Stability of Nonlinear Systems

CMPT 419/983

16/09/2019

Duffing's Equation

- Damped and no forcing:

$$\dot{x} = y$$
$$\dot{y} = x - y - x^3$$

- Equilibrium points:

$$\dot{x} = 0 \Rightarrow y = 0$$
$$\dot{y} = 0 \Rightarrow x - y - x^3 = 0$$
$$\Rightarrow x(1 - x^2) = 0$$
$$\Rightarrow x = -1,0,1$$

- Linearization:

$$\frac{\partial f}{\partial(x,y)} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial(x,y)}\right|_{(\pm 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Eigenvalues:

$$\lambda^2 + \lambda + 2 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1-8}}{2}$$

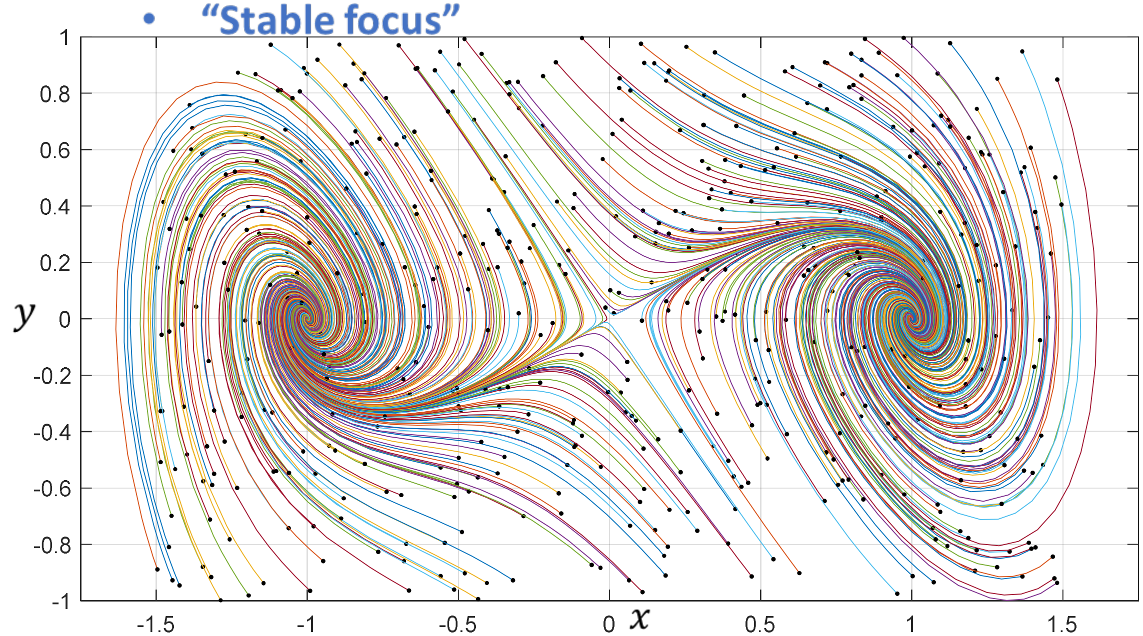
- Complex conjugate pairs
- Negative real part
- **"Stable focus"**

$$\left.\frac{\partial f}{\partial(x,y)}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Eigenvalues:

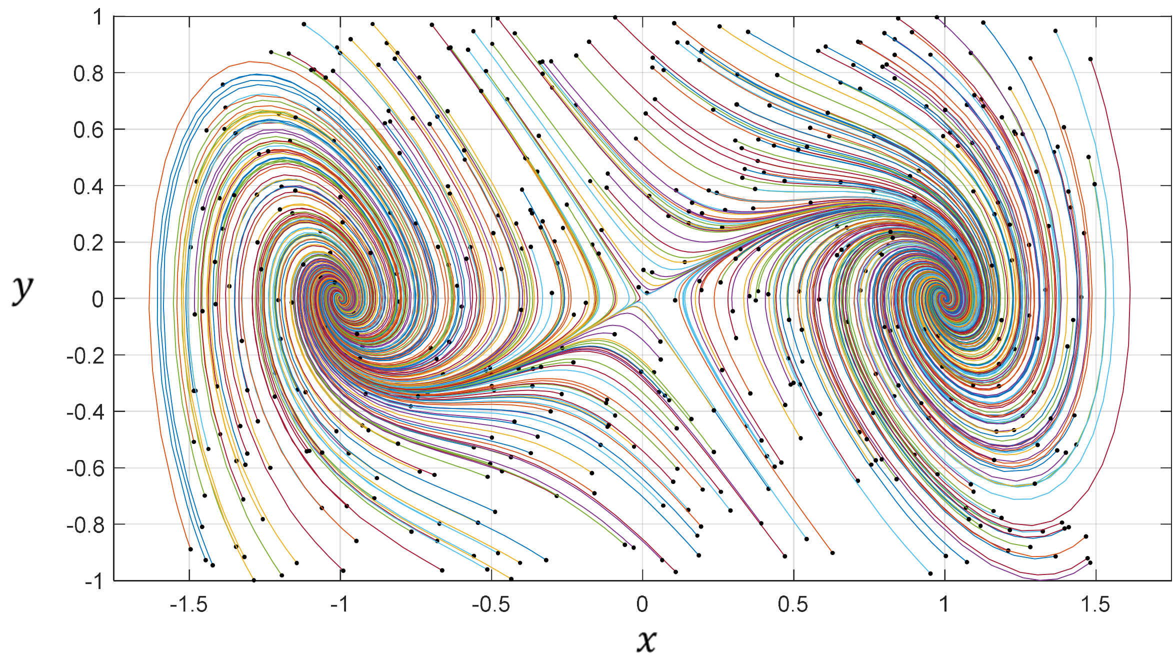
$$\lambda^2 + \lambda - 1 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1+4}}{2}$$

- Real and opposite sign
- **"Saddle"**



Phase Portraits

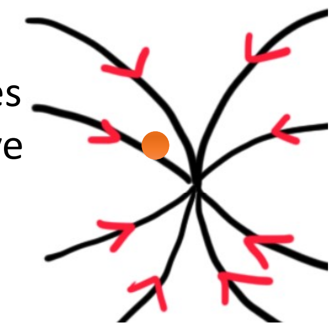
- Phase portraits: Graphs of $y(t)$ vs. $x(t)$ for 2D systems

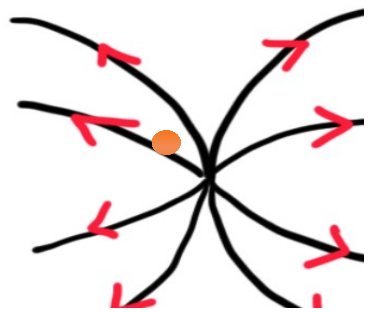


Phase Portraits

- Phase portraits: Graphs of $y(t)$ vs. $x(t)$ for 2D systems

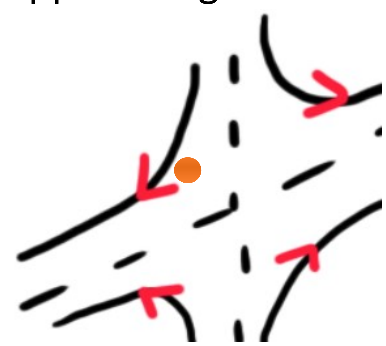
Stable node
- Both eigenvalues real and negative



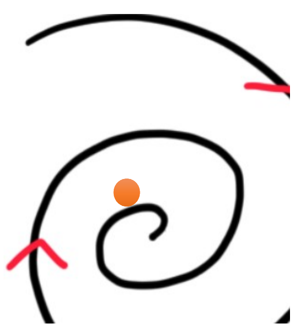


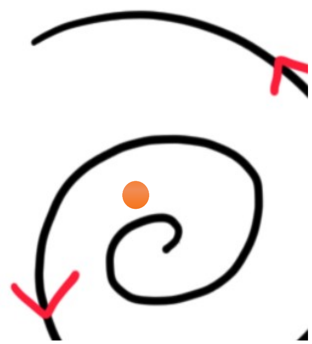
Unstable node
- Both eigenvalues real and positive

Saddle
- Real eigenvalues with opposite signs



Stable focus
- Complex eigenvalues pairs
- Negative real part



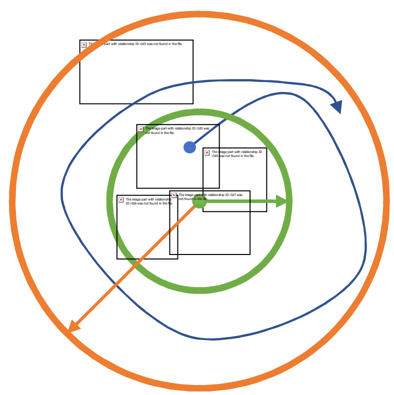


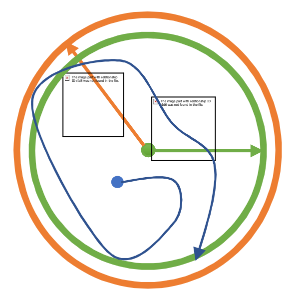
Unstable focus
- Complex eigenvalues pairs
- Positive real part

Lyapunov Stability

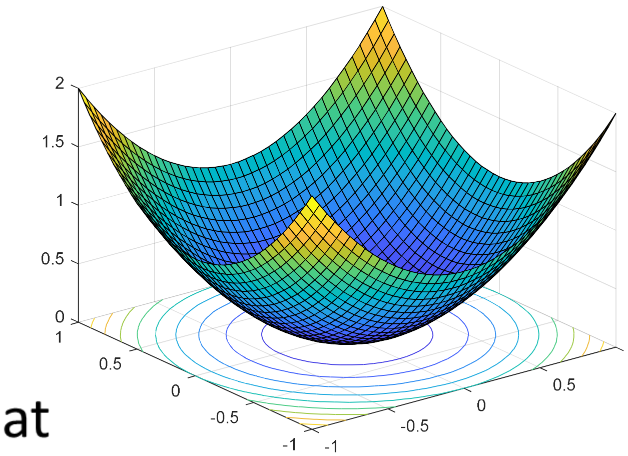
- A system is **stable in the sense of Lyapunov** if $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that

$$\|x_0 - x_e\| < \delta(\epsilon) \Rightarrow \forall t \geq t_0, \|x(t) - x_e\| < \epsilon$$

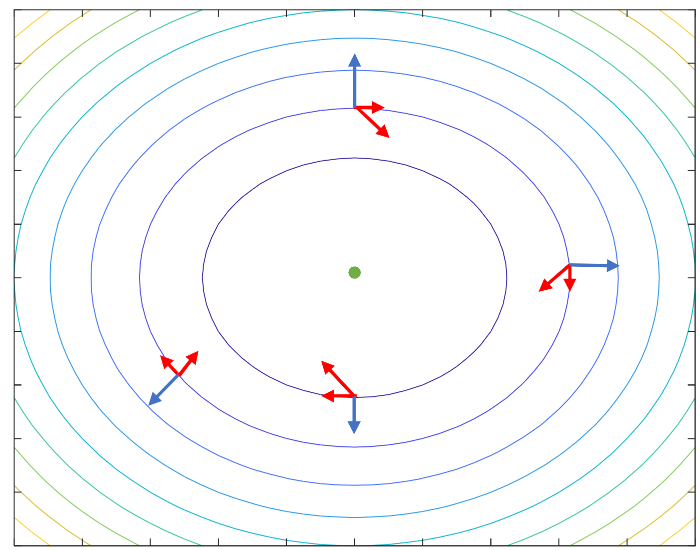




Lyapunov Stability Main Result

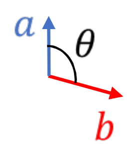


- Let $x = 0$ be an equilibrium point
- Suppose there is a function $V(x)\colon \mathbb{R}^n \rightarrow \mathbb{R}$ such that
  $V(x) = 0$ if and only if $x = 0$,
  $V(x) > 0$ if and only if $x \neq 0$.
  - If for all $x \neq 0, \dot{V}(x) = \nabla V^\top f(x) \leq 0$, then $x = 0$ is **stable in the sense of Lyapunov**
  - If for all $x \neq 0, \dot{V}(x) = \nabla V^\top f(x) < 0$, then $x = 0$ is **asymptotically stable**
- $V(x)$ is called a **Lyapunov function**



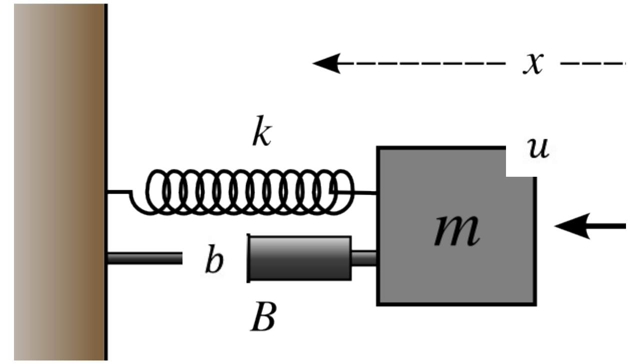
Key linear algebra fact: Given two vectors a and b, separated by an angle of θ,

$$a^\top b = \sum_i^n a_i b_i = \|a\|_2 \|b\|_2 \cos\theta$$



Proof: https://en.wikipedia.org/wiki/Dot_product

Lyapunov Stability Example in $\mathbb{R}^2$

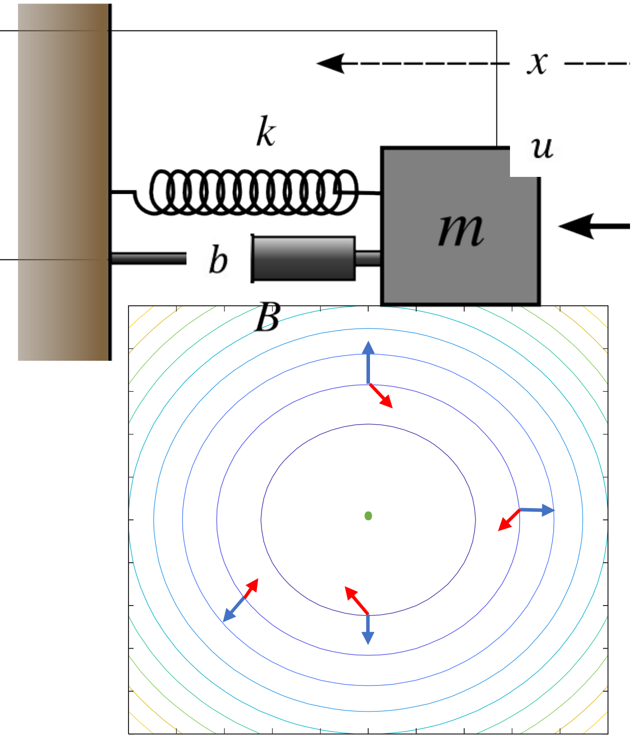


- Damped mass spring system
  - Newton's laws: $F = ma = m\ddot{x}$
  - Assume $m = 1, u = 0$
  - Define state space representation: Let $x_1 = x, x_2 = \dot{x}$

$$m\ddot{x} = -kx - b\dot{x} + u$$
$$\ddot{x} = -kx - b\dot{x}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -kx_1 - bx_2$$

- Equilibrium point: $\dot{x}_1 = 0 \Rightarrow x_2 = 0$
  $\dot{x}_2 = 0 \Rightarrow x_1 = 0$
  - Is this stable?
  - Intuition:
    - The origin is stable because there is friction
    - Friction causes the energy of the system to decrease, until no energy remains



- Let $V(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$
    - Potential energy plus kinetic energy

$$\begin{aligned} \Rightarrow \quad \dot{V}(x_1, x_2) &= \nabla V^\top f(x) \\ &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\ &= kx_1\dot{x}_1 + x_2\dot{x}_2 \\ &= kx_1x_2 - kx_1x_2 - bx_2^2 \\ &= -bx_2^2 \\ &< 0 \text{ for all } x \neq (0,0) \end{aligned}$$

Lyapunov Stability: Discussion

- What if there is control? $\dot{x} = f(x, u)$
  - Need at least one control that makes V non-increasing
- Advantages
  - Direct nonlinear analysis
  - "Global" result
  - "Region of attraction": the region where $\dot{V}(x) \leq 0$
- How to find a Lyapunov function?
  - Intuition → Guess something that works
  - Computational techniques
    - Optimization
    - Optimal control