ENSC327
Communications Systems
22: Gaussian Processes and White Noise

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Outline

- Required Background:
  - pdf of a Gaussian Random Variable:
  - Covariance of two R.V.s :
- Gaussian Random Processes
  - N-dimensional Gaussian random vector
  - Gaussian random processes
- Noise (Normally referred to an unwanted R.P.)
  - White Noise
  - Guassian Noise
  - White Guassian Noise
  - Filtered White Noise

Joint Gaussian pdf

Assume X_i (i=1, …, N) are n Gaussian random variables with means μ_{X_i}.

The <u>Gaussian Random Vector</u> $\mathbf{X} = [X_1, X_2, \ldots, X_N]$ is completely specified by

The <u>**means**</u> of individual R.V.s and all <u>**pairwise covariances**</u> among X_i.

- The mean vector:

- Covariance matrix:

$$\mathbf{\Lambda} = E\left\{[\mathbf{X}-\mu_{\mathbf{X}}]^T[\mathbf{X}-\mu_{\mathbf{X}}]\right\} = E\left\{\begin{bmatrix} X_1-\mu_{X_1} \\ X_2-\mu_{X_2} \\ \vdots \\ X_N-\mu_{X_N} \end{bmatrix}\begin{bmatrix} X_1-\mu_{X_1} & X_2-\mu_{X_2} & \ldots & X_N-\mu_{X_N} \end{bmatrix}\right\}$$

Joint Gaussian pdf (Cont.)

- The (i,j)-th entry of the covariance matrix is the covariance between Xi and Xj:

$\Rightarrow$ $\mathbf{\Lambda}_{ij} =$

- If X_i and X_j are uncorrelated for all i $\neq j$:

$$\mathbf{\Lambda} = \begin{bmatrix} \sigma_{X_1}^2 & & & \\ & \sigma_{X_2}^2 & & \\ & & \ddots & \\ & & & \sigma_{X_N}^2 \end{bmatrix}$$

Important Reminder: In general, independent ➔ uncorrelated, but converse is not true. **For Guasian R.V.s, independent** $\leftrightarrow$ **uncorrelated.**

Joint Gaussian pdf (Cont.)

- The joint pdf of the N-Dim Gaussian random vector X:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{|2\pi\mathbf{\Lambda}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\mathbf{\Lambda}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T} = \frac{1}{(2\pi)^{\frac{N}{2}}|\mathbf{\Lambda}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\mathbf{\Lambda}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T}$$

$$|\mathbf{\Lambda}|: \text{ determinant of matrix } \mathbf{\Lambda}.$$

- Important Property: the N-Dim Gaussian pdf only depends on the means of all Xi's and covariances between any two components. No higher order cross-correlations are needed.

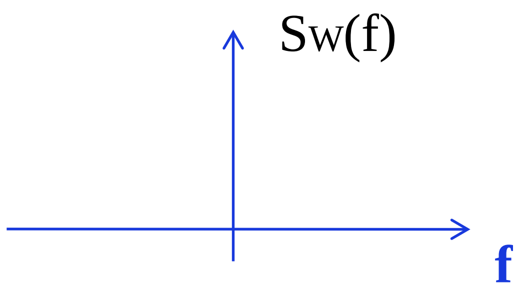
Gaussian Random Processes

- A random process X(t) is a Gaussian process if for all k and all $\{t_1, t_2, \ldots, t_k\}$, $\{X(t_1), X(t_2), \ldots., X(t_k)\}$ are jointly Gaussian distributed.

- Properties of Gaussian Random Processes:
  1. The mean $\mu_{X(t)}$ and autocorrelation $R_X(t_1, t_2)$ give a complete description of a Gaussian process, regardless of its stationarity.
  2. If a Gaussian random process is WSS ($R_X(t_1, t_2) = R_X(t_1 - t_2)$), then it is also strictly stationary, because the Gaussian pdf is only related to means and covariances.
  3. If a Gaussian random process is applied to an LTI system, the output is also a Gaussian process.

White Noise (8.10)

- White noise (or white process): A random process $W(t)$ is called white noise if it has a constant power spectral density for all f.
- What's the power of white noise?



As long as the bandwidth of the noise is much wider than that of the signal, we can treat the noise as white noise.

- Significance of white noise:
  - Thermal noise is close to white in a large range of freqs.
  - Many processes can be modeled as output of LTI systems driven by a white noise.

White Noise (Cont.)

- The psd of white noise is usually denoted as

$$S_W(f) = \frac{N_0}{2}.$$

- The 1/2 factor emphasizes that the spectrum extends to both positive and negative frequencies.
- The autocorrelation function of WSS white noise:

→ Different samples of white noise in time domain are

Gaussian Noise

- A noise process (random process), $X(t)$, is called Gaussian noise (Gaussian R.P.) if the pdf of $X(t)$ is Gaussian for all t.
- This says nothing of the correlation of the noise in time or of the spectral density of the noise:
  - Gaussian noise and white noise are two different concepts. Neither implies the other.

White Gaussian Noise

- White Gaussian noise: A white noise (constant power spectral density) with Gaussian distributed amplitude.

- Gaussian white noise is a good approximation of many real-world situations and generates mathematically tractable models.

- Samples of Gaussian white noise are independent:
  - Uncorrelated and independent are same for Gaussian pdf.

Filtered White Noise –Example 1

A white noise with zero mean and psd $N_0/2$ is filtered by an ideal lowpass filter of bandwidth B and unit gain. Find the auto-correlation and average power of the output noise.

Filtered White Noise – Example 2

- A white noise with zero mean and psd $\frac{N_0}{2}$ is filtered by an RC lowpass filter with:

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Find the psd and autocorrelation of the output.