Continuous Time LQR and Robotic Safety via Reachability

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References

- Dynamic Programming:
  - Bertsekas, Dynamic Programming and Optimal Control. Athena Scientific, 2017, 1886529434.

- Reachability Analysis:
  - Chen & Tomlin. “Hamilton-Jacobi Reachability: Some Recent Theoretical Advances and Applications in Unmanned Airspace Management”. *Annual Review*.

Hamilton-Jacobi Equation

- Hamilton-Jacobi partial differential equation

$$\frac{\partial V}{\partial t} + \min_u \left[c(x,u) + \frac{\partial V}{\partial x} \cdot f(x,u) \right] = 0, \qquad V(T,x) = l(x)$$

- Minimization over u is typically easy
  - Most systems are control affine: $f(x,u)$ has the form $f(x) + g(x)u$
  - Control constraints are typically “box” constraints, e.g. $|u_i| \leq 1$

- PDE is solved on a grid
  - $x \in \mathbb{R}^n$ means $V(t,x)$ is computed on an $(n+1)$-dimensional grid

- $V(t,x)$ is often not differentiable (or continuous)
  - Viscosity solutions
  - Lax Friedrichs numerical method

Example: Continuous LQR

$$J(t, x(t)) = \int_t^T c(x(s), u(s))ds + l(x(T))$$

- Linear system: $\dot{x} = Ax + Bu$
- Cost involving quadratic expressions:

$$J(t, x) = \frac{1}{2}\int_t^T \left(x(t)^\top Q x(t) + u(t)^\top R u(t)\right)dt + \frac{1}{2}x(T)^\top L x(T)$$

  - L, Q, R are symmetric positive semidefinite
  - T is given
  - $x(t)$ and $u(t)$ are unconstrained
- The Hamilton-Jacobi equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\min_u \Big[\underbrace{x(t)^\top Q x(t) + u(t)^\top R u(t) + \frac{\partial V}{\partial x} \cdot \left(Ax(t) + Bu(t)\right)}_{\text{Pre-Hamiltonian: } H\left(x, u, \frac{\partial V}{\partial x}\right)}\Big] = 0$$

Continuous Time LQR

- Pre-Hamiltonian: $H\left(x, u, \frac{\partial V}{\partial x}\right) = x(t)^\top Q x(t) + u(t)^\top R u(t) + \frac{\partial V}{\partial x} \cdot \left(Ax(t) + Bu(t)\right)$
  - Take Jacobian to optimize H: $\frac{\partial H}{\partial u} = Ru(s) + B \cdot \frac{\partial V}{\partial x}$
  - Observe that $\frac{\partial^2}{\partial u^2} = R \succcurlyeq 0$, so first order condition is sufficient
- Setting $\frac{\partial H}{\partial u}$ to zero, we get $u^*(t) = -R^{-1} B^\top \frac{\partial V}{\partial x}$
  - Plugging this back into H, we get the Hamiltonian:

$$H^*\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2} x(t)^\top Q x(t) + \frac{1}{2}\left(R^{-1} B^\top \frac{\partial V}{\partial x}\right)^\top R R^{-1} B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top \left(Ax - B R^{-1} B^\top \frac{\partial V}{\partial x}\right)$$

$$H^*\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2} x(t)^\top Q x(t) + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top B R^{-1} B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top \left(Ax - B R^{-1} B^\top \frac{\partial V}{\partial x}\right)$$

$$H^*\left(x, \frac{\partial V}{\partial x}\right) = \frac{1}{2} x(t)^\top Q x(t) - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top B R^{-1} B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top Ax$$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}x(t)^\top Q x(t) - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top B R^{-1} B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top A x = 0, \qquad V\big(T, x(T)\big) = \frac{1}{2}x(T)^\top L x(T)$$

- Strategy for obtaining solution: guess something that works
  - $V(t,x) = \frac{1}{2}x^\top K(t) x,\ K(t) \geq 0$

$$\frac{\partial V}{\partial t} = \frac{1}{2}x^\top \dot{K}(t) x, \qquad \frac{\partial V}{\partial x} = K(t)x$$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Q x(t) - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^\top BR^{-1}B^\top \frac{\partial V}{\partial x} + \left(\frac{\partial V}{\partial x}\right)^\top Ax = 0, \qquad V\left(x(t_f), t_f\right) = \frac{1}{2}x(t_f)^\top L x(t_f)$$

- Strategy for obtaining solution: guess something that works
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Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Qx(t) - \frac{1}{2}x^\top K(t)^\top BR^{-1}B^\top K(t)x + x^\top K(t)^\top Ax = 0, \qquad V\left(x\left(t_f\right), t_f\right) = \frac{1}{2}x\left(t_f\right)^\top Lx\left(t_f\right)$$

- Strategy for obtaining solution: guess something that works
  - $V(t, x) = \frac{1}{2}x^\top K(t)x,\ K(t) \geq 0$

$$\frac{\partial V}{\partial t} = \frac{1}{2}x^\top \dot{K}(t)x, \qquad \frac{\partial V}{\partial x} = K(t)x$$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Q x(t) - \frac{1}{2}x^\top K(t)^\top B R^{-1} B^\top K(t)x + \underbrace{x^\top K(t)^\top A x}_{\frac{1}{2}x^\top K(t)^\top A x + \frac{1}{2}x^\top A^\top K(t) x} = 0$$

Scalar, therefore symmetric

  - Collect like terms

$$\frac{1}{2}x^\top \left(\dot{K}(t) + Q - K(t)^\top B R^{-1} B^\top K(t) + K(t)^\top A + A^\top K(t)\right) x = 0$$

  - This equation must hold for all $x(t)$, so

$$\dot{K}(t) + Q - K(t)^\top B R^{-1} B^\top K(t) + K(t)^\top A + A^\top K(t) = 0$$

  - Boundary condition: $V\big(T, x(T)\big) = \frac{1}{2}x(T)^\top L x(T)$
    - Therefore $K(T) = L$

Continuous Time LQR

- Hamilton-Jacobi equation:

$$\frac{1}{2}x^\top \dot{K}(t)x + \frac{1}{2}x(t)^\top Q x(t) - \frac{1}{2}x^\top K(t)^\top B R^{-1} B^\top K(t) x + x^\top K(t)^\top A x = 0, \qquad V(x(T),T) = \frac{1}{2}x(T)^\top L x(T)$$

- PDE becomes ODE:

$$\dot{K}(t) + Q - K(t)^\top B R^{-1} B^\top K(t) + K(t)^\top A + A^\top K(t) = 0, \qquad K(T) = L$$

  - “Riccati equation”
  - Integrate backwards in time
  - Optimal control is linear state feedback!

$$u^*(t,x) = -R^{-1}B^\top \frac{\partial V}{\partial x} = -R^{-1}B^\top K(t)x$$

Comments

- No control constraint

- What if there is control constraint?
  - Easy: Let controllers saturate
  - Difficult but proper: Explicitly treat it in the minimization of J

Suppose $|u(t)| \leq 1$ is a constraint
- Use $u(t) = -R^{-1}B^\top K(t)x$ if $|-R^{-1}B^\top K(t)x| \leq 1$
- Use $u(t) = 1$ if $-R^{-1}B^\top K(t)x > 1$
- Use $u(t) = -1$ if $-R^{-1}B^\top K(t)x < -1$

- MATLAB commands
  - Discrete time: `dlqr`; continuous time: `lqr`

- In general, need to solve

$$\frac{\partial V}{\partial t} + \min_u \left[C(x,u) + \frac{\partial V}{\partial x} \cdot f(x,u) \right] = 0, \qquad V(x,T) = l(x)$$

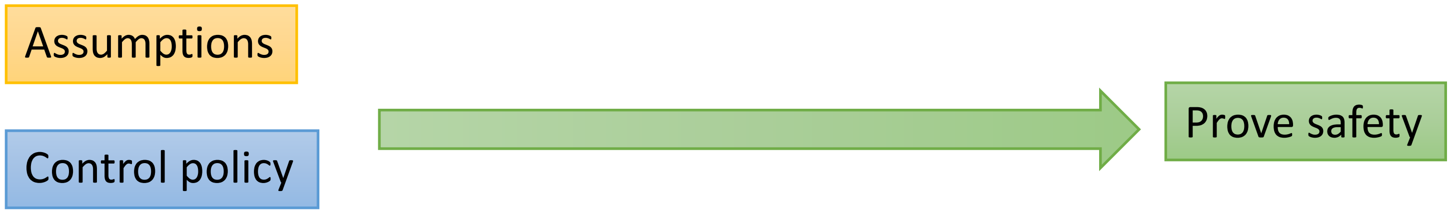
  - $V(x,t)$ is $(n+1)$-dimensional, if $x \in \mathbb{R}^n$
  - Optimal state feedback control: $u^*(t,x) = \arg\min_u \left[C(x,u) + \frac{\partial V}{\partial x} \cdot f(x,u) \right]$

Time Horizon

- Finite time horizon problems
  - Time-varying value function $V(t, x)$: optimal cost from some time and state
  - Time-varying control policy $u^*(t, x)$: achieves optimal cost
- Infinite time horizon problems
  - Let final time be 0, and apply DP backwards until convergence
  - Convergence not guaranteed; if $V(t, x)$ converges, then we have a time-invariant value function and control policy
    - $V_\infty(x), u^*_\infty(x)$

Robotic Safety

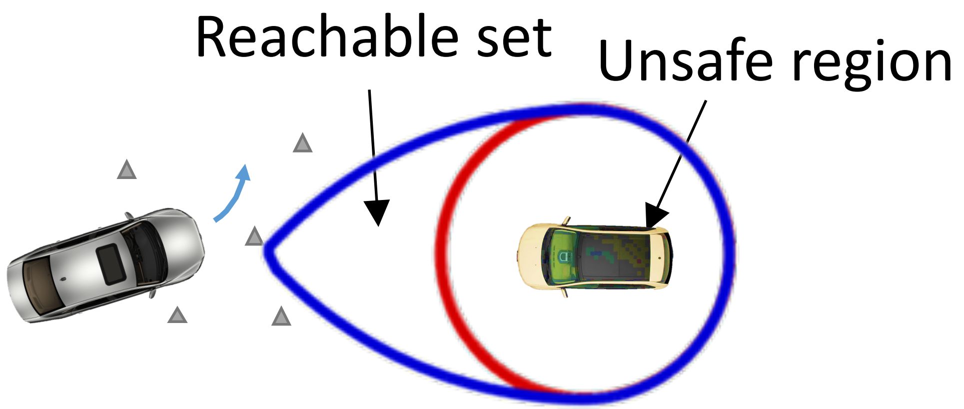
- Verification methods



  - Considers all possible system behaviours, given assumptions

- Can be written as an optimal control problem

Reachability Analysis: Avoidance



Assumptions:
- Model of robot
- Unsafe region: Obstacle

→

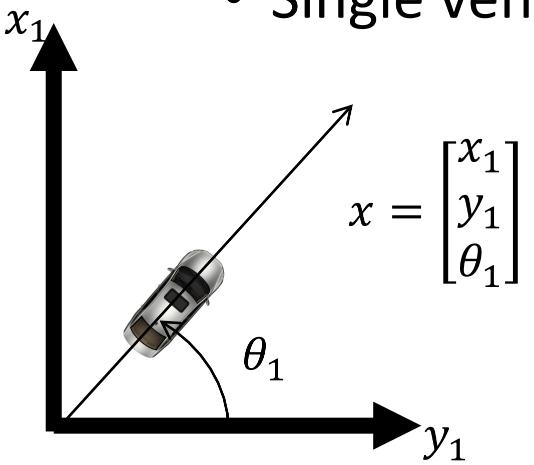
Control policy

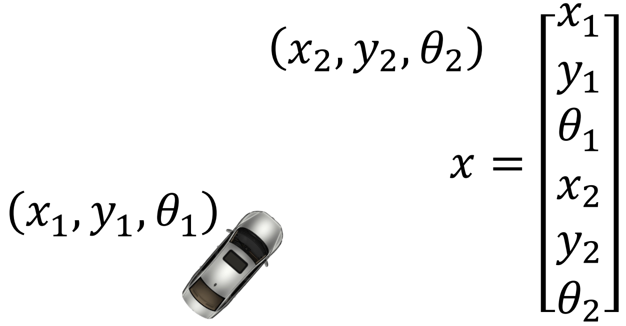
Backward reachable set
(States leading to danger)

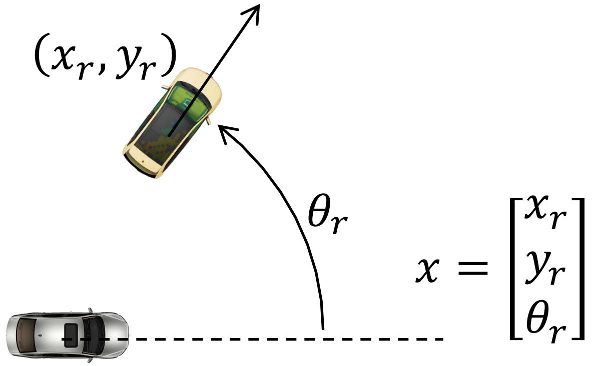
Assumptions

- System dynamics: $\dot{x} = f(x, u, d), t \leq 0$ (by convention, final time is 0)

- State x
  - Single vehicle, multiple vehicle, relative coordinates



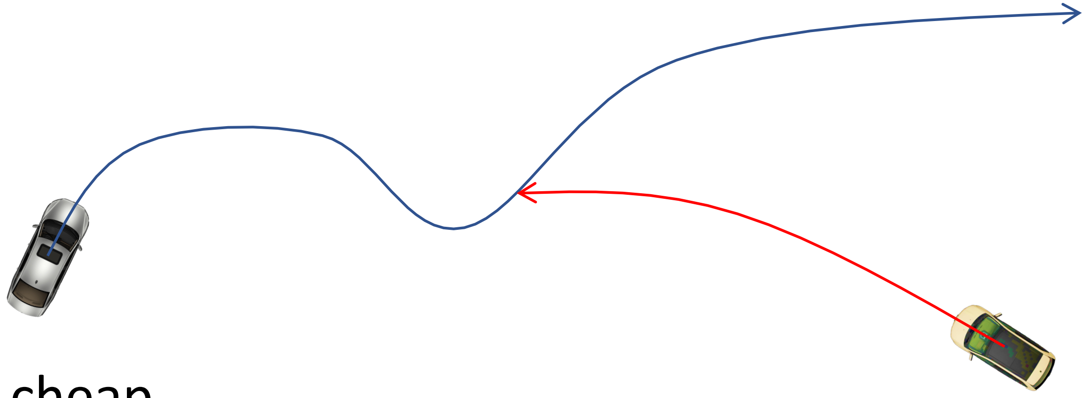




- Disturbance d: uncontrolled factors that affect the system, such as wind
  - Can be used to model other agents, when state includes them
  - Assume worst case

Information Pattern

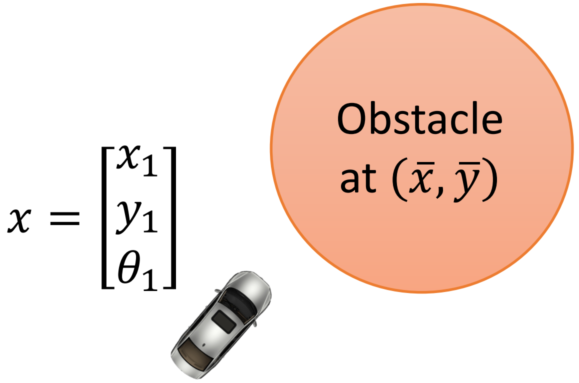
- Control: chosen by “ego” robot
- Disturbances: chosen by other robot (or weather gods)
  - Assume worst case



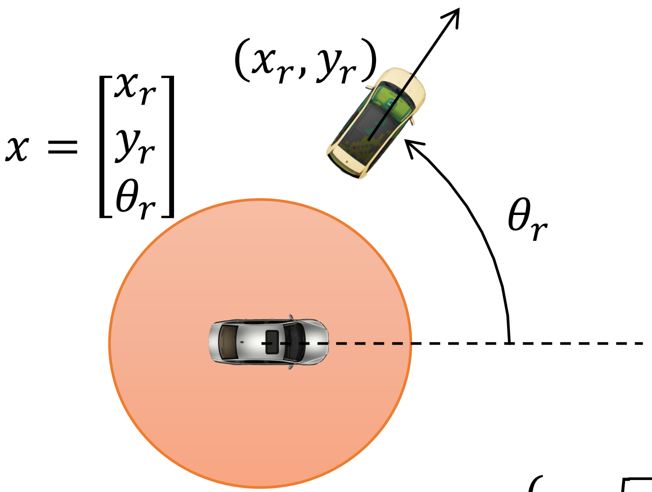
- “Open-loop” strategies
  - Ego robot declares entire plan
  - Other robot responds optimally (worst-case)
  - Conservative, unrealistic, but computationally cheap
- “Non-anticipative” strategies
  - Other robot acts based on state and control trajectory up current time
  - Notation: $d(\cdot) = \Gamma[u](\cdot)$
  - Disturbance still has the advantage: it gets to react to the control!

Assumptions

- "Target set", $\mathcal{T}$
  - Can specify set of states leading to danger
  - Expressed through set notation

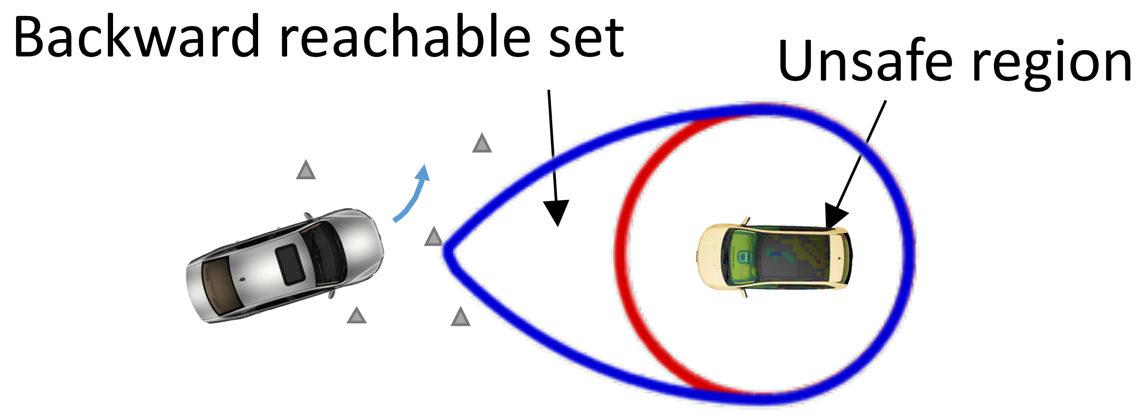


$$\mathcal{T} = \left\{x: \sqrt{(x_1 - \bar{x})^2 + (y_1 - \bar{y})^2} \leq r\right\} \subseteq \mathbb{R}^3$$



$$\mathcal{T} = \left\{x: \sqrt{x_r^2 + y_r^2} \leq R\right\} \subseteq \mathbb{R}^3$$

Reachability Analysis: Avoidance

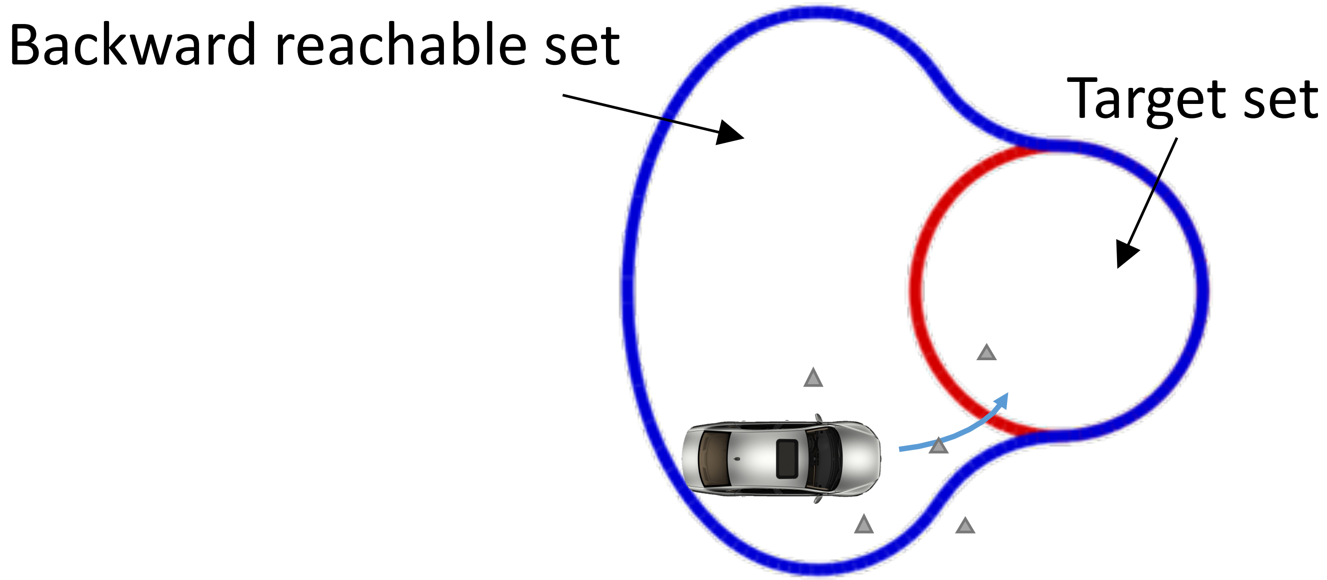


- Model of robot
- Unsafe region

Control policy

Backward reachable set
(States leading to danger)

Reachability Analysis: Goal Reaching



- Model of robot
- Goal region

Control policy

Backward reachable set
(States leading to goal)

Reachability Analysis

States at time t satisfying the following:

there exists a disturbance such that for all control, system enters target set at $t = 0$

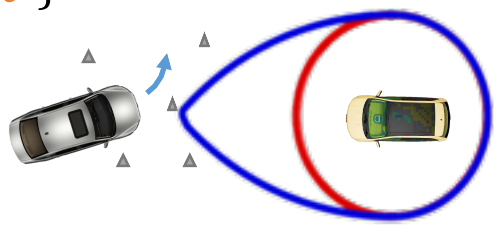
- $\mathcal{A}(t) = \{\bar{x}: \exists\Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x,u,d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$

- Model of robot
- Unsafe region

→

Backward reachable set (States leading to danger)

Control policy



- $\dot{x} = f(x,u,d)$
- $\mathcal{T}$

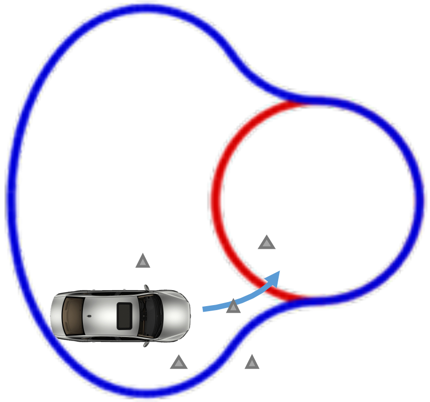
- $u^*(t,x)$

- Model of robot
- Goal region

→

Control policy

Backward reachable set (States leading to goal)



- $\mathcal{R}(t) = \{\bar{x}: \forall\Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x,u,d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$

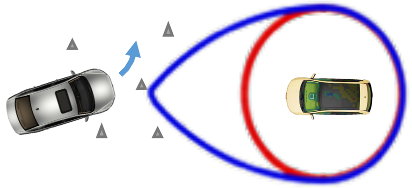
States at time t satisfying the following:

for all disturbances, there exists a control such that system enters target set at $t = 0$

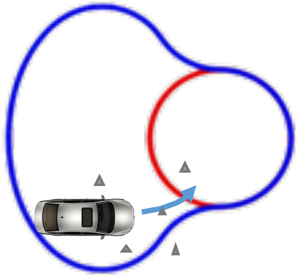
Terminology

- Minimal backward reachable set
  - $\mathcal{A}(t) = \{\bar{x}\colon \exists \Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x,u,d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$
  - Control minimizes size of reachable set

- Maximal backward reachable set
  - $\mathcal{R}(t) = \{\bar{x}\colon \forall \Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x,u,d), x(t) = \bar{x}, x(0) \in \mathcal{T}\}$
  - Control maximizes size of reachable set



- Minimal and maximal backward reachable tube
  - $\bar{\mathcal{A}}(t) = \{\bar{x}\colon \exists \Gamma[u](\cdot), \forall u(\cdot), \dot{x} = f(x,u,d), x(t) = \bar{x}, \exists s \in [t,0], x(s) \in \mathcal{T}\}$
  - $\bar{\mathcal{R}}(t) = \{\bar{x}\colon \forall \Gamma[u](\cdot), \exists u(\cdot), \dot{x} = f(x,u,d), x(t) = \bar{x}, \exists s \in [t,0], x(s) \in \mathcal{T}\}$



Computing Reachable Sets

- Start from continuous time dynamic programming
- Observe that disturbances do not affect the procedure
- Remove running cost
- Pick final cost intelligently