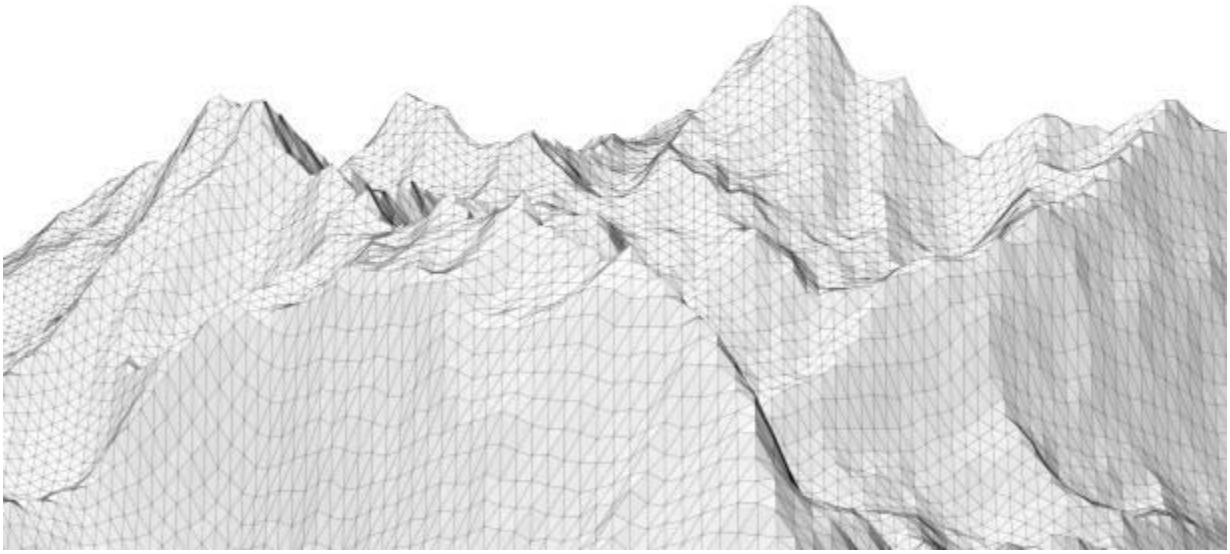
Fractals



A **fractal** is an object with the two properties:

- there is infinite detail at every point
- there is **self-similarity** between the object parts and the overall object

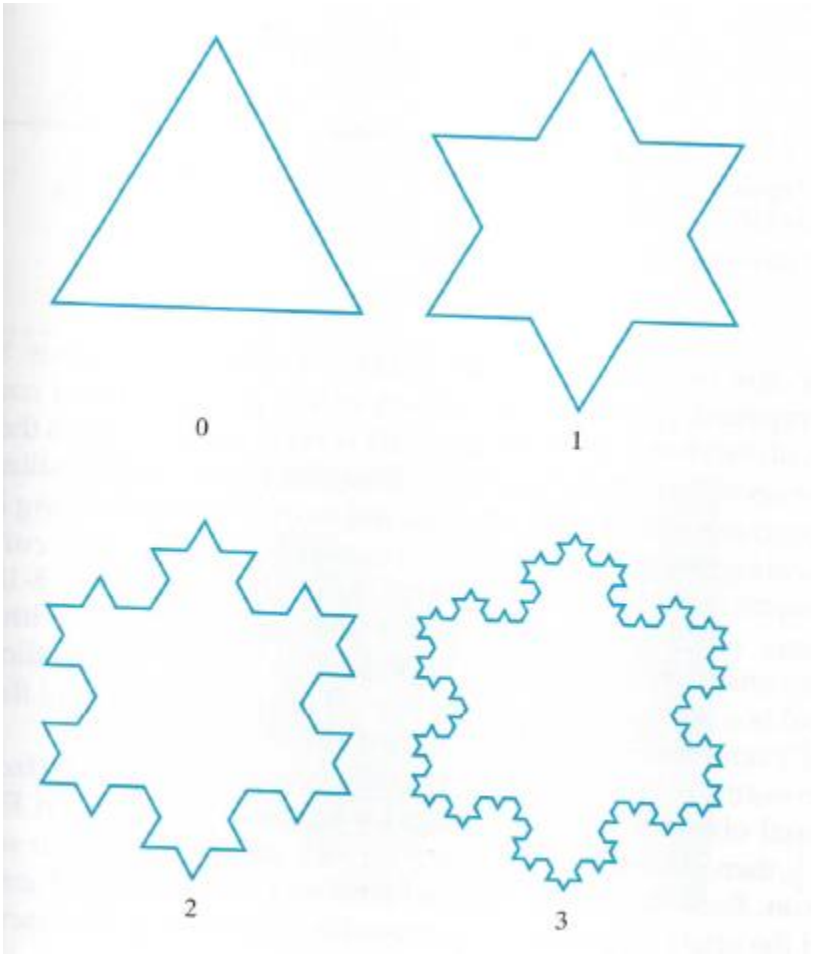
Alternatively, we can say that a fractal is an object with **fractional dimension**. (We'll get to what that is soon.)

We typically describe fractals with procedures that get repeated to produce the detail in the object subparts. Theoretically, these procedures repeat an infinite number of times. Pragmatically, we stop after the detail gets too small for us to worry about.

Here's a simple example, the Koch snowflake. To produce the detail, we take every straight line segment in the snowflake and replace it with a pattern of 4 segments, each one 1/3 of the length of the original.

The fractal object does this transformation **infinitely.** So whenever we think we have a straight line segment of it pinned down, it goes and subdivides again.

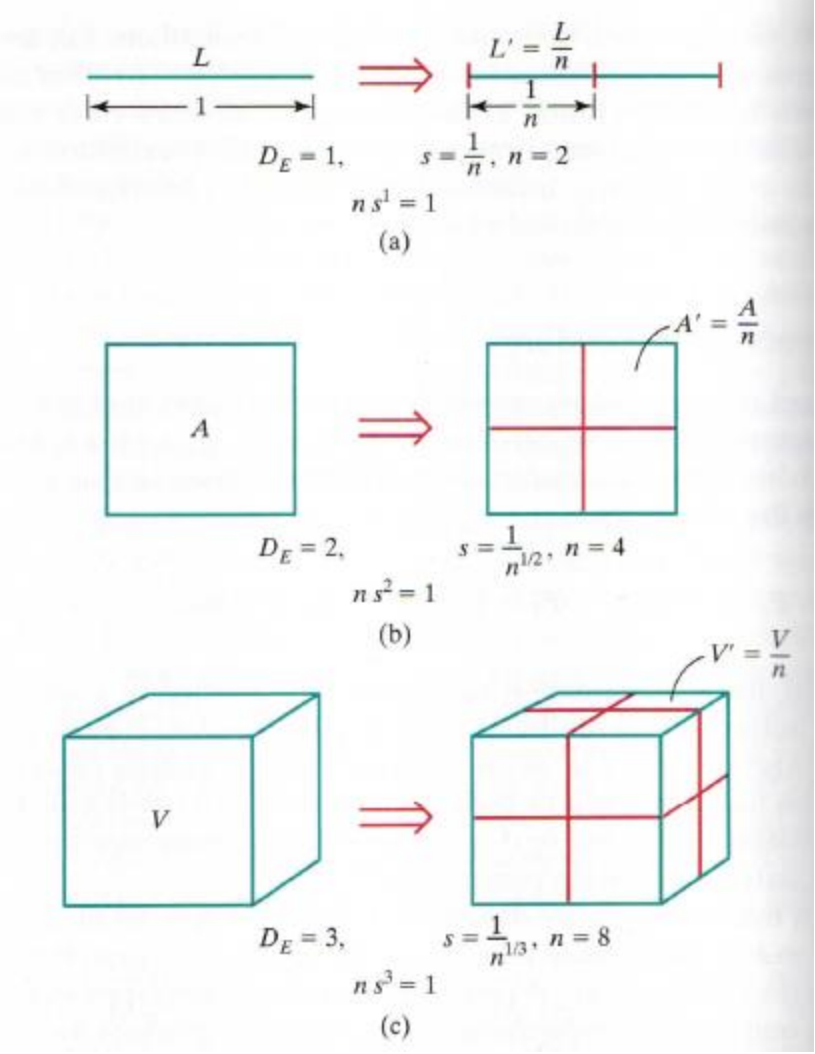
Here's an image of the first three subdivisions of the Koch curve, starting with a triangle **initiator**.



What we work with in computer graphics are often called **pseudofractals**, because we don't keep subdividing an infinite number of times. But then, neither does nature.

Fractal Dimension

The **fractal dimension** is a number that characterizes the amount of variation in the structure of a fractal object. It is defined analagously to the dimension of an ordinary object.



If we divide an ordinary object up into pieces that are scaled by ½, we get 2 pieces in 1 dimension, 4 pieces in two dimensions, and 8 pieces in 3 dimensions. The relationship is that $ns^D = 1$, where D is the dimension, s is the scale factor, and n is the number of pieces we get.

We can solve this equation for D:

$$D = \frac{\log n}{\log(1/s)}$$

If we apply this formula to our Koch snowflake curve, we have n = 4 pieces, at a scale factor of s = 1/3. So its fractal dimension is

$$D = \frac{\log 4}{\log(1/(1/3))} = \frac{\log 4}{\log 3} \cong 1.2619$$

If we have a fractal where the subparts use different scaling factors s_1, s_2, etc. then the formula is the more-difficult to solve

$$\sum_{k=1}^{n} S_k^D = 1$$

which becomes our simple $ns^D = 1$ when all scale factors are the same.

For example, suppose the **generator** we used for the Koch curve is replaced with:

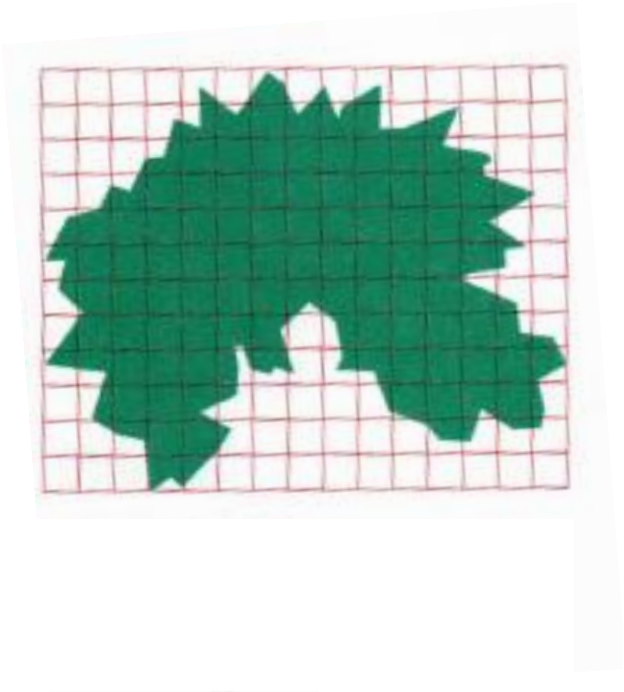
which has 1 segment of length 1/2 and 4 segments of length 1/4.

$$\sum_{k=1}^{n} S_k^D = \left(\frac{1}{2}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D = 1$$

$$\left(\frac{1}{2}\right)^D + 4\left(\frac{1}{4}\right)^D = 1$$

or D is approximately 1.357.

For more general object shapes (ones with curves and nonplanar faces, for example) we can use the notion of **box covering** to obtain a fractional dimension. Basically, we count the number of boxes necessary to cover the object at some scale factor of boxes.



And then use

$$D = \frac{\log n}{\log(1/s)}$$

My guess for this example is n = 92 with a scale factor of 1/15, so we get

$$D = \frac{\log 92}{\log 15} \approx 1.67$$

for the **box counting dimension** of the object.

For a fractal curve that lies in the plane, the fractal dimension D is greater than 1. The closer it is to 1, the smoother the curve. If D=2, we have a 2D-space-filling **Peano curve**. It will completely fill some finite region of 2D. If D > 2, the curve covers some finite region, self-intersects, and may cover the region an infinite number of times!

Deterministic self-similar fractals can be constructed in the way we constructed the Koch curve, starting with an **initiator** (the triangle) and subdividing using a **generator** (the 4-segment replacement curve).

Note that curves with fractal dimension > 1 that are constructed this way have infinite length. For example, at each level of subdivision, the Koch curve increases its length by a factor of 4/3. Since there are infinite levels of subdivision, there's infinite length.

There's no rule saying that generators have to be connected. For instance, consider the generator:

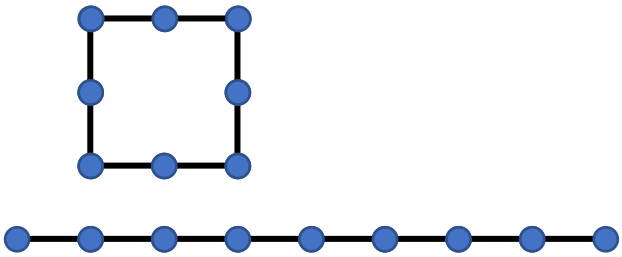
Which generates a "curve" with two parts of scale factor 1/3.

The fractal dimension of this object is

$$D = \frac{\log n}{\log(1/s)} = \frac{\log 2}{\log 3} \approx 0.631$$

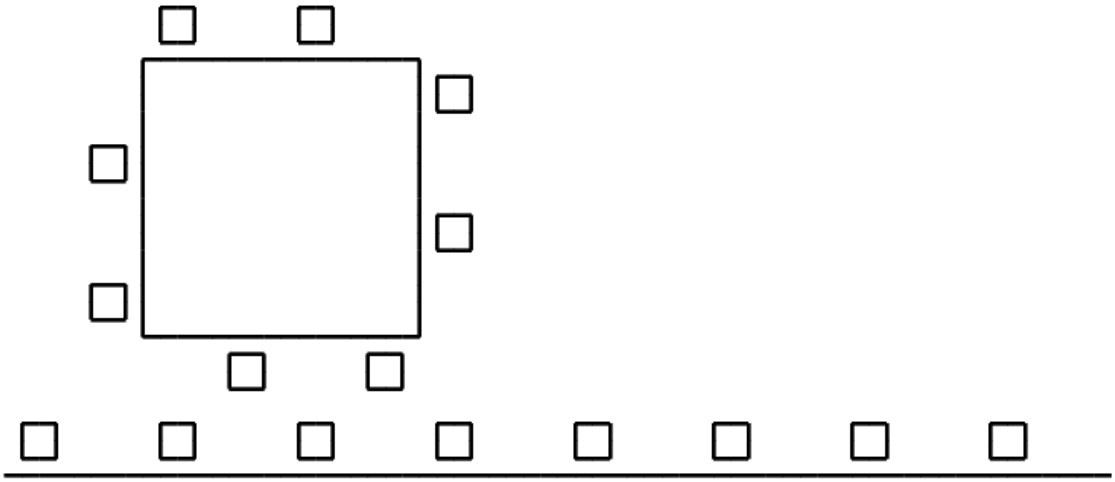
Here the length keeps getting smaller by a factor of 2/3 at each level of subdivision, so the total length of it is 0! This is called the **Cantor set**.

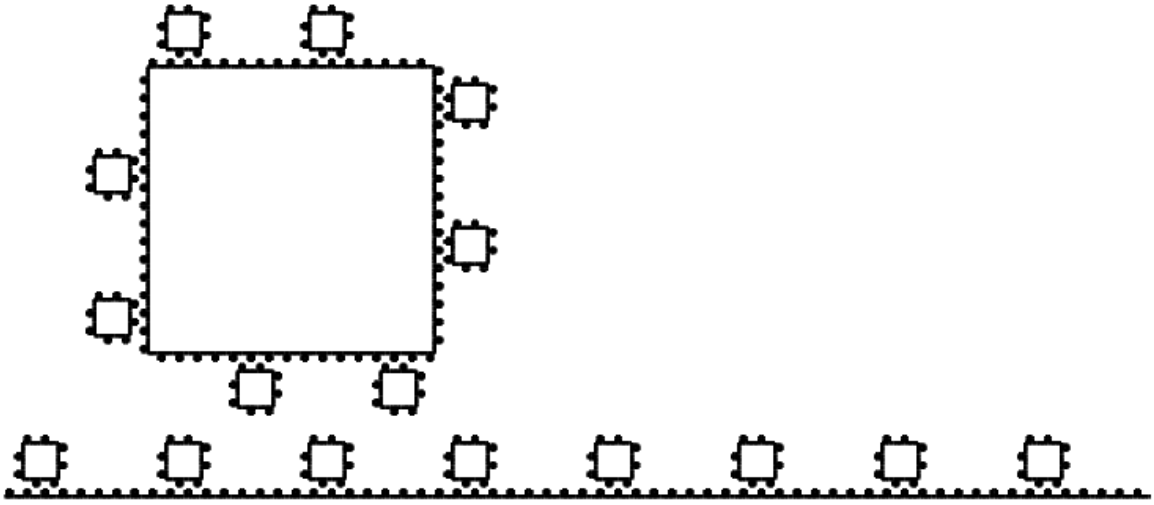
Another example of a non-connected generator is



where the segment length is 1/8. This has a fractal dimension of $\frac{\log 16}{\log 8} = \frac{4}{3} \approx 1.333$

Here's the second and third levels for that generator:





The generators we've looked at so far are mainly one-dimensional (i.e. curves). This means we can easily encode them as **strings**. Suppose we give the following characters meanings:

S : a segment of unit length

l : a left turn of 60 degrees

r : a right turn of 60 degrees

Then we can encode the Koch generator as

S => SlSrrSlS

And the Koch initiator (the triangle) by

SrrSrrS

except that we want the scale factor on the segments to go down by a factor of 1/3 each time we use the generator. So let's suppose we redefine **S** to do that

S: a length $(1/3)^k$ segment, where k is the number of subdivisions done.

So let's start with the initiator **S** (it's simpler than **SrrSrrS**) and do a couple of levels of subdivision. So 0 levels of subdivision gives us just

S

To get to the next level of subdivision, we apply the generator to every **S** in the previous level. So for level 1 we get:

SlSrrSlS

Again we apply the generator to every **S** to get the second level of subdivision:

SlSrrSlS l SlSrrSlS rr SlSrrSlS l SlSrrSlS

And again for the third level:

SlSrrSlS l SlSrrSlS rr SlSrrSlS l SlSrrSlS l

SlSrrSlS l SlSrrSlS rr SlSrrSlS l SlSrrSlS rr

SlSrrSlS l SlSrrSlS rr SlSrrSlS l SlSrrSlS l

SlSrrSlS l SlSrrSlS rr SlSrrSlS l SlSrrSlS

These are essentially driving directions for someone who wants to drive a Koch curve.

For other curves, we may need a letter that stands for moving a distance without drawing a segment. Consider the Cantor set generator:

We can model this as **S => STS** , **T => TTT** where **S** is a segment of length $(1/3)^k$ as before, and **T** is a non-drawing movement of length $(1/3)^k$**.** We can get the first few levels of subdivision as follows:

S => STS => STS TTT STS =>

STSTTTSTS TTTTTTTTT STSTTTSTS

Those of you who have had compilers or programming languages may recognize that we are doing something very much like a context-free grammar with the strings here. Our system has a set of **nonterminal symbols** such as **S** and **T**, a set of **terminal symbols** such as **l** and **r**, a start string, and a set of **productions**, each of which look like:

N => some string of terminals and nonterminals

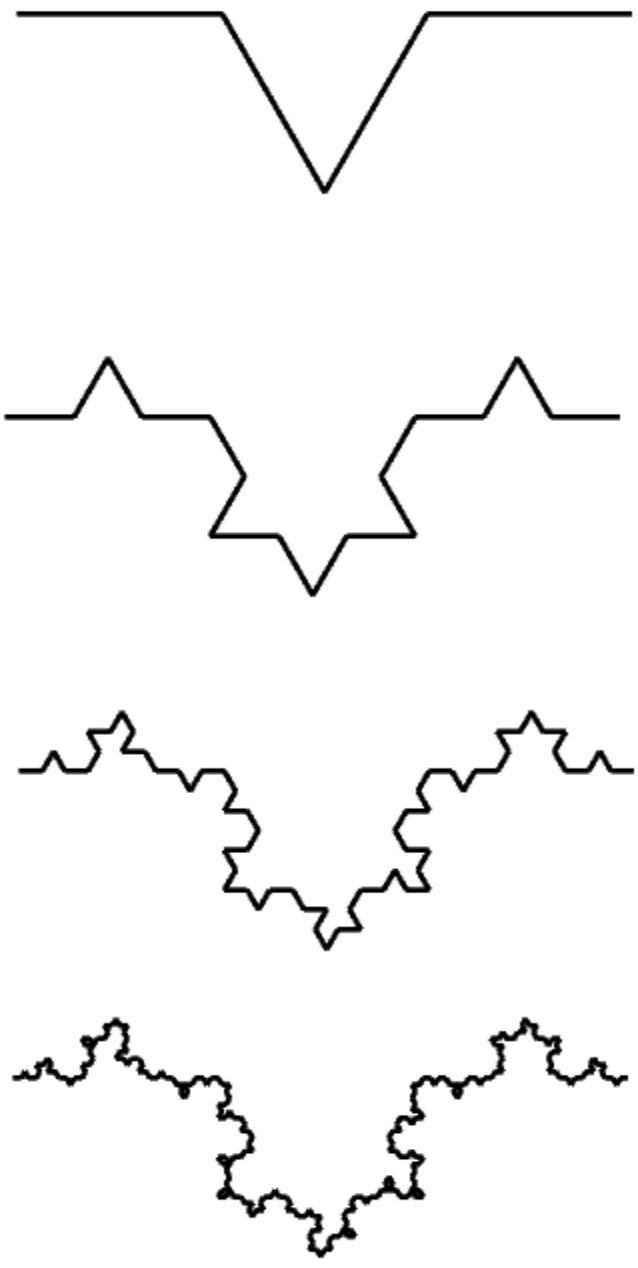
The difference with grammars is that at each stage of a derivation, we replace **all** of the nonterminals rather than just **one**. This rule switch makes this a **Lindenmeyer system** (or **L-system** for short).

In an L-system, as with grammars, we can have more than one production for a given nonterminal. For instance, we could make a variation on the Koch curve with the productions

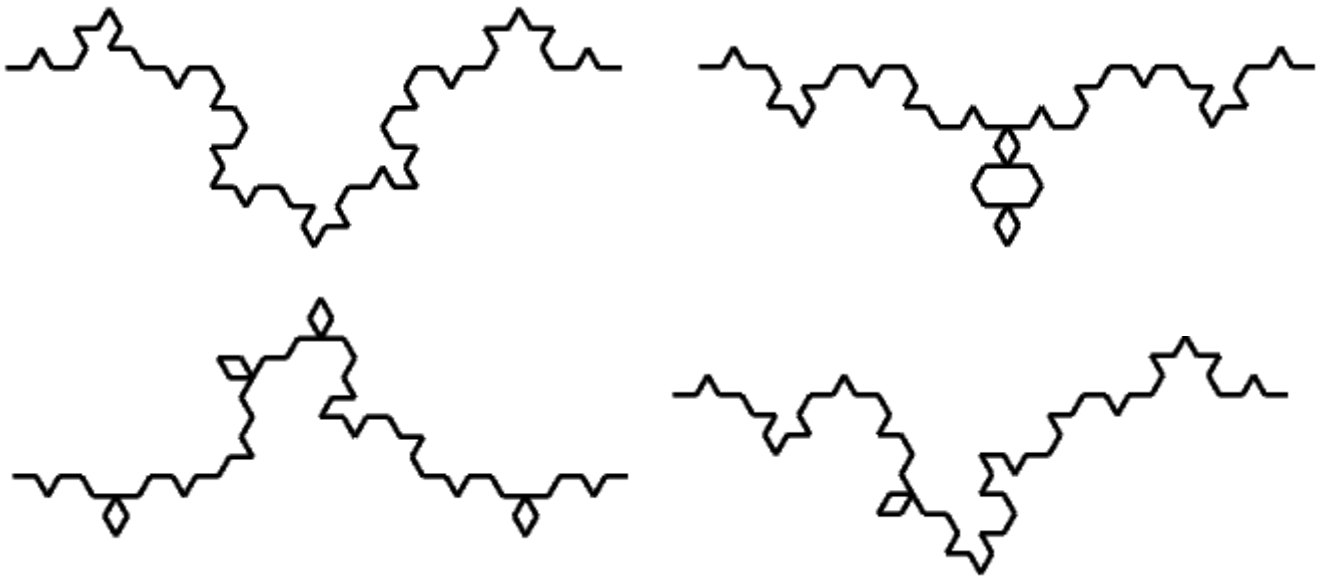
S => SlSrrSlS

S => SrSllSrS

At each step of the derivation, for each **S**, the L-system would choose one of these two productions to use to replace **S** with.

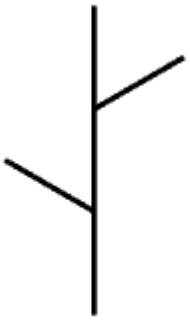


With this addition, there's no longer a single (deterministic) fractal curve being described; we're describing a whole **family** of **stochastic fractal curves**.

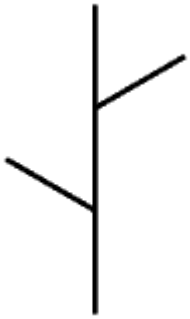


Stochastic means **random**, and indeed we've introduced the random choice of which of the two productions to use for **S**.

Yet one more thing we can do with L-systems helps when we have a **branching** generator.



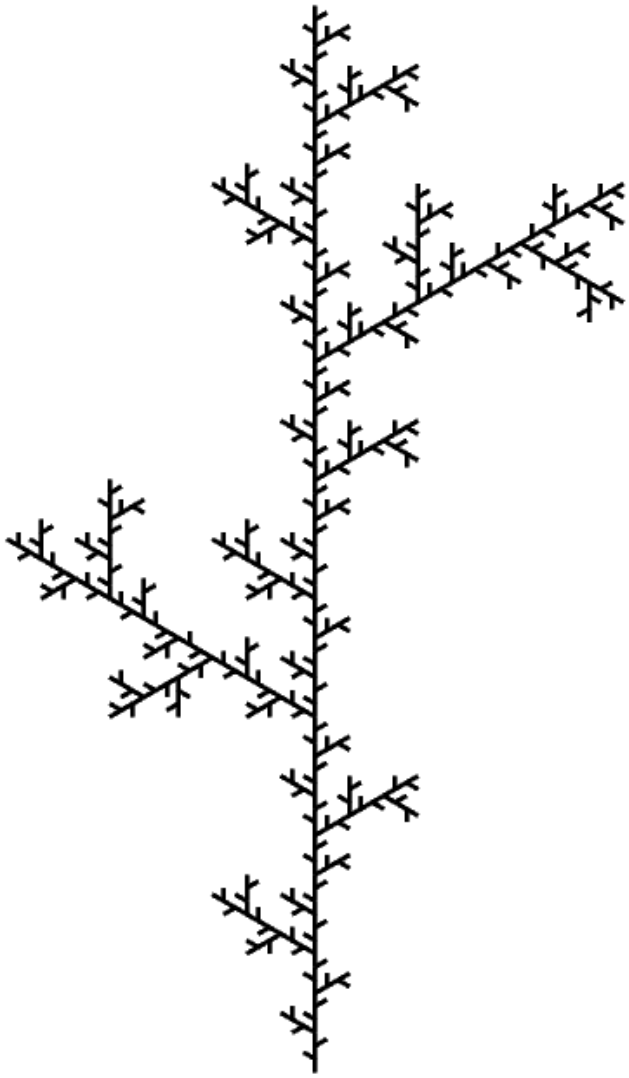
It's a technique called **bracketing**. Basically, you include a set (or sets) of brackets as terminals; these brackets are always introduced in pairs, and open bracket is like a OpenGL **pushmatrix** and close bracket is like **popMatrix**.



The production for this generator is

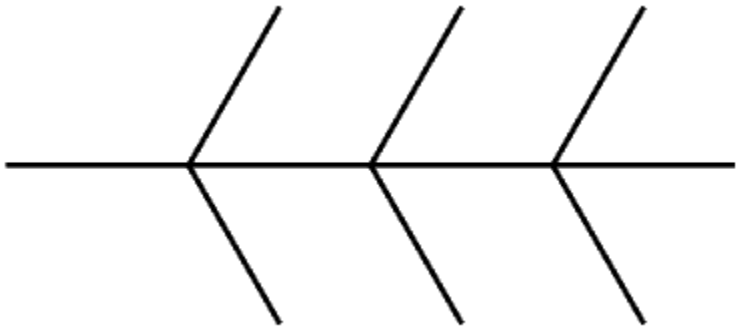
S => S{lS}S{rS}S

And here's the fourth subdivision using that generator.



Here's another branching generator:

S => S{lS}{rS}S{lS}{rS}S{lS}{rS}S

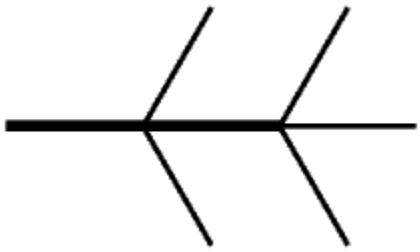


And its third subdivision:

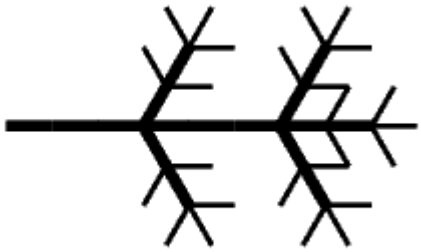
We can introduce a nonterminal **U** that draws a segment like **S** but give it a different subdivision rule:

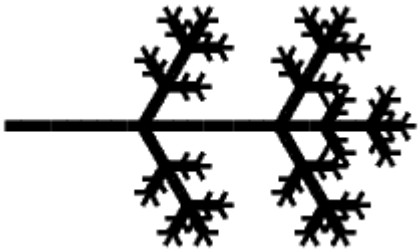
S => U{lS}{rS}U{lS}{rS}S

U => UUU

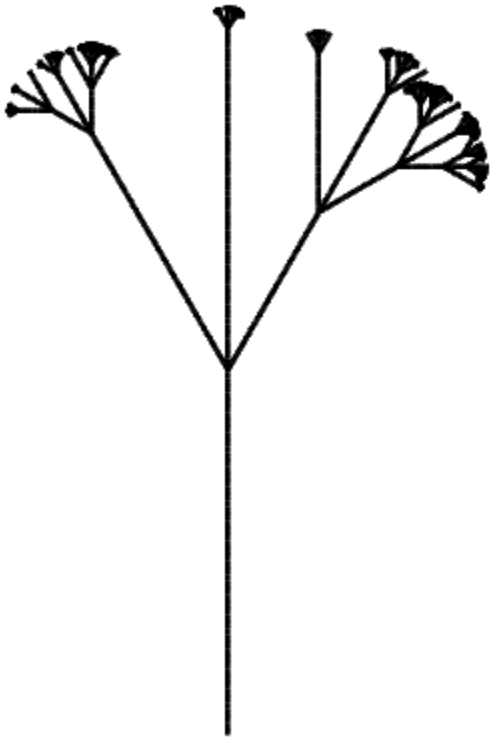


I've made the **U** lines thicker so you can see them. At the next level, all the **S** segments subdivide, but the **U** segments don't. (A **U** technically subdivides into 3 segments in a straight line, but the net effect of that is to leave it unchanged.)





Even adding a little randomness can make more natural-looking objects:

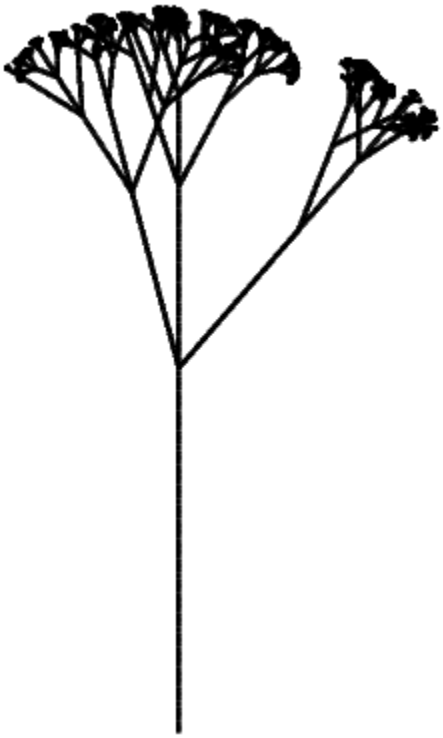


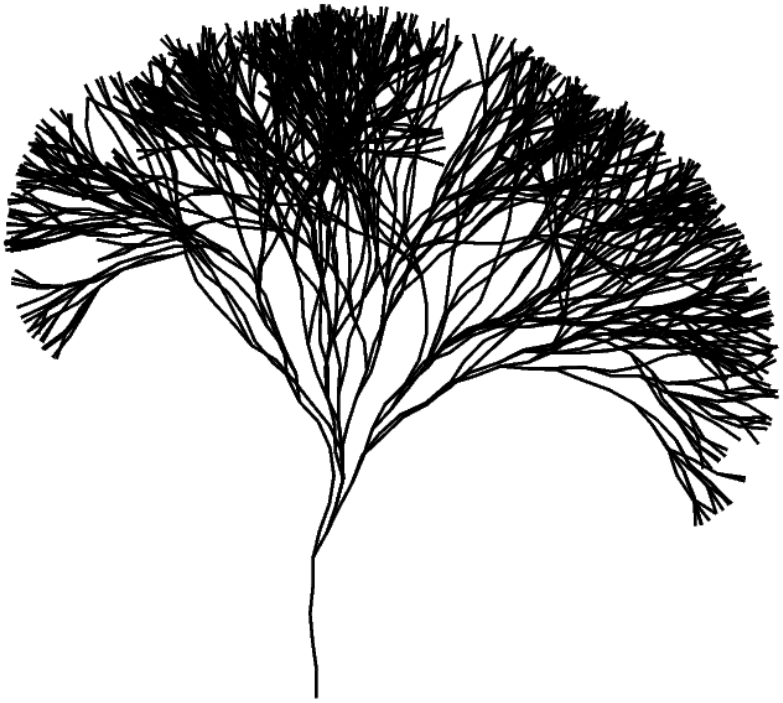
S => U{lS}{rS}S

S => US

U => UU

where **l** and **r** rotate 30 degrees each.





S => Ur{rS}S

S => Ul{lS}S

S => UrS

S => UlS

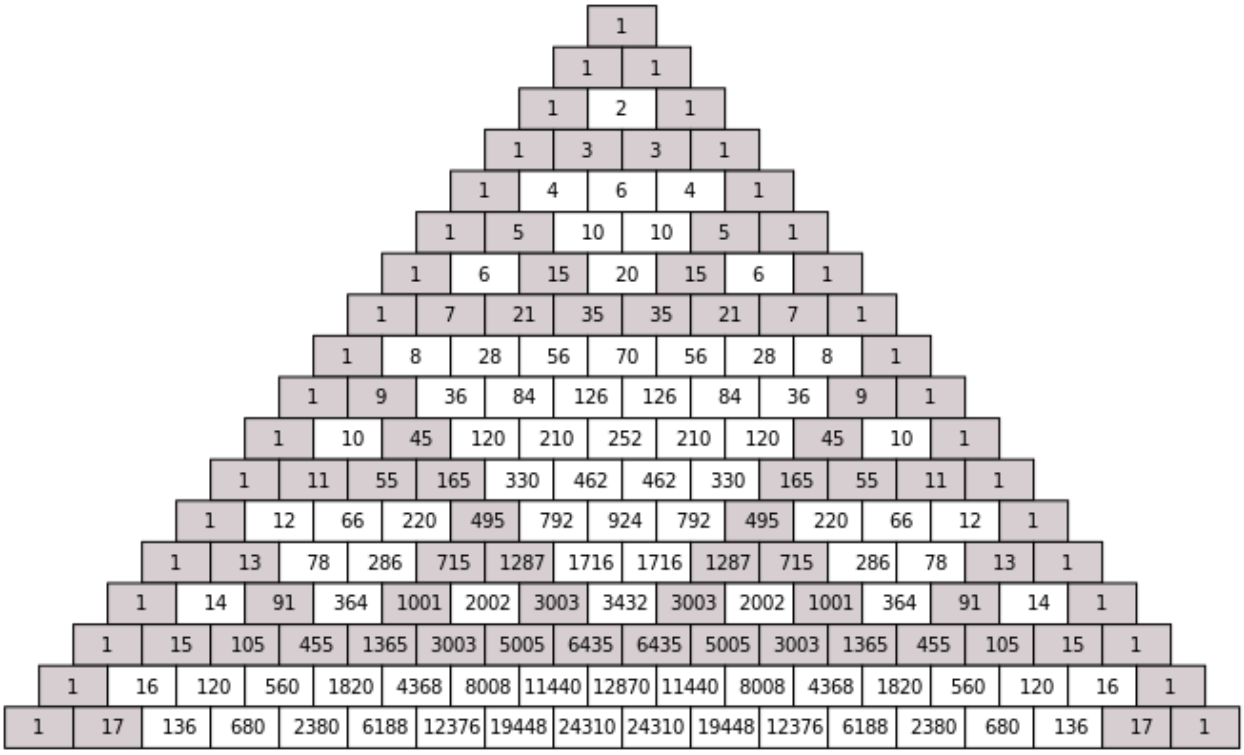
U => U

l and **r** rotate randomly from 0 to 20 degrees.

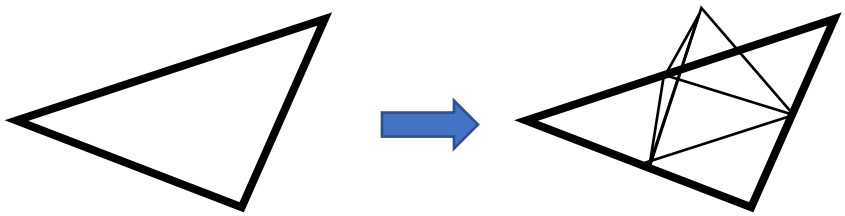
We have mainly been looking at 1D (line, curve) generators in the 2D plane. (The exception is the Cantor set, which is a 1D generator in 1D.) We can also look at 1D generators in 3-space.

We could look at 2D generators in 2-space.

That fractal is known as the **Sierpinski gasket**. It has a well-known relationship to Pascal's triangle:


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1
1 17 136 680 2380 6188 12376 19448 24310 24310 19448 12376 6188 2380 680 136 17 1

Or we could look at 2D generators in 3-space. Consider the following generator:



where we replace an equilateral triangle by 6 equilateral triangles scaled by a factor of 1/2. That gives this fractal surface a fractal dimension of

$$D = \frac{\log n}{\log(1/s)} = \frac{\log 6}{\log 2} \approx 2.585$$

The surface that is generated is the 3D equivalent of the 2D Koch curve.

And it doesn't stop there. We can look at k-dimensional generators in any space of at least k dimensions.