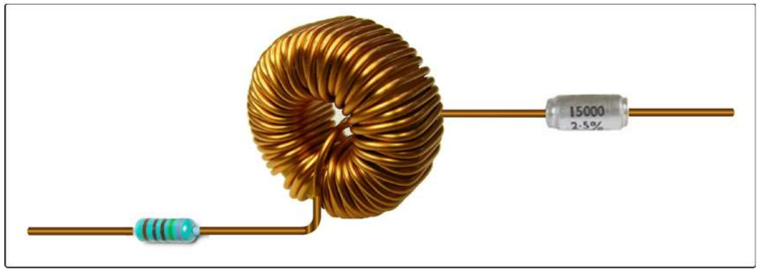
Announcements

- Course website: https://coursys.sfu.ca/2019fa-cmpt-419-x1/pages/

- Instructor office hours, TASC 1 8225
  - This week: 13:00 – 14:30
  - In the future: Mondays 14:00 – 15:30

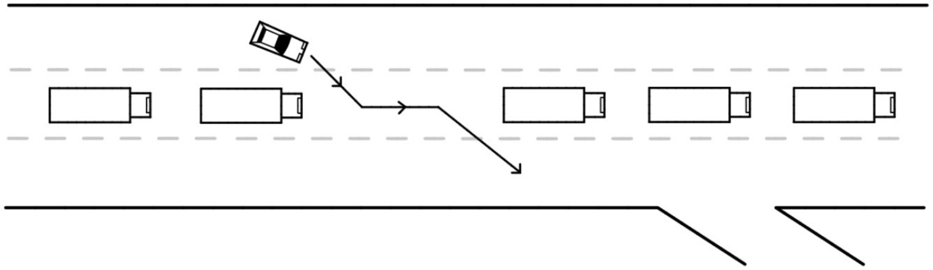
- TA (Shubam Sachdeva) office hours, ASB 9808
  - Thursdays 12:00 – 13:00

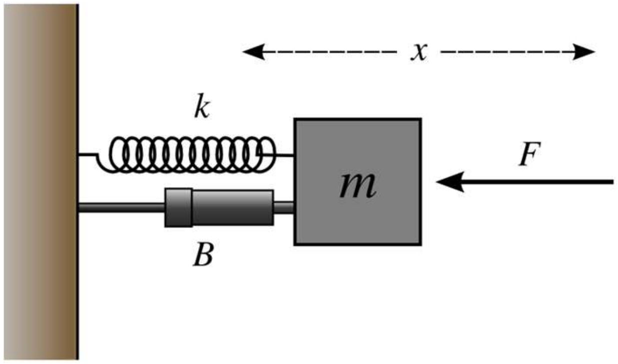


Linear Systems

CMPT 419/983

09/09/19

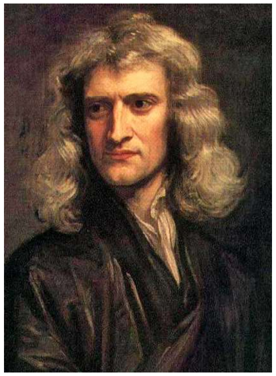




References for Linear Systems

- F. Callier & C. A. Desoer, Linear System Theory, Springer-Verlag, 1991.
- W. J. Rugh, Linear System Theory, Prentice-Hall, 1996.

Differential Equations



- Continuous time model of robotic systems
  - In general, nonlinear systems
  - One may construct discrete time models from continuous time models

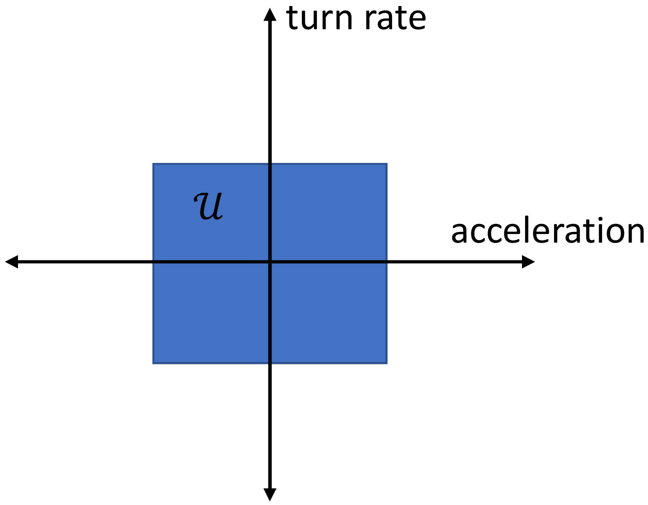
- **Dynamics:** $\dot{x} = f(t, x, u, d), x \in \mathbb{R}^n, t \geq t_0$
  - Specifies how the robot state or configuration changes over time
  - In some ways, the most “natural” model, since $F = ma = m\ddot{x}$
  - Defining $x_1 = x, x_2 = \dot{x}$, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{F}{m} \end{bmatrix}$$

Differential Equations

- **State:** $x(t) \in \mathbb{R}^n, x(t_0) = x_0$
  - Contains all information needed to specify the configuration of the robot
  - Most common: position, velocity, angular position, angular velocity

- **Control:** $u(t) \in \mathcal{U}$
  - Examples: steering, accelerating, decelerating
  - Usually constrained to be within some set

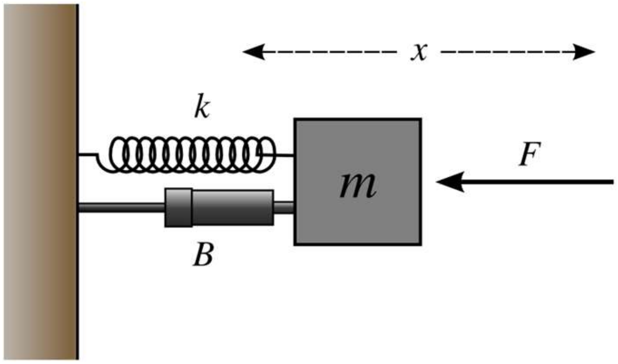


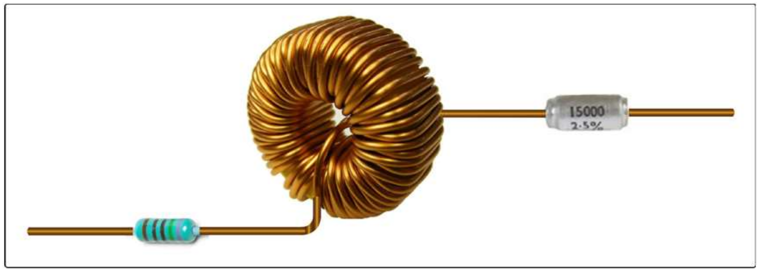
- **Disturbance:** $d(t) \in \mathcal{D}$
  - Examples: wind, input noise, another agent

Linear Systems

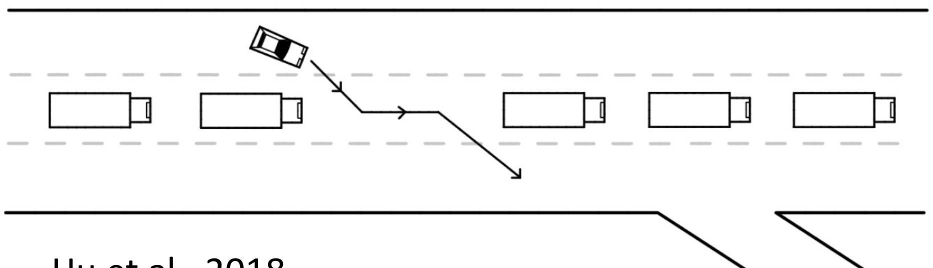
- Differential equations generally do not have closed-form solutions
  - Numerical methods can be used to obtain approximate solutions
  - Other analysis techniques offer insight into the solutions

- Linear time-invariant (LTI) systems: $\dot{x} = Ax + Bu$
  - Damped mass spring systems
  - Circuits involving resistors, capacitors, inductors
  - Approximations of nonlinear systems





Linear Systems



Hu et al., 2018

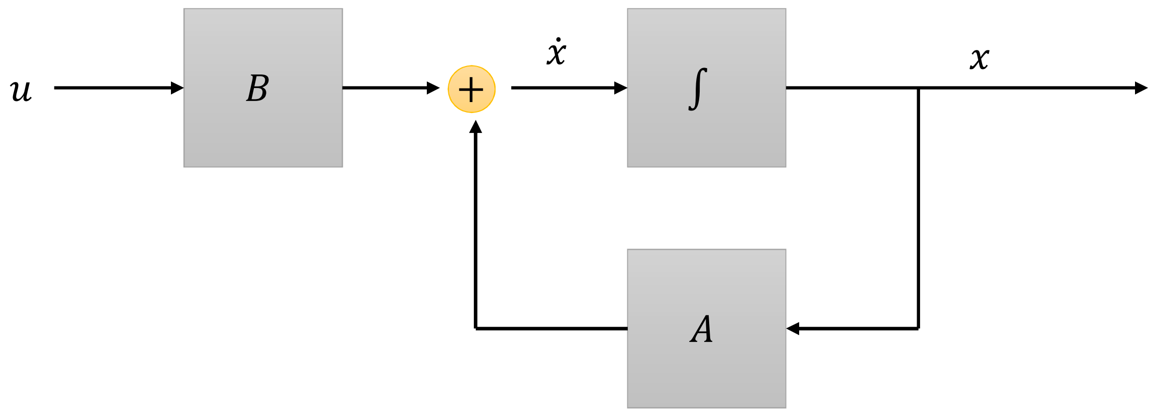
(If flying near hover, and slowly)
Bouffard, 2012

Road Map

- Basic properties and closed form solution
- Stability
- Linear state feedback control

LTI Systems

- Linear time-invariant (LTI) systems: $\dot{x} = Ax + Bu$



LTI Systems: Closed Form Solution

- $\dot{x} = Ax + Bu, \;\; x(0) = x_0$
  - $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

Matlab: `expm`

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$$

Solution to LTI System: Proof

- If $\dot{x} = Ax + Bu,\ \ x(0) = x_0$, then $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

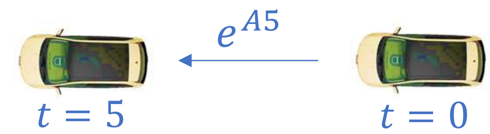
- Initial conditions:
  - $x(0) = e^{A(0)}x_0 + \int_0^0 e^{A(t-\tau)}Bu(\tau)d\tau = x_0$

$$\frac{d}{dt}\int_a^t g(\tau)d\tau = g(t)$$

- Differentiate:
  - $\dot{x} = \frac{d}{dt}(e^{At}x_0) + \frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)$
  - $\dot{x} = Ae^{At}x_0 + A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)$
  - $\dot{x} = Ax(t) + Bu(t)$

$$\begin{aligned}
\frac{d}{dt}\left(\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) &= \frac{d}{dt}\left(\int_0^t e^{At}e^{-A}\ Bu(\tau)d\tau\right) \\
&= \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau\right) \\
&= Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-A}\ Bu(t) \\
&= A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)
\end{aligned}$$

Matrix Exponential Properties

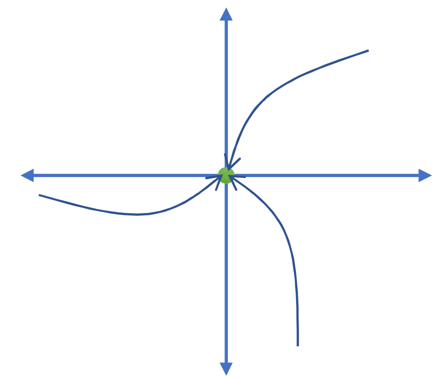


- If $\dot{x} = Ax,\ x(0) = x_0$, then $x(t) = e^{At}x_0$
- $e^0 = I$ (follows from the above)
- $e^{A(t+s)} = e^{At}e^{As}$
  - $x(t+s) = e^{A(t+s)}x_0 = e^{At}\overbrace{e^{As}x_0}^{x(s)}$
- $e^{(A+B)t} = e^{At}e^{Bt}$ if and only if $AB = BA$
- $(e^{At})^{-1} = e^{-At}$
  - So $e^{At}e^{-At} = I$
- $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$
  - From definition: $e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$

e^{At} "propagates" a state forward by a duration of t, according to the system dynamics A

- **State transition matrix**

LTI System: Stability of $\dot{x} = Ax$



- **Equilibrium point** of $\dot{x} = f(x)$ is where $f(x) = 0$
  - For $\dot{x} = Ax$, in general $\mathbf{0}_n$ is an equilibrium point: $x_e = 0_n$
  - Also, $x_e \in$ the nullspace of A

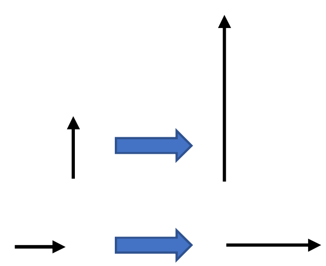
- **Stable**: $x(t)$ is bounded for all $t \geq 0$, for all initial conditions x_0
- **Asymptotically stable**: $x(t) \to x_e$ as $t \to \infty$
- **Exponentially stable**: $\exists M, \alpha > 0$ such that $\|x(t)\| \leq Me^{-\alpha t}\|x_0\|$

- The system $\dot{x} = Ax$ is exponentially stable if and only if all eigenvalues of A are in the *open* left half plane, i.e. $\forall k, \mathrm{Re}(\lambda_k) < 0$

Eigenvalues and Eigenvectors

- Eigenvalues:
  - If there is some vector e and scalar λ such that $Ae = \lambda e$, then e is called the eigenvector corresponding to eigenvalue λ of the matrix A

- Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$
  - $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  - $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - When a matrix is applied to eigenvectors, the effect is simple!



Eigenvalues and Eigenvectors

- Define $T^{-1} = [e_1 \quad e_2 \quad \cdots \quad e_n]$
  - Then, $AT^{-1} = T^{-1}\Lambda$, where $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
  - $A = T^{-1}\Lambda T$. This is a **similarity transform**.
  - Define $z = Tx$, and we have $Ax = T^{-1}\Lambda Tx = T^{-1}\Lambda z$
    - In the coordinate system obtained from applying transformation T, the map A is diagonal
    - To obtain the result of applying A in the original coordinate system, transform back with T^{-1}

Obtaining Eigenvalues and Eigenvectors

- Hand calculation: $A = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$
  - Eigenvalues

$$Ae = \lambda e$$
$$Ae - \lambda I e = 0$$
$$(A - \lambda I)e = 0$$

This means the matrix $A - \lambda I$ has an eigenvalue of 0

Solve for λ in $\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)(2-\lambda) - 9 = 0$$

$$2 - \lambda = \pm 3$$
$$\lambda = 2 \pm 3 = -1{,}5$$

  - Eigenvectors

$$\lambda = -1: \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} e = 0 \Rightarrow e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 5: \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} e = 0 \Rightarrow e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- Matlab: `eig(A)`

Jordan Form

- Not all matrices are diagonalizable

$$J = TAT^{-1} = \begin{bmatrix} \lambda & 1 & 0 & & & \\ 0 & \lambda & 1 & & & \\ 0 & 0 & \lambda & & & \\ & & & \lambda & 1 & \\ & & & 0 & \lambda & \\ & & & & & \lambda \end{bmatrix}$$

$$\begin{aligned} Ae_1 &= \lambda e_1 \qquad & Av_1 &= \lambda v_1 + e_1 \qquad & Aw_1 &= \lambda w_1 + v_1 \\ Ae_2 &= \lambda e_2 & Av_2 &= \lambda v_2 + e_2 & & \\ Ae_3 &= \lambda e_3 & & & & \end{aligned}$$

$$T^{-1} = [e_1 \quad v_1 \quad w_1 \quad e_2 \quad v_2 \quad e_3]$$

  - This is the matrix structure for one eigenvalue
  - There may be more than one such blocks in general

- All matrices can be put into **Jordan form**
  - Matlab: `jordan(A)`
  - Note that the eigenvalues of J are the same as those of A

Imagine $\det(J - sI) = 0$

Functions of Matrices

- Consider a polynomial of a matrix, $f(A) = A^k$
  - $A^k = (T^{-1}JT)^k = (T^{-1}JT)\,(T^{-1}JT)\,(T^{-1}JT)\ldots(T^{-1}JT) = T^{-1}J^kT$
  - Adjacent T matrices and inverse cancel!

- This motivates general functions of matrices, like $f(A) = \sin A$ or $f(A) = e^{At}$, defined through Taylor series
  - $\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots$
  - $e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$

Functions of Matrices

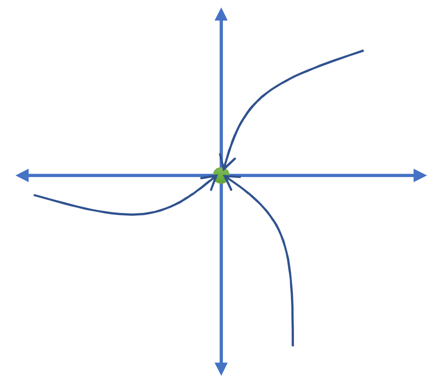
- Suppose $J = \begin{array}{cc} n & \\ \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} & n \end{array}$

- Then, $f(J) = \begin{bmatrix} f(\lambda) & f'(\lambda) & \cdots & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ & f(\lambda) & \ddots & \vdots \\ & & \ddots & f'(\lambda) \\ & & & f(\lambda) \end{bmatrix}$

- And $f(A) = T^{-1}f(J)T$, where $A = T^{-1}JT$
- **Spectral theorem**: the eigenvalues of $f(A)$ are $\{f(\lambda)\}$, where $\{\lambda\}$ are eigenvalues of A

Imagine $\det(f(J) - sI) = 0$

LTI System: Stability of $\dot{x} = Ax$



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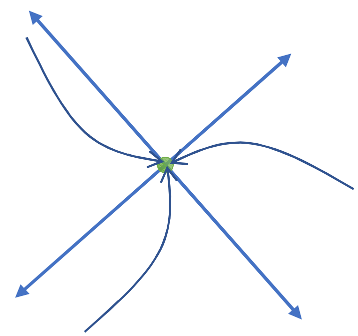
LTI System: Stability

- The system $\dot{x} = Ax$ is exponentially stable if and only if all eigenvalues of A are in the open left half plane, i.e. $\forall k, \mathrm{Re}(\lambda_k) < 0$
  - $z = Tx \Rightarrow \ \dot{z} = TAT^{-1}z = \Lambda z,\ z_0 = Tx_0$
  - $\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}$
  - If $\mathrm{Re}(\lambda_k) < 0$, $e^{\lambda_k t} \to 0$, so $z_k(t) = e^{\lambda_k t} z_{k0} \to 0$



- If $\max \mathrm{Re}(\lambda_k) = 0$, $z(t)$ stays bounded only if $\bar{\lambda}_k$ has Jordan block of size 1

Eigenvalue with largest real part

LTI System: Stability

- If $\max \mathrm{Re}(\lambda_k) = 0$, $z(t)$ stays bounded only if $\bar{\lambda}_k$ has Jordan block of size 1

$$e^{Jt}z_0 = \begin{bmatrix} e^{\lambda_1 t} & & & & & \\ & e^{\lambda_1 t} & te^{\lambda_1 t} & & & \\ & & e^{\lambda_1 t} & & & \\ & & & e^{\lambda_2 t} & te^{\lambda_2 t} & \frac{1}{2}t^2 e^{\lambda_2 t} \\ & & & & e^{\lambda_2 t} & te^{\lambda_2 t} \\ & & & & & e^{\lambda_2 t} \end{bmatrix} z_0$$

- When $\lambda_i = 0$…

LTI System: Stability

- If $\max \operatorname{Re}(\lambda_k) = 0$, $z(t)$ stays bounded only if $\bar{\lambda}_k$ has Jordan block of size 1

$$e^{Jt} z_0 = \begin{bmatrix} 1 & & & & & \\ & 1 & t & & & \\ & & 1 & & & \\ & & & 1 & t & \frac{1}{2}t^2 \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix} z_0$$

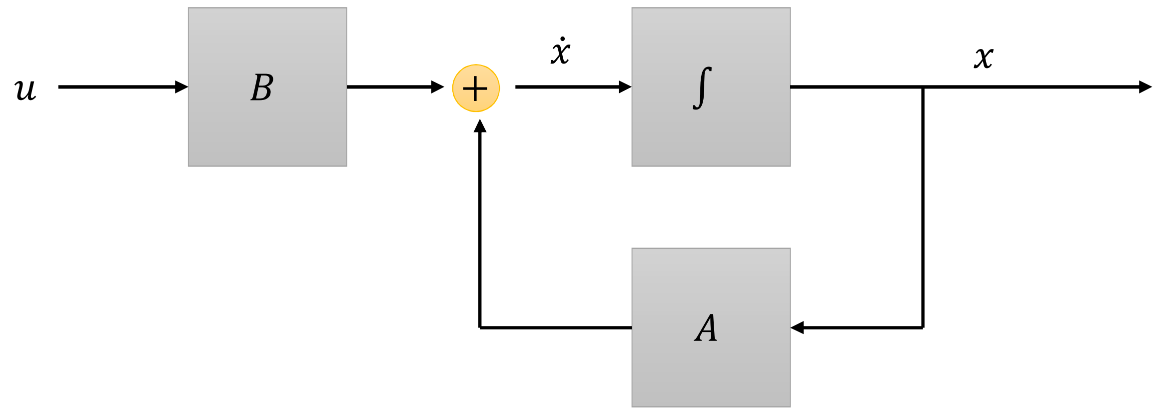
- When $\lambda_i = 0$…
- Not stable!

State Feedback Control

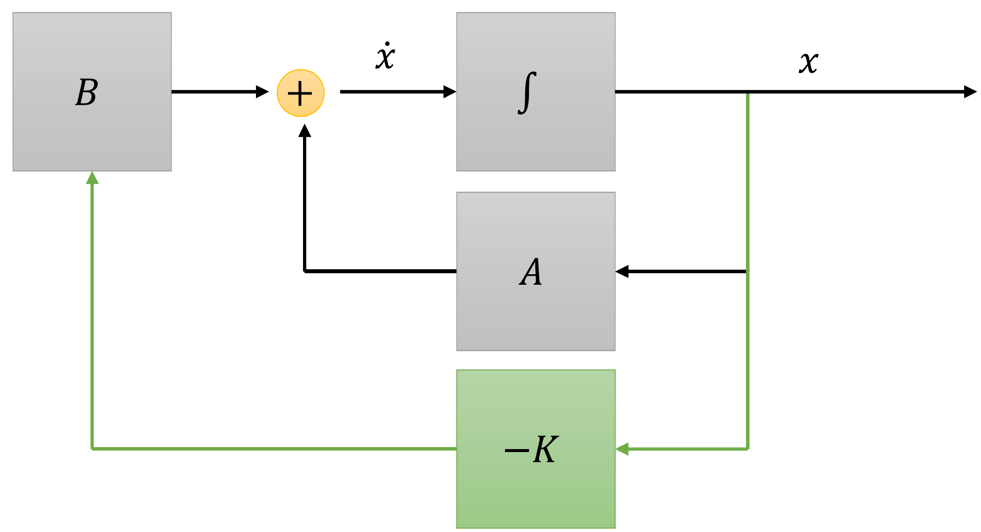
- Suppose $\dot{x} = Ax + Bu$, can we design u to make $x = \mathbf{0}_n$ stable?

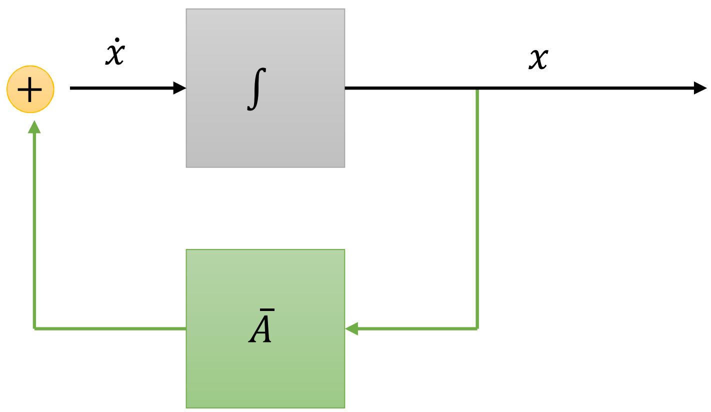
- Try linear state feedback: $u = -Kx \Rightarrow \dot{x} = (A - BK)x$
  - Define $\bar{A} = A - BK$, and we have $\dot{x} = \bar{A}x$
  - We can try to choose the elements of K, such that the eigenvalues of $\bar{A}$ are in the left half-plane

State Feedback



- System: $\dot{x} = Ax + Bu$
- **Open-loop** control: $u = u(t)$
- **Closed-loop** (linear state feedback) control: $u = -Kx$
  - $\dot{x} = Ax - BKx$
  - $\dot{x} = (A - BK)x$
  - $\dot{x} = \bar{A}x$, where $\bar{A} = A - BK$





Stabilization by State Feedback

- Suppose $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - Is the system stable when $u(t) \equiv 0$? No!
    - $Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$
    - $\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda = 2,2$

  - Choose K so that $u = -Kx$ stabilizes the system.
    - Let $K = [k_1 \quad k_2]$, then $\bar{A} \coloneqq A - BK = \begin{bmatrix} 2-k_1 & 2-k_2 \\ -k_1 & 2-k_2 \end{bmatrix}$
    - $\det(\bar{A} - \lambda I) = (2 - k_1 - \lambda)(2 - k_2 - \lambda) + (2 - k_2)k_1$
    - Choose k_1, k_2 such that $\det(\bar{A} - \lambda I) = 0$ gives λ in the open left half plane

Stabilization by State Feedback

- Choose K so that $u = -Kx$ stabilizes the system.
  - Let $K = [k_1 \quad k_2]$, then $\bar{A} \coloneqq A - BK = \begin{bmatrix} 2 - k_1 & 1 - k_2 \\ 1 - k_1 & 2 - k_2 \end{bmatrix}$
  - $0 = (2 - k_1 - \lambda)(2 - k_2 - \lambda) + (2 - k_2)k_1$
  - $0 = \lambda^2 + (k_1 + k_2 - 4)\lambda + (2 - k_1)(2 - k_2) + (2 - k_2)k_1$
  - $0 = \lambda^2 + (k_1 + k_2 - 4)\lambda + 4 - 2k_1 - 2k_2 + k_1k_2 + 2k_1 - k_1k_2$
  - $0 = \lambda^2 + (k_1 + k_2 - 4)\lambda + 4 - 2k_2$

- One choice: $\lambda = -2, -2$
  - $0 = (\lambda + 2\,)(\lambda + 2) = \lambda^2 + 4\lambda + 4$
  - $k_2 = 0, k_1 = 8$

State Feedback Control

- Suppose $\dot{x} = Ax + Bu$, can we design u to make $x = \mathbf{0}_n$ stable?

- Try linear state feedback: $u = -Kx \Rightarrow \dot{x} = (A - BK)x$
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  - We can try to choose the elements of K, such that the eigenvalues of $\bar{A}$ are in the left half-plane

- Issues
  - Controller saturation
  - Full state information required