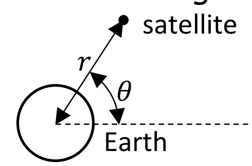
CMPT 419/983 Fall 2019 Assignment 1

- Due date: Sept. 30
- Submit zip file to CourSys

1) A simplified set of equations of motion for the Earth and an orbiting satellite is given by



$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1,$$
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{1}{r}u_2,$$

where r represents the Earth-satellite distance measured from their centres, and θ represents the phase of the orbit. k is a positive constant.

a) Derive a state space model of this system in the form of a first-order ordinary differential equation.

b) What are the equilibrium points of the state space model, under zero control input, $u_1 = u_2 = 0$? Give a physical interpretation of the result.

Equilibrium points and physical interpretation, zero-input case

c) What are the equilibrium points of the state space model, under $u_1 = \frac{k}{x_1^2}, u_2 = 0$? Give a physical interpretation of this control set point and of the equilibrium points.

d) Linearize the model with respect to a reference orbit given by $r(t) \equiv \rho, \theta(t) = \omega t, u_1 = u_2 = 0$

e) Numerically integrate the ODE, with $u_1(t), u_2(t) \equiv 0$, using your own implementation of RK4. Plot the state trajectory and intuitively explain the behaviour. Please attach your code.

Use parameters for the international space station orbiting the Earth:

- $r(0) = 410\ \mathrm{km} + 6378\ \mathrm{km}$, which represents an orbit $410\ \mathrm{km}$ above the earth's surface,
- $\dot{r}(0) = 0\ \mathrm{m/s}$,
- $\theta(0) = 0$,
- $\dot{\theta}(0) = 2\pi/T$, where $T = 92.68\ \mathrm{minutes}$, the orbital period
- $k = GM$, where $G = 6.67 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}, M = 5.97 \times 10^{24}\ \mathrm{kg}$.

2) The Lotka-Volterra Predator-Prey Model describes the interaction between a population of predators and a population of prey. Let x be the number of prey, and y be the number of predators, and the model describes the evolution of x and y as follows:

$$\dot{x} = ax - bxy$$
$$\dot{y} = -dy + cxy$$

The interpretation of the parameters a, b, c, d are as follows:

- a represents the birth rate of prey,
- d represents the death rate of predators,
- b is the prey's susceptibility to predators, and
- c is the ability of predators to hunt prey.

Note that all parameters and states are strictly positive.

a) Find the (non-trivial) equilibrium of the system.

b) Let $V(x, y) = y^a e^{-by} x^d e^{-cx}$. Show that $\dot{V}(x, y) = 0$.

c) Prove that the system is stable around the equilibrium point.

Hint: Find the maximum of $V(x, y)$, by using the fact that $\log V(x, y)$ has the same maximum. In addition, consider the convexity of $\log V(x, y)$

3) Given the system $\dot{x} = Ax + Bu$, with $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, construct a linear state feedback controller so that the closed loop system is stable.

4) Consider the linear system $\dot{x} = Ax$, with $A = \begin{bmatrix} 0 & 1 \\ -500 & -501 \end{bmatrix}$.

a) Determine conditions on the time step size that must be satisfied for the forward Euler method to be stable.

b) Repeat the above two steps for the backward Euler method.