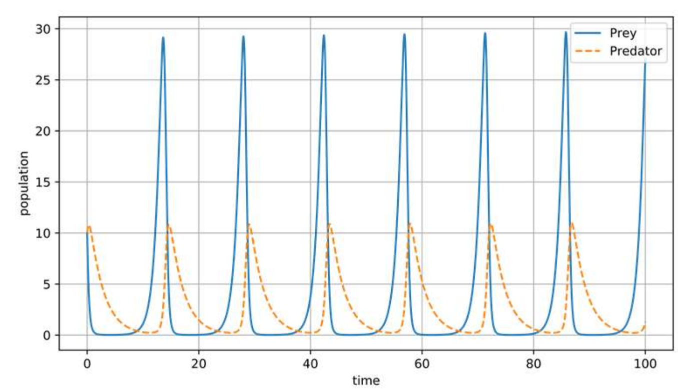
- Closed orbits and limit cycles

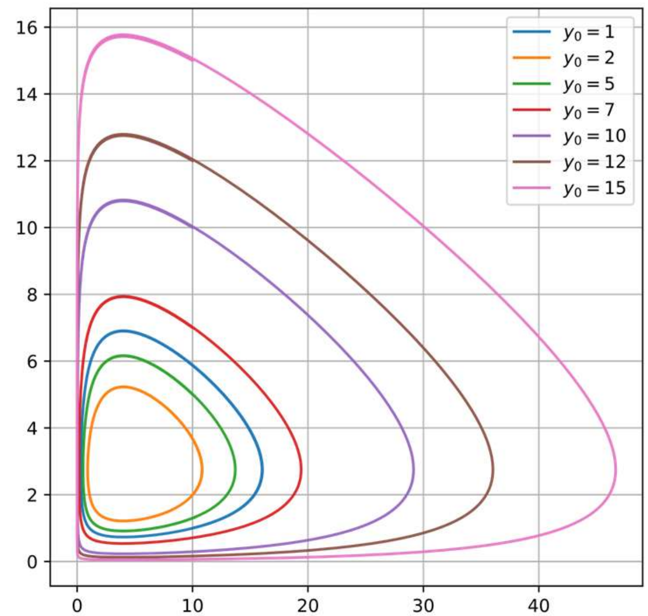
Predator-Prey Model

- Predator-prey model: x is number of preys, y is number of predators

$$\dot{x} = \alpha x - \beta xy$$
$$\dot{y} = \delta xy - \gamma y$$

  - α: prey natural growth rate
  - β: prey decline rate due to interaction with predator
  - δ: predator growth rate due to interaction with prey
  - γ: prey natural decline rate



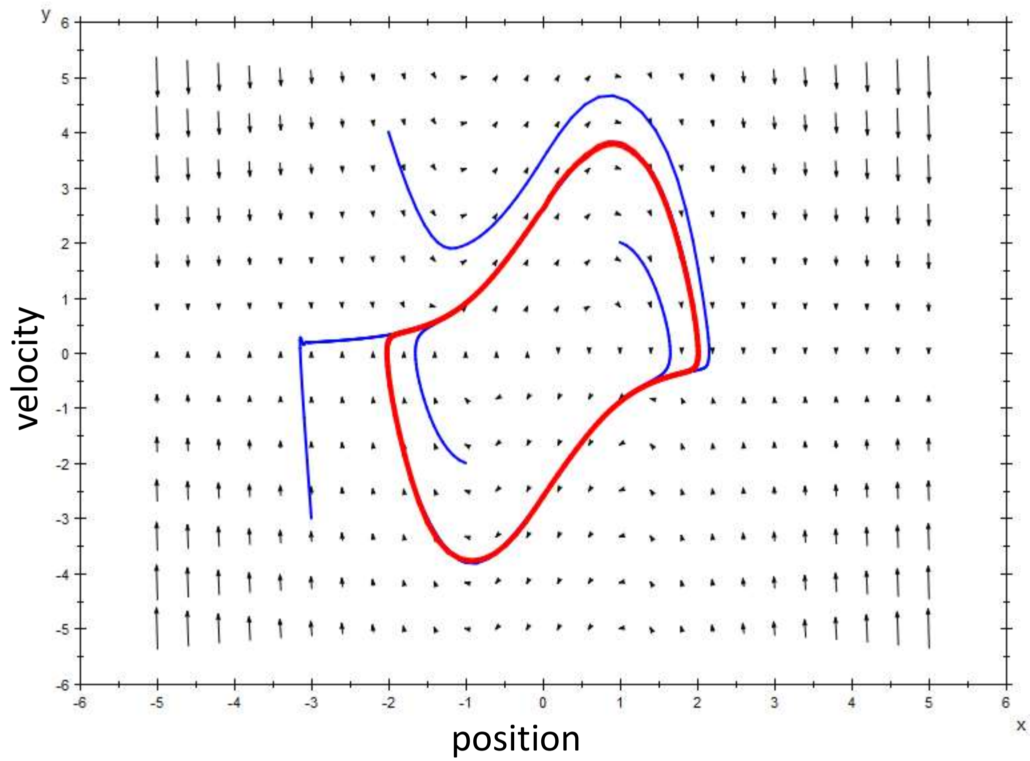


Van der Pol Oscillator

$$\dot{x} = \mu\left(x - \frac{1}{3}x^3 - y\right)$$
$$\dot{y} = \frac{1}{\mu}x$$

- Model for several natural phenomena
  - Neuron action potentials
  - Geological fault
  - Heart beat

- Limit cycle
  - No matter the initial state, trajectories converge to the cycle



Features of Nonlinear Systems

- Almost all real-world robots are modelled by nonlinear systems

- Closed orbits and limit cycles

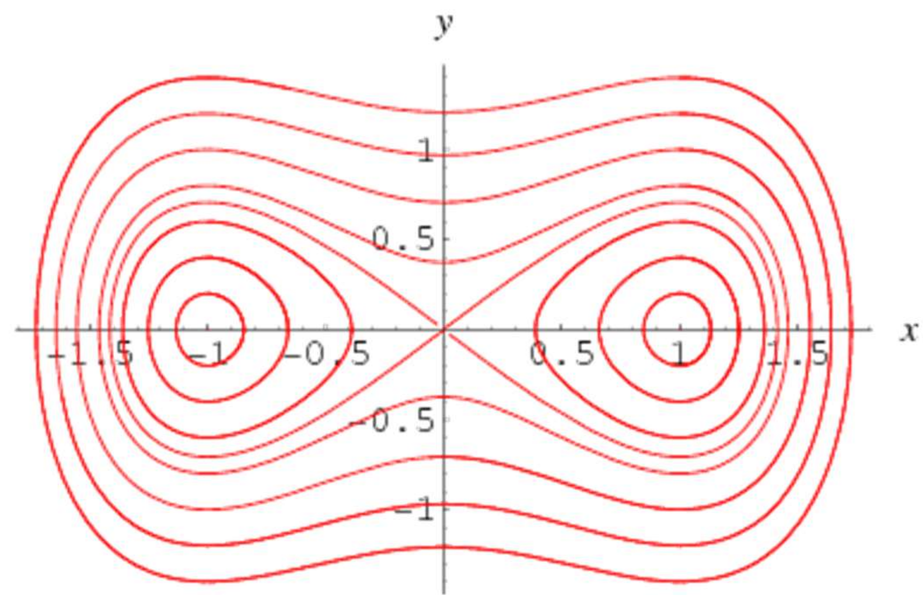
- Multiple isolated equilibrium points

Duffing's Equation

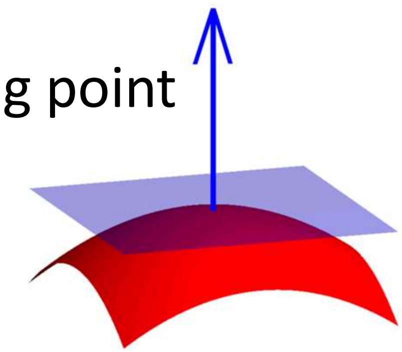
- More complex model of oscillators compared to the simple harmonic oscillator, which is a linear system

- No damping and no forcing:

$$\dot{x} = y$$
$$\dot{y} = x - x^3$$



Linearization



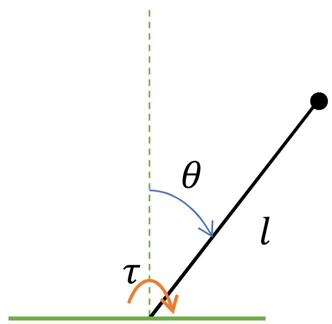
- Local behaviour of nonlinear system $\dot{x} = f(x,u)$ at operating point $(x,u) = (\bar{x},\bar{u})$
  - At the operating point, $\dot{\bar{x}} = f(\bar{x},\bar{u})$
  - Define new variables $\tilde{x} = x - \bar{x}, \tilde{u} = u - \bar{u}$

- Taylor approximation:
  - $f(x,u) = f(\bar{x}+\tilde{x},\bar{u}+\tilde{u}) \approx f(\bar{x},\bar{u}) + \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \tilde{u}$
  - $\dot{x} = \dot{\bar{x}} + \dot{\tilde{x}} \approx f(\bar{x},\bar{u}) + \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \tilde{u}$
  - $\dot{\tilde{x}} = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \tilde{u}$

Linearization

- From previous slide: $\dot{\tilde{x}} = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \tilde{u}$
  - $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$
- $\frac{\partial f}{\partial x} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\bar{x},\bar{u})}}_{\tilde{A}} \in \mathbb{R}^{n\times n}, \frac{\partial f}{\partial u} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_k} \end{bmatrix}_{(\bar{x},\bar{u})}}_{\tilde{B}} \in \mathbb{R}^{n\times k}$

Linearization



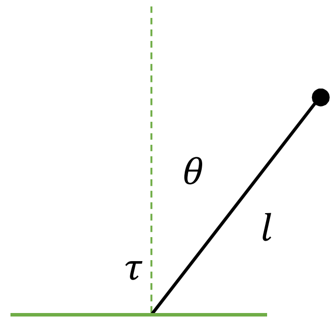
- Inverted pendulum
  - Newton's laws: $\ddot{\theta} = \frac{\tau}{ml^2} + \frac{g}{l}\sin\theta$
  - Let $x_1 = \theta, x_2 = \dot{\theta}, u = \frac{\tau}{ml^2}$ ("normalized control")

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g}{l}\sin x_1 + u$$

- Linearize around $\theta = x_1 = 0, \dot{\theta} = x_2 = 0, u = 0$

Linearization

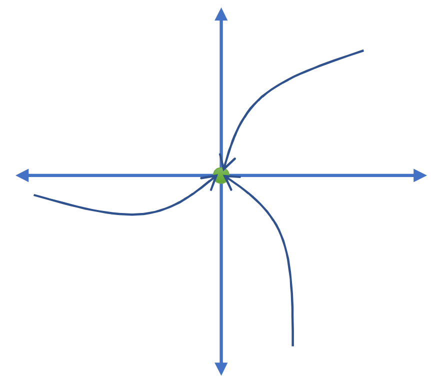
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g}{l}\sin x_1 + u$$



- Linearize around $\theta = x_1 = 0, \dot{\theta} = x_2 = 0, u = 0$
  - $\dot{\tilde{x}} \approx \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} \tilde{x} + \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} \tilde{u}$
  - $\left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\mathbf{0},0)} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos x_1 & 0 \end{bmatrix}_{(\mathbf{0},0)} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}$
  - $\left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{(\mathbf{0},0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- So $\dot{x} \approx \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \Rightarrow$

$$\dot{x}_1 \approx x_2$$
$$\dot{x}_2 \approx \frac{g}{l} x_1 + u$$

LTI System: Stability of $\dot{x} = Ax$



- **Equilibrium point** of $\dot{x} = f(x)$ is where $f(x) = 0$
  - For $\dot{x} = Ax$, in general $\mathbf{0}_n$ is an equilibrium point: $x_e = 0_n$
  - Also, $x_e \in N(A)$

- **Stable**: $x(t)$ is bounded for all $t \geq 0$, for all initial conditions x_0
- **Asymptotically stable**: $x(t) \rightarrow x_e$ as $t \rightarrow \infty$
- **Exponentially stable**: $\exists M, \alpha > 0$ such that $\|x(t)\| \leq Me^{-\alpha t}\|x_0\|$

- The system $\dot{x} = Ax$ is exponentially stable if and only if all eigenvalues of A are in the *open* left half plane, i.e. $\forall k, \mathrm{Re}(\lambda_k) < 0$

Equilibrium Points and Stability: Nonlinear Systems

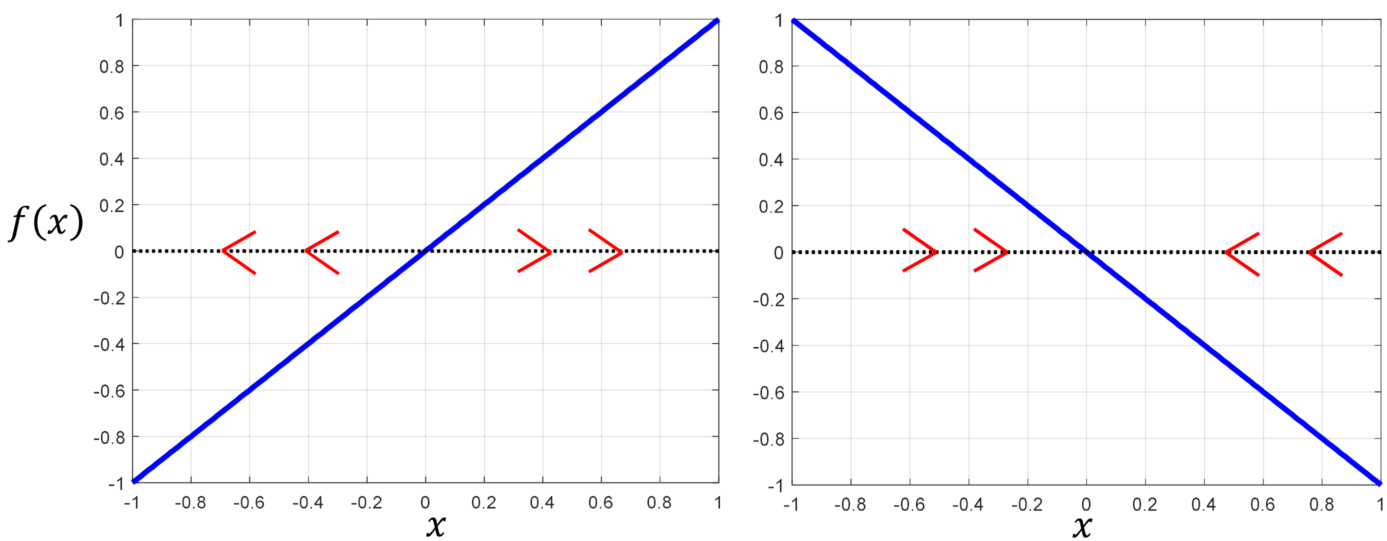
- 1D: Determine stability pictorially
- In general: eigenvalues of linearization around equilibrium point

- Example:
  - $\dot{x} = ax$

Linearization:

$$\frac{\partial f}{\partial x} = a$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = a$$



$a > 0$: unstable

$a < 0$: stable

Equilibrium Points and Stability: Nonlinear Systems

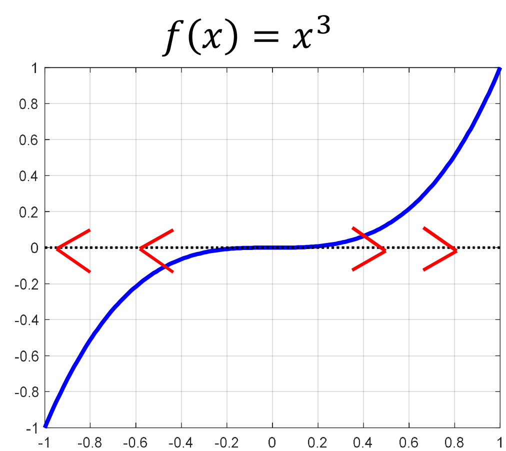
- 1D: Determine stability pictorially
- In general: eigenvalues of linearization around equilibrium point

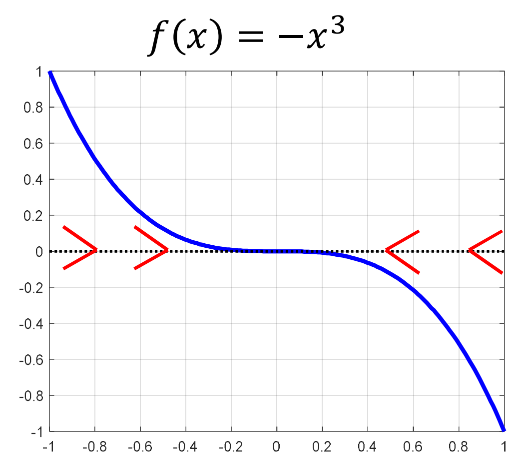
- Examples:
  - $\dot{x} = x^3$
  - $\dot{x} = -x^3$

Linearization:

$$\frac{\partial f}{\partial x} = \pm 3x^2$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = 0$$





Duffing's Equation

- Damped and no forcing:

$$\dot{x} = y$$
$$\dot{y} = x - y - x^3$$

- Equilibrium points:

$$\dot{x} = 0 \Rightarrow y = 0$$
$$\dot{y} = 0 \Rightarrow x - y - x^3 = 0$$
$$\Rightarrow x(1 - x^2) = 0$$
$$\Rightarrow x = -1,0,1$$

- Linearization:

$$\frac{\partial f}{\partial(x,y)} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial(x,y)}\right|_{(\pm 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Eigenvalues:

$$s^2 + s + 2 = 0$$
$$s = \frac{-1 \pm \sqrt{1-8}}{2}$$

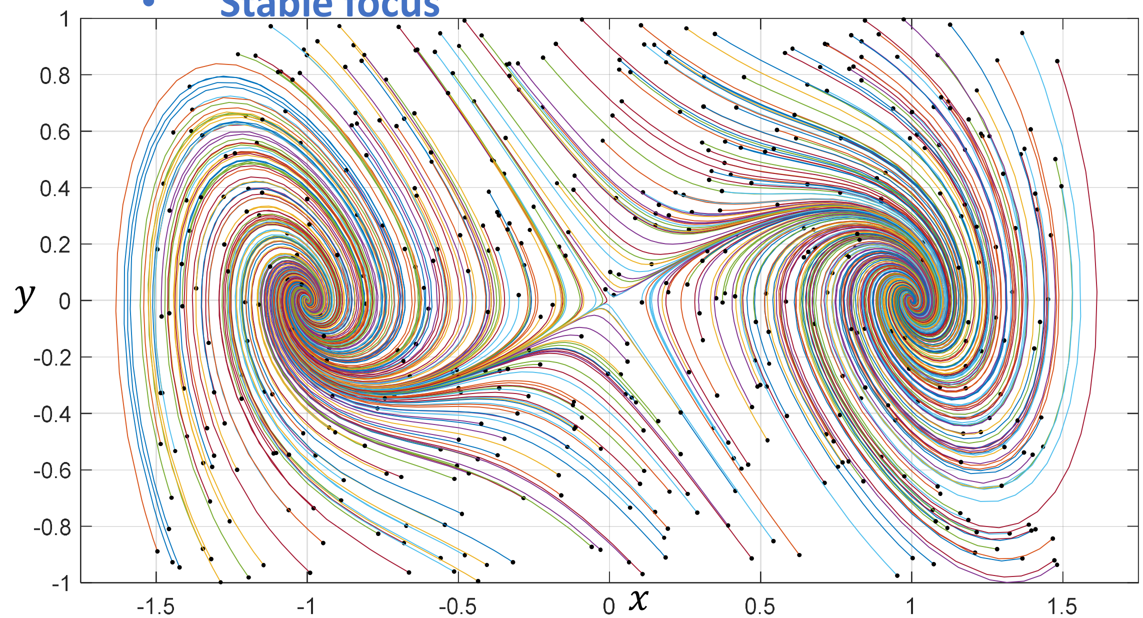
- Complex conjugate pairs
- Negative real part
- **"Stable focus"**

$$\left.\frac{\partial f}{\partial(x,y)}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigenvalues:

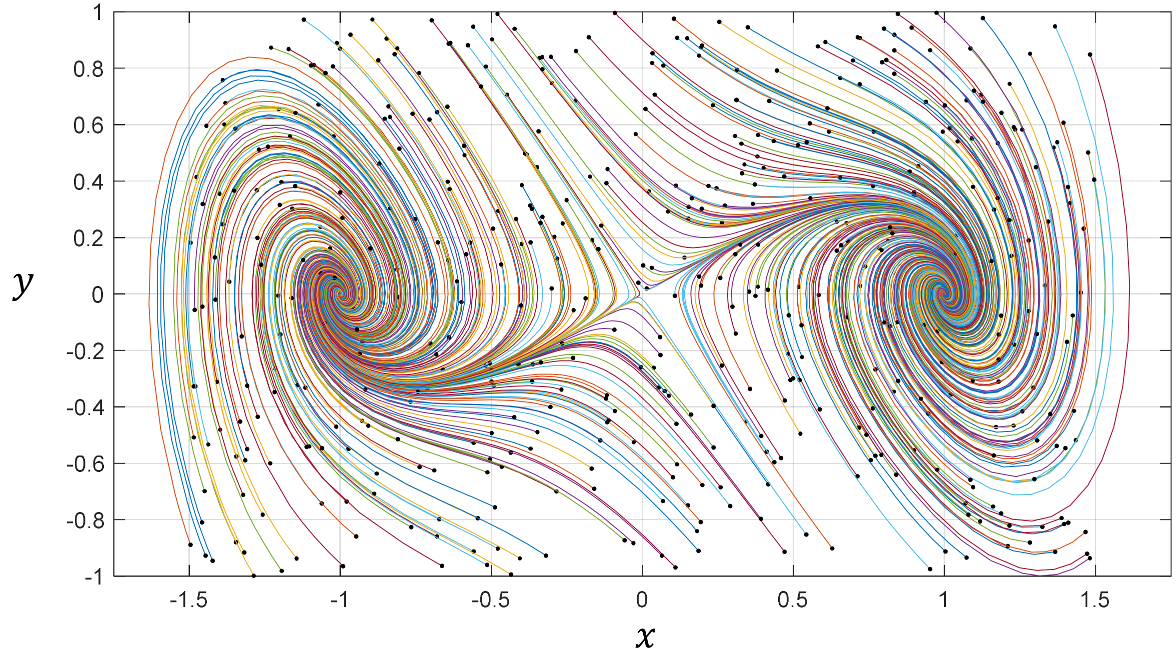
$$s^2 + s - 1 = 0$$
$$s = \frac{-1 \pm \sqrt{1+4}}{2}$$

- Real and opposite sign
- **"Saddle"**



Phase Portraits

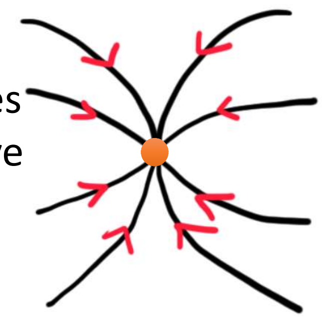
- Phase portraits: Graphs of $y(t)$ vs. $x(t)$ for 2D systems

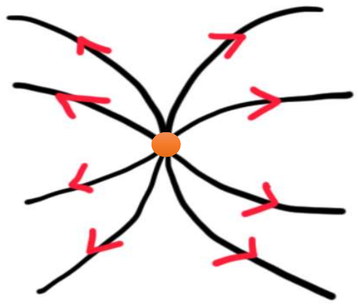


Phase Portraits

- Phase portraits: Graphs of $y(t)$ vs. $x(t)$ for 2D systems

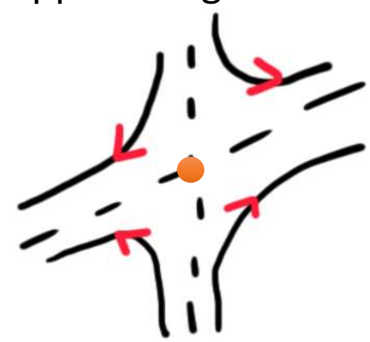
Stable node
- Both eigenvalues real and negative



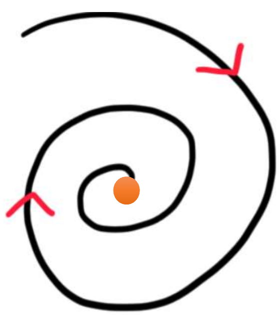


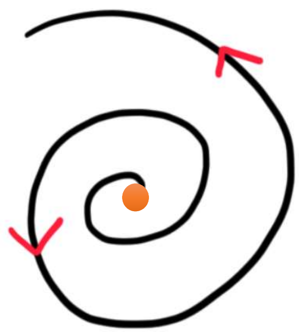
Unstable node
- Both eigenvalues real and positive

Saddle
- Real eigenvalues with opposite signs



Stable focus
- Complex eigenvalues pairs
- Negative real part





Unstable focus
- Complex eigenvalues pairs
- Positive real part