1) A simplified set of equations of motion for the Earth and an orbiting satellite is given by

\[
\ddot{r} = r \dot{\theta}^2 - \frac{k}{r^2} + u_1, \\
\ddot{\theta} = -\frac{2r}{r^2} \dot{r} \dot{\theta} + \frac{1}{r} u_2,
\]

where \( r \) represents the Earth-satellite distance measured from their centres, and \( \theta \) represents the phase of the orbit. \( k \) is a positive constant.

a) Derive a state space model of this system in the form of a first-order ordinary differential equation. (1 mark)

b) What are the equilibrium points of the state space model, under zero control input, \( u_1 = u_2 = 0 \)? Give a physical interpretation of the result. (2 marks)

Equilibrium points and physical interpretation, zero-input case

c) What are the equilibrium points of the state space model, under \( u_1 = \frac{k}{x_1^2}, u_2 = 0 \)? Give a physical interpretation of this control set point and of the equilibrium points. (2 marks)

d) Linearize the model with respect to a reference orbit given by \( r(t) \equiv \rho, \theta(t) = \omega t, u_1 = u_2 = 0 \) (4 marks)

e) Numerically integrate the ODE, with \( u_1(t), u_2(t) \equiv 0 \), using your own implementation of RK4. Plot the state trajectory and intuitively explain the behaviour. Please attach your code. (5 marks)

Use parameters for the international space station orbiting the Earth:
- \( r(0) = 410 \text{ km} + 6378 \text{ km} \), which represents an orbit 410 km above the earth’s surface,
- \( \dot{r}(0) = 0 \text{ m/s} \),
- \( \theta(0) = 0 \),
- \( \dot{\theta}(0) = 2\pi/T \), where \( T = 92.68 \text{ minutes} \), the orbital period
- \( k = GM \), where \( G = 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}, M = 5.97 \times 10^{24} \text{ kg} \).
2) The Lotka-Volterra Predator-Prey Model describes the interaction between a population of predators and a population of prey. Let $x$ be the number of prey, and $y$ be the number of predators, and the model describes the evolution of $x$ and $y$ as follows:

$$\begin{align*}
\dot{x} &= ax - bxy \\
\dot{y} &= -dy + cxy
\end{align*}$$

The interpretation of the parameters $a, b, c, d$ are as follows:

- $a$ represents the birth rate of prey,
- $d$ represents the death rate of predators,
- $b$ is the prey's susceptibility to predators, and
- $c$ is the ability of predators to hunt prey.

Note that all parameters and states are strictly positive.

a) Find the (non-trivial) equilibrium of the system. (2 marks)

b) Let $V(x, y) = y^a e^{-b} x^d e^{-cx}$. Show that $\dot{V}(x, y) = 0$. (2 marks)

c) Prove that the system is stable around the equilibrium point. (3 marks)

Hint: Find the maximum of $V(x, y)$, by using the fact that $\log V(x, y)$ has the same maximum. In addition, consider the convexity of $\log V(x, y)$

3) Given the system $\dot{x} = Ax + Bu$, with $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, construct a linear state feedback controller so that the closed loop system is stable. (4 marks)

4) Consider the linear system $\dot{x} = Ax$, with $A = \begin{bmatrix} 0 & 1 \\ -500 & -501 \end{bmatrix}$.

a) Determine conditions on the time step size that must be satisfied for the forward Euler method to be stable. (3 marks)

b) Repeat the above two steps for the backward Euler method. (2 marks)