ENSC327
Communications Systems
8: Complex Envelope (3.8)

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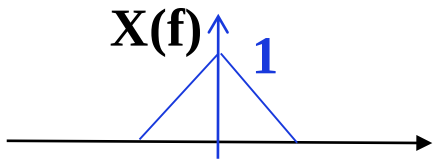
Outline

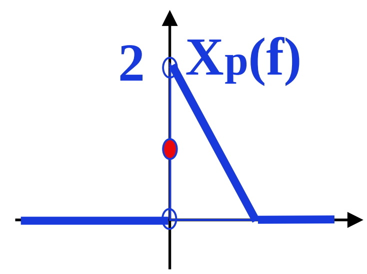
- Required Background
- Complex Envelope Representation of bandpass signals
- Complex Envelope Representation of bandpass systems
- Application of complex envelope?
  - Will be used to analyze the spectrum of FM (ch. 4)
  - Can be used to represent a high-freq bandpass system by a low-freq baseband system, which is easier to analyze and simulate with software.

Required Background

- Hilbert Transform of $x(t)$:



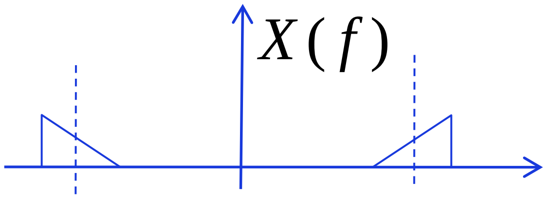
- Analytical signal of the positive frequencies of $x(t)$:



- If $x(t)$ is real, then $x(t) =$
- Shift in frequency
- LTI systems

Complex Envelope (Defined for Real and Bandpass Signals)

- A signal $x(t)$ is called a Bandpass signal if its frequency spectrum, X(f), is centered around a frequency f_c and has a limited bandwidth. In general X(f) may not be symmetric around f_c.
- If $x(t)$ is also real, then X(f) is Conjugate Symmetric:



- The "complex envelope" , defined for a *real and bandpass* signal, $x(t)$, is defined as the signal $\tilde{x}$(t), such that:

$$x(t) = Re\left\{ \tilde{x}(t)e^{j2\pi f_c t} \right\}$$

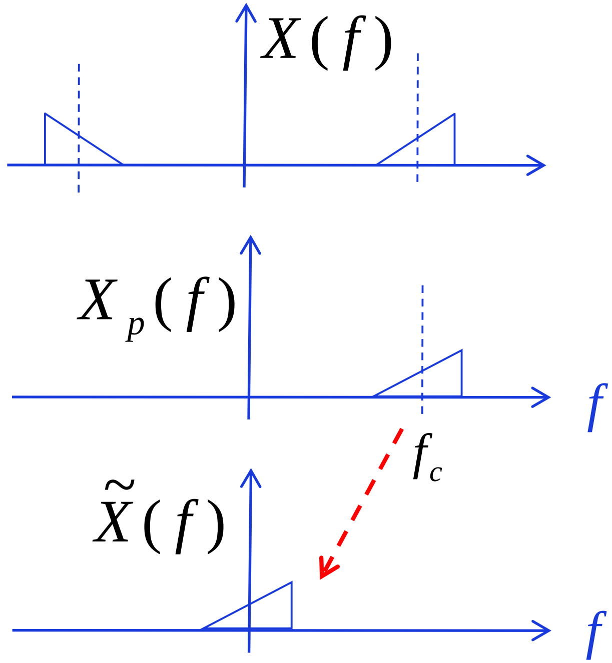
Finding $\tilde{x}$(t) in terms of $x(t)$

- We want to find $\tilde{x}$(t) such that $x(t) = Re\{\tilde{x}(t)e^{j2\pi f_c t}\}$

- Recall the pre-envelope-positive frequencies of $x(t)$:

$$x(t) = Re\{x_p(t)\}$$

- Compare the above equations , it suffices to have



- Notice that the complex-envelope is a low-pass signal.

Example 1

$$x(t) = \cos(2\pi f_c t).$$

What is the complex-envelope of $x(t)$ with respect to f_c as the center frequency.

Example 2

$$x(t) = \cos(2\pi(f_c + f_0)t). \quad f_c >> f_0 \quad \tilde{x}(t)?$$

Find the complex envelope of $x(t)$ with respect to center frequency f_c.

Example 3

$$x(t) = m(t)\cos\left(2\pi f_c t\right)$$

Find the complex envelope of $x(t)$ with respect to center frequency f_c (Assume $m(t)$ to be a lowpass signal).

In-phase and Quadrature Components of a Bandpass signal

- Assume $\tilde{x}(t)$ is the complex envelope of the bandpass signal $x(t)$.
- Since $\tilde{x}(t)$ in general is a "complex" signal, it can be written as

  $\tilde{x}(t) = x_I(t) + j\, x_Q(t)$, where

  - $x_I(t)$: is called the **In-Phase component** of $x(t)$
  - $x_Q(t)$: is called the **Quadrature component** of $x(t)$
- Any bandpass signal $x(t)$ can thus be written as:

Envelope and Phase of a Bandpass Signal

- We just showed that any bandpass signal can be written as:

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)$$

- $x(t)$ can also be written in terms of its envelope and phase as:

$$x(t) = A(t)\cos(2\pi f_c t + \theta(t)),$$ where

- $A(t)$: is called the **Envelope** of $x(t)$ and
- $\theta(t)$: is called the **Phase** of $x(t)$.
- Formulas:

$$A(t) = \sqrt{{x_I}^2(t) + {x_Q}^2(t)}$$

$$\theta(t) = \tan^{-1}\frac{x_Q(t)}{x_I(t)}$$

Summary of the Three Envelopes

- Bandpass signal:

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) = A(t)\cos(2\pi f_c t + \theta(t)),$$

- Pre-envelope (analytic signal) applies to any signal (lowpass or bandpass)

$$x_p(t) = x(t) + j\,\hat{x}(t)$$

(Twice of the positive frequency components of $x(t)$.)

- Complex Envelope applies to bandpass signals only

$$\tilde{x}(t) = x_p(t)e^{-j2\pi f_c t}$$

($X_p(f)$ shifted from f_c to 0)

- Envelope and phase of a bandpass signal $x(t)$:

$$A(t) = \sqrt{x_I^2(t) + x_Q^2(t)} = \left|x_p(t)\right| = \left|\tilde{x}(t)\right|$$

$$\theta(t) = \tan^{-1}\frac{x_Q(t)}{x_I(t)}$$

Complex Envelope Representation of Bandpass Systems

- The complex envelope can be used to represent a bandpass (BP) system by a lowpass system. This makes theoretical analysis and computer simulations easier.

Bandpass s(t) $\longrightarrow \boxed{h(t)} \longrightarrow$ **Bandpass y(t)**

- Find the complex envelope of $s(t)$ (with respect to f_c), call it $\tilde{s}(t)$.
- Find the complex envelope of $h(t)$ (with respect to f_c), call it $\tilde{h}(t)$.
- Then we will have:

$$\tilde{s}(t) \longrightarrow \boxed{\tilde{h}(t)} \longrightarrow 2\tilde{y}(t)$$

- Why the scaling factor of 2 ?
- Once the lowpass complex envelope $\tilde{y}(t)$ is known, the bandpass signal $y(t)$ can be obtained by:

Application of Complex Envelope in Angle Modulation

- Chapter 4: The general form for phase modulation (PM) or frequency modulation (FM):

$$s(t) = A_c \cos(2\pi f_c t + \phi(t))$$

$\phi(t)$ is related to the message.

- As we will see, finding the Fourier transform and bandwidth of $s(t)$ is difficult.
- This becomes less complicated if we work with the complex envelope of $s(t)$.