Optimal Control and Differential Flatness

CMPT 419/983

Mo Chen
SFU Computing Science

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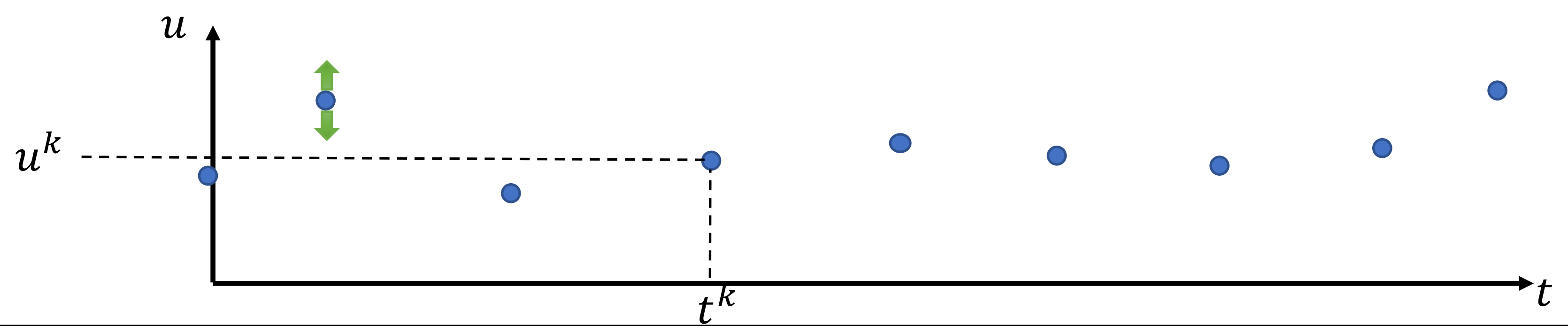
Nonlinear Optimization

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, i = 1, \ldots, n \\ & h_j(x) = 0, j = 1, \ldots, m \end{aligned}$$

- Nonlinear optimization:
  - Decision variable is $x \in \mathbb{R}^n$
  - $x \coloneqq (u^0, u^1, \ldots, u^n)$ could be the control



Optimal Control

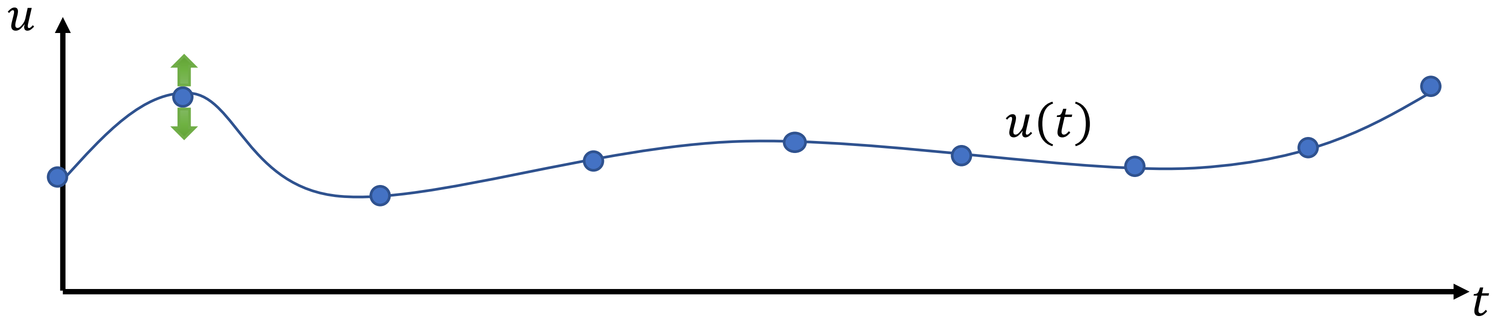
$$
\begin{aligned}
\underset{u(\cdot)}{\text{minimize}}\ & \overbrace{l(x(T),T)}^{\text{Final cost}} + \overbrace{\int_0^T c(x(t),u(t),t)dt}^{\text{Running cost}} \\
\text{subject to}\ & \dot{x}(t) = f\big(x(t),u(t)\big) \\
& g\big(x(t),u(t)\big) \geq 0 \\
& x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, x(0) = x_0
\end{aligned}
$$

Cost functional, $J\big(x(\cdot),u(\cdot)\big)$

Dynamic model

Additional constraints
- Eg. actuation limits

- Nonlinear optimization:
  - Decision variable is $x \in \mathbb{R}^n$
- Optimal control:
  - Decision variable is a **function** $u(\cdot)$



Optimal Control

$$\underset{u(\cdot)}{\text{minimize}}\ \overbrace{l(x(T),T)}^{\text{Final cost}} + \overbrace{\int_0^T c(x(t),u(t),t)dt}^{\text{Running cost}}$$

$$\text{subject to } \dot{x}(t) = f\big(x(t),u(t)\big)$$

$$g\big(x(t),u(t)\big) \geq 0$$

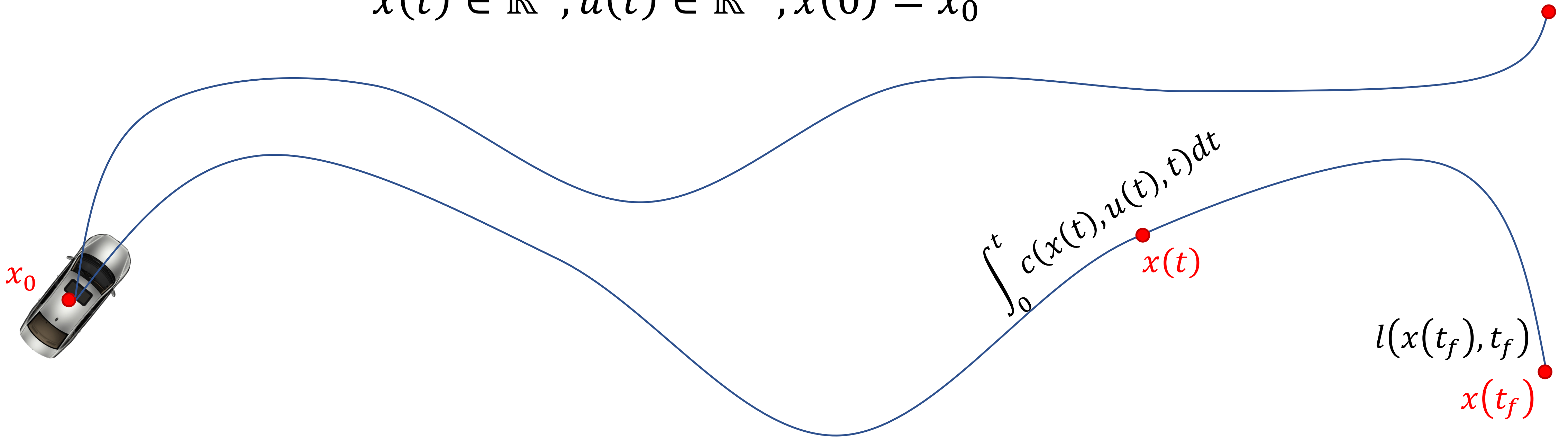
$$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, x(0) = x_0$$

Cost functional, $J\big(x(\cdot),u(\cdot)\big)$

Dynamic model

Additional constraints

- Eg. actuation limits



Optimal Control: Facts

$$\underset{u(\cdot)}{\text{minimize}}\ l(x(T),T) + \int_0^T c(x(t),u(t),t)dt$$

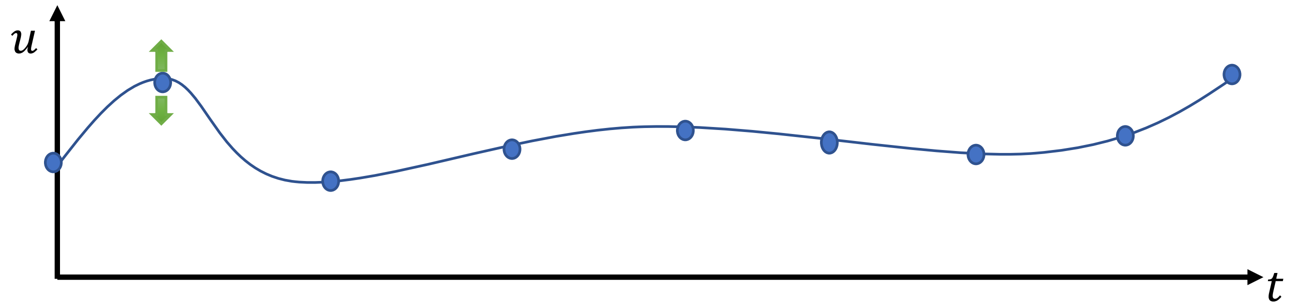
$$\text{subject to } \dot{x}(t) = f\big(x(t),u(t)\big)$$

- Observation 1: Discretize time → nonlinear optimization problem
- Fact 1: Minimizing “cost” is same as maximizing “reward”
- Fact 2: Discretize time + maximizing reward → “reinforcement learning problem”

Optimal Control: Types of Solutions

$$\underset{u(\cdot)}{\text{minimize}}\ l(x(T),T) + \int_0^T c(x(t),u(t),t)dt$$

$$\text{subject to } \dot{x}(t) = f\big(x(t),u(t)\big)$$



- Open-loop control
  - Find $u(t)$ for $t \in [0,T]$
  - Scalable, but errors will add up
- Closed-loop control
  - Find $u(t,x)$ for $t \in [0,T]$, $x \in \mathbb{R}^n$
  - Not scalable, but robust
  - "Special" techniques needed (eg. Reinforcement learning) for large n
- Receding horizon control:
  - Find $u(t)$ for $t \in [0,T]$, use $u(t)$ for $t \in [0,h]$, then find $u(t)$ for $t \in [h,T+h]$ and repeat
  - Has features of both open- and closed-loop control

Optimal Control: Variants

- For now: Deterministic systems, continuous time, continuous state

- Other variations:
  - Stochastic
  - Discrete time
  - Discrete state

Outline: Open-Loop Control

- Optimal Control Problems
- Differential flatness
- Direct Methods (Numerical Methods)
  - Shooting methods
  - Collocation
  - CasADi Matlab toolbox

Optimal Control

$$\underset{u(\cdot)}{\text{minimize}}\ \overbrace{l(x(T),T)}^{\text{Final cost}} + \overbrace{\int_0^T c(x(t),u(t),t)dt}^{\text{Running cost}}$$

$$\text{subject to } \dot{x}(t) = f\big(x(t),u(t)\big)$$

$$g\big(x(t),u(t)\big) \geq 0$$

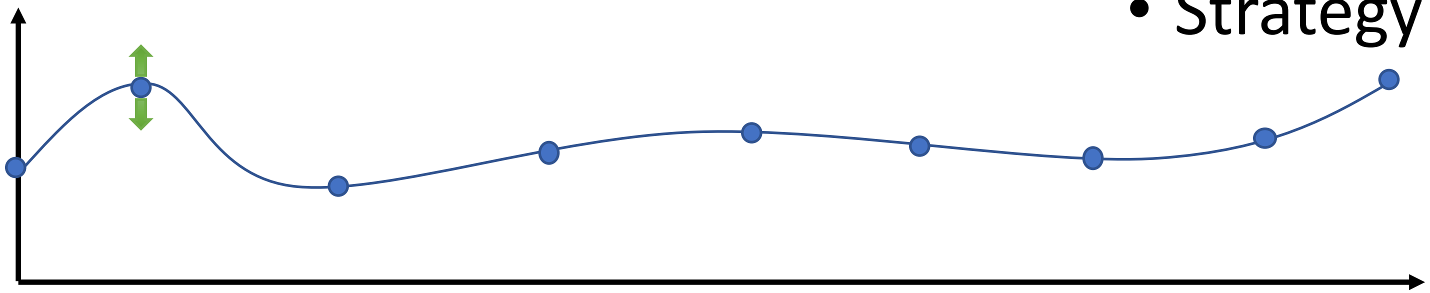
$$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, x(0) = x_0$$

Cost functional, $J\big(x(\cdot),u(\cdot)\big)$

Dynamic model

Additional constraints
- Eg. actuation limits

- Optimal control:
  - Decision variable is a function $u(\cdot)$
  - Other constraints are possible
    - E.g. $x(T) = x_f$



- Strategy 1: Optimality conditions
- Strategy 2: Discretize first → nonlinear optimization
- Strategy 3: Use differential flatness (if lucky)

Differential Flatness

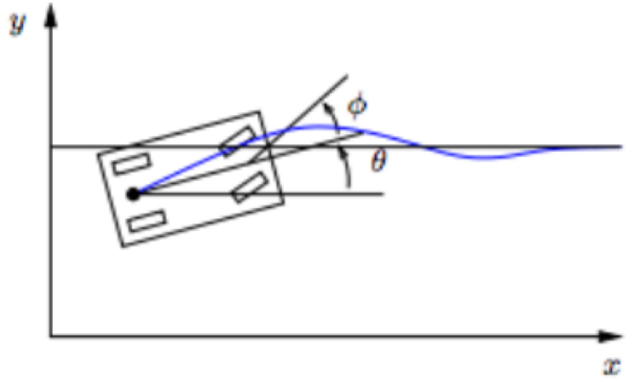
- Problem: find a $u(\cdot)$ such that

$$\dot{x}(t) = f(x,u)$$
$$x(0) = x_0$$
$$x(T) = x_f$$

  - Worry about feasibility for now, and ignore cost

- Example: vehicle steering
  - State: (x, y, θ)
  - Inputs: (v, ϕ)

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \frac{v}{l}\tan\phi$$



Use Special Structure

Dynamics:

$$\dot{x} = v\cos\theta$$

$$\dot{y} = v\sin\theta$$

$$\dot{\theta} = \frac{v}{l}\tan\phi$$

- First, suppose $x(t), y(t)$ are smooth and **given**.

1. Obtain heading: $$\frac{\dot{y}}{\dot{x}} = \frac{\sin\theta}{\cos\theta} \Rightarrow \theta = \arctan\left(\frac{\dot{y}}{\dot{x}}\right)$$

2. Obtain speed $$\begin{aligned}\dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta\end{aligned} \quad \Rightarrow v = \sqrt{(\dot{x})^2 + (\dot{y})^2}$$

3. Obtain steering angle $$\dot{\theta} = \frac{v}{l}\tan\phi \Rightarrow \phi = \arctan\left(\frac{l\dot{\theta}}{v}\right)$$

- All state variables and control inputs can be determined from the given trajectory!

Differential Flatness Definition

A nonlinear system $\dot{x} = f(x, u)$ is differentially flat if there exists a function α such that

$$z = \alpha\left(x, u, \dots, u^{(p)}\right)$$

and we can write the solutions of the nonlinear system as functions of z and a finite number of derivatives

$$x = \beta\left(z, \dot{z}, \dots, z^{(q)}\right)$$
$$u = \gamma\left(z, \dot{z}, \dots, z^{(q)}\right)$$

Differential Flatness Definition

Generic system		**Kinematic car** $x(t), y(t)$ are smooth and given
$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u)$	$\longrightarrow$	$\dot{x} = v\cos\theta$ $\dot{y} = v\sin\theta$ $\dot{\theta} = \frac{v}{l}\tan\phi$
$z = \alpha\left(\boldsymbol{x}, u, \dots, u^{(p)}\right)$	$\longrightarrow$	$z = (x, y)$
$\boldsymbol{x} = \beta\left(z, \dot{z}, \dots, z^{(q)}\right)$	$\longrightarrow$	$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} x \\ y \\ \arctan\left(\frac{\dot{x}}{\dot{y}}\right) \end{bmatrix}$
$u = \gamma\left(z, \dot{z}, \dots, z^{(q)}\right)$	$\longrightarrow$	$\begin{bmatrix} v \\ \phi \end{bmatrix} = \begin{bmatrix} \sqrt{(\dot{x})^2 + (\dot{y})^2} \\ \arctan\left(\frac{l\dot{\theta}}{v}\right) \end{bmatrix}$

Trajectory Generation

- Before: $x(t)$ was given
- Now: find a feasible trajectory $x(t)$ that satisfies

$$\dot{x}(t) = f\big(x(t), u(t)\big)$$
$$x(0) = x_0$$
$$x(T) = x_f$$

- Differential flatness: $\boldsymbol{x} = \beta\big(z, \dot{z}, \dots, z^{(q)}\big)$

$$\Rightarrow x(0) = \beta\left(z(0), \dot{z}(0), \dots, z^{(q)}(0)\right) = x_0$$
$$x(T) = \beta\left(z(T), \dot{z}(T), \dots, z^{(q)}(T)\right) = x_f$$

- Let $z(t) = \sum_{i=1}^{N} b_i \psi_i(t) \Rightarrow \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t)$

$$\vdots$$
$$z^{(q)}(t) = \sum_{i=1}^{N} b_i \psi_i^{(q)}(t)$$

ψ_i: basis functions

- Your choice!
- You can choose N too!
- E.g. $\psi_i = x^{i-1}$ -- monomial basis

Trajectory Generation

- Differential flatness:

$$x(0) = \beta\left(z(0), \dot{z}(0), \dots, z^{(q)}(0)\right) = x_0$$
$$x(T) = \beta\left(z(T), \dot{z}(T), \dots, z^{(q)}(T)\right) = x_f$$

$$z(t) = \sum_{i=1}^{N} b_i \psi_i(t), \quad \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t), \quad \cdots \quad z^{(q)}(t) = \sum_{i=1}^{N} b_i \phi_i^{(q)}(t)$$

$$\Rightarrow \begin{bmatrix} \psi_1(0) & \psi_2(0) & \cdots & \psi_N(0) \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \\ \\ \\ \end{bmatrix}$$

Trajectory Generation

- Differential flatness:

$$x(0) = \beta\left(z(0), \dot{z}(0), \dots, z^{(q)}(0)\right) = x_0$$
$$x(T) = \beta\left(z(T), \dot{z}(T), \dots, z^{(q)}(T)\right) = x_f$$

$$z(t) = \sum_{i=1}^{N} b_i \psi_i(t), \quad \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t), \quad \cdots \quad z^{(q)}(t) = \sum_{i=1}^{N} b_i \phi_i^{(q)}(t)$$

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Trajectory Generation

- Differential flatness:

$$x(0) = \beta\left(z(0), \dot{z}(0), \dots, z^{(q)}(0)\right) = x_0$$
$$x(T) = \beta\left(z(T), \dot{z}(T), \dots, z^{(q)}(T)\right) = x_f$$

$$z(t) = \sum_{i=1}^{N} b_i \psi_i(t)\,, \quad \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t)\,, \qquad \cdots \qquad z^{(q)}(t) = \sum_{i=1}^{N} b_i \phi_i^{(q)}(t)$$

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What to do with $\boldsymbol{b}$?

$$\begin{bmatrix} \psi_1(0) & \psi_2(0) & \cdots & \psi_N(0) \\ \dot{\psi}_1(0) & \dot{\psi}_2(0) & \cdots & \dot{\psi}_N(0) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(q)}(0) & \psi_2(0)^{(q)} & \cdots & \psi_N^{(q)}(0) \\ \psi_1(T) & \psi_2(T) & \cdots & \psi_N(T) \\ \dot{\psi}_1(T) & \dot{\psi}_2(T) & \cdots & \dot{\psi}_N(T) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(q)}(T) & \psi_2^{(q)}(T) & \cdots & \psi_N^{(q)}(T) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ \vdots \\ z_1^{(q)}(0) \\ z_1(T) \\ \dot{z}_1(T) \\ \vdots \\ z_1^{(q)}(T) \end{bmatrix}$$

$$z(t) = \sum_{i=1}^{N} b_i \psi_i(t), \quad \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t), \quad \cdots \quad z^{(q)}(t) = \sum_{i=1}^{N} b_i \phi_i^{(q)}(t)$$

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$$z(t) = \sum_{i=1}^{N} b_i \psi_i(t), \quad \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t), \quad \cdots \quad z^{(q)}(t) = \sum_{i=1}^{N} b_i \phi_i^{(q)}(t)$$

$$\boldsymbol{x} = \boldsymbol{\beta}\left(\boldsymbol{z}, \dot{\boldsymbol{z}}, \dots, \boldsymbol{z}^{(\boldsymbol{q})}\right)$$

$$\boldsymbol{u} = \boldsymbol{\gamma}\left(\boldsymbol{z}, \dot{\boldsymbol{z}}, \dots, \boldsymbol{z}^{(\boldsymbol{q})}\right)$$

What to do with $\boldsymbol{b}$?

$$(**)\qquad \begin{bmatrix} \psi_1(0) & \psi_2(0) & \cdots & \psi_N(0) \\ \dot{\psi}_1(0) & \dot{\psi}_2(0) & \cdots & \dot{\psi}_N(0) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(q)}(0) & \psi_2(0)^{(q)} & \cdots & \psi_N^{(q)}(0) \\ \psi_1(T) & \psi_2(T) & \cdots & \psi_N(T) \\ \dot{\psi}_1(T) & \dot{\psi}_2(T) & \cdots & \dot{\psi}_N(T) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(q)}(T) & \psi_2^{(q)}(T) & \cdots & \psi_N^{(q)}(T) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} z_1(0) \\ \dot{z}_1(0) \\ \vdots \\ z_1^{(q)}(0) \\ z_1(T) \\ \dot{z}_1(T) \\ \vdots \\ z_1^{(q)}(T) \end{bmatrix}$$

$$z(t) = \sum_{i=1}^{N} b_i \psi_i(t), \quad \dot{z}(t) = \sum_{i=1}^{N} b_i \dot{\psi}_i(t), \quad \cdots \quad z^{(q)}(t) = \sum_{i=1}^{N} b_i \phi_i^{(q)}(t)$$

$$\boldsymbol{x} = \boldsymbol{\beta}\left(\boldsymbol{z}, \dot{\boldsymbol{z}}, \dots, \boldsymbol{z}^{(q)}\right)$$

$$\boldsymbol{u} = \boldsymbol{\gamma}\left(\boldsymbol{z}, \dot{\boldsymbol{z}}, \dots, \boldsymbol{z}^{(q)}\right)$$

ψ_i: basis functions

- Your choice!
- You can choose N too!
- E.g. $\psi_i = x^{i-1}$ -- monomial basis

q is from dynamics

- Determines number of rows
  - i.e. number of equations
- Can't choose this

N is chosen

- Determines number of columns
  - i.e. number of variables in b

- N too small: no solutions
- N very large: many solutions

Optimal Control Problem

$$\underset{u(\cdot)}{\text{minimize}}\ \underbrace{l(x(T),T)}_{\text{Final cost}} + \underbrace{\int_0^T c(x(t),u(t),t)dt}_{\text{Running cost}}$$

$$\text{subject to } \dot{x}(t) = f\big(x(t),u(t)\big)$$

$$g\big(x(t),u(t)\big) \geq 0$$

$$x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m,$$

$$x(0) = x_0, x(T) = x_f$$

Cost functional, $J\big(x(\cdot),u(\cdot)\big)$

Dynamic model

Additional constraints

- Eg. actuation limits

Optimal Control Problem

$$\underset{u(\cdot)}{\text{minimize}}\ \underbrace{l(x(T),T)}_{\text{Final cost}} + \underbrace{\int_0^T c(x(t),u(t),t)dt}_{\text{Running cost}}$$

$$\text{subject to } \dot{x}(t) = f\big(x(t),u(t)\big)$$

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$$x(0) = x_0, x(T) = x_f$$

Cost functional, $J\big(x(\cdot),u(\cdot)\big)$

Dynamic model

Additional constraints

- Eg. actuation limits

$$\underset{b}{\text{minimize}}\ l(x(T),T) + \int_0^T c(x(t),u(t),t)dt$$

$$\text{subject to } (**)$$

$$g\big(x(t),u(t)\big) \geq 0$$

$$u(t) \in \mathbb{R}^m, x(0) = x_0$$

Optimal Control Problem

$$\underset{\boldsymbol{u}(\cdot)}{\text{minimize}} \quad \underbrace{l(x(T),T)}_{\text{Final cost}} + \underbrace{\int_0^T c(x(t),\boldsymbol{u}(\boldsymbol{t}),t)dt}_{\text{Running cost}}$$

Cost functional, $J(x(\cdot),u(\cdot))$

$$\text{subject to } \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{x}(\boldsymbol{t}),\boldsymbol{u}(\boldsymbol{t}))$$

Dynamic model

$$g(x(t),u(t)) \geq 0$$

Additional constraints

- Eg. actuation limits

$$\boldsymbol{x}(\boldsymbol{t}) \in \mathbb{R}^{\boldsymbol{n}}, u(t) \in \mathbb{R}^m,$$

$$\boldsymbol{x}(\boldsymbol{0}) = \boldsymbol{x_0}, \boldsymbol{x}(\boldsymbol{T}) = \boldsymbol{x_f}$$

$$\underset{\boldsymbol{b}}{\text{minimize}} \quad l(x(T),T) + \int_0^T c(x(t),u(t),t)dt$$

$$\text{subject to } (**)$$

$$g(x(t),u(t)) \geq 0$$

$$u(t) \in \mathbb{R}^m, x(0) = x_0$$

$$\boldsymbol{z} = \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{u},\ldots,\boldsymbol{u}^{(p)})$$

$$\boldsymbol{x} = \boldsymbol{\beta}(\boldsymbol{z},\dot{\boldsymbol{z}},\ldots,\boldsymbol{z}^{(q)})$$

$$\boldsymbol{u} = \boldsymbol{\gamma}(\boldsymbol{z},\dot{\boldsymbol{z}},\ldots,\boldsymbol{z}^{(q)})$$

Key Points

$$\underset{\boldsymbol{b}}{\text{minimize}} \; l(x(T), T) + \int_0^T c(x(t), u(t), t)dt$$

subject to $(**)$

$$g\big(x(t), u(t)\big) \geq 0$$

$$u(t) \in \mathbb{R}^m, x(0) = x_0$$

$$\boldsymbol{z} = \boldsymbol{\alpha}\big(\boldsymbol{x}, \boldsymbol{u}, \dots, \boldsymbol{u}^{(p)}\big)$$

$$\boldsymbol{x} = \boldsymbol{\beta}\big(\boldsymbol{z}, \dot{\boldsymbol{z}}, \dots, \boldsymbol{z}^{(q)}\big)$$

$$\boldsymbol{u} = \boldsymbol{\gamma}\big(\boldsymbol{z}, \dot{\boldsymbol{z}}, \dots, \boldsymbol{z}^{(q)}\big)$$

- Trajectory generation via solving algebraic equations $(**)$
- Other constraints can be transformed into z space
- Cost/performance index also transformed into z space
- After obtaining b, we can obtain x and u

- Quadrotors are differentially flat
  - D. Mellinger and V. Kumar. *Minimum snap trajectory generation and control for quadrotors*, ICRA 2011.