ENSC327 Communications Systems 2: Fourier Representations

School of Engineering Science
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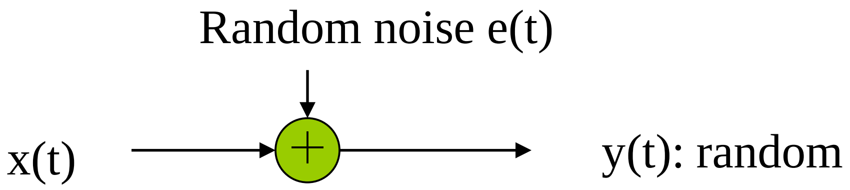
Outline

- Chap 2.1 – 2.5:
  - Signal Classifications
  - Fourier Transform
  - Dirac Delta Function (Unit Impulse)
  - Fourier Series
  - Bandwidth
  - (Chap 2.6-2.9 will be studied together with Chap. 8)

Signal Classification: Deterministic vs Random

- Deterministic signals: can be modeled as completely specified functions, no uncertainty at all
  - Example: x(t) = sin(a t)

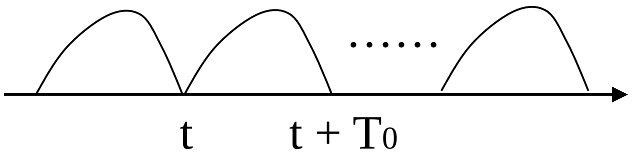
- Random signals: take a random value at any time
  - Example: Noise-corrupted channel output
  - Probability distribution is needed to analyze the signal
  - It is more useful to look at the statistics of the signal:
    - Average, variance …



Signal Classification: Periodic vs Aperiodic

- Periodic: A signal $x(t)$ is periodic if and only if we can find some constant T_0 such that

$$x(t+T_0) = x(t), \; -\infty < t < \infty.$$



- Fundamental period: the smallest T_0 satisfying the equation above.

- Aperiodic: Any signal not satisfying the equation is called aperiodic.

Signal Classification: Energy Signals vs Power Signals

Power and energy of arbitrary signal x(t):

- Energy: $$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$
- Power: The average amount of energy per unit of time. $$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$
- For a periodic signal: $$P = \frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 \, dt$$
- What's the energy of a periodic signal?
- What's the power of an aperiodic and time limited signal?

Signal Classification: Energy Signals vs Power Signals

- A signal is called an Energy Signal if its energy is finite

$$0 < E < \infty$$

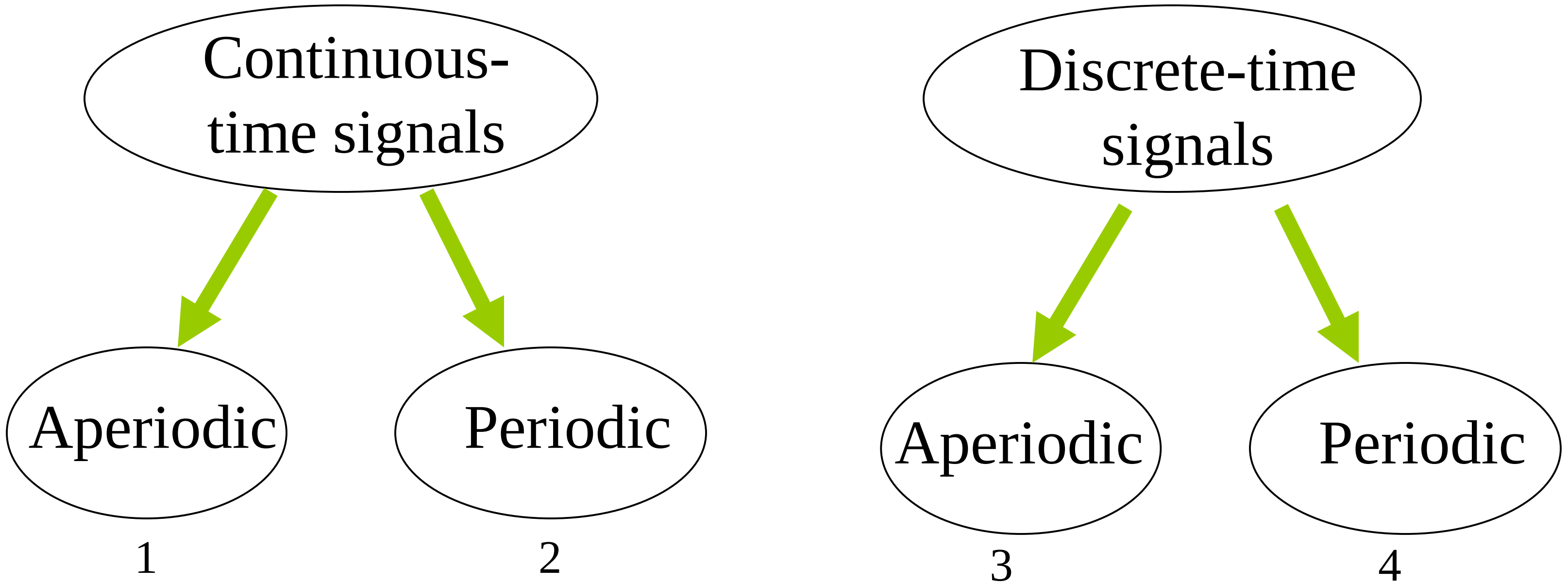
➔x(t) = 0 at $\pm\infty$
➔P = 0

- A signal is called a Power Signal if its power is finite

$$0 < P < \infty$$

- Periodic Signal are

Types of Fourier Series and Transforms



- Continuous-time signals:
  - 1. Aperiodic: Fourier transform
  - 2. Periodic: Fourier series (and Fourier transform)
- Discrete-time signals:
  - 3. Aperiodic: Discrete-time Fourier transform
  - 4. Periodic: Discrete-time Fourier series (and Fourier transform)

Fourier Transform (FT)

- For aperiodic, continuous-time signal:

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt, \qquad g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

ω: continuous angular frequency

- In terms of frequency f (Recall $\omega = 2\pi f$)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt, \qquad g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

Amplitude and Phase Spectra

$$G(f) = |G(f)| e^{j\theta(f)}$$

$|G(f)|$: amplitude spectrum

$\theta(f)$: phase spectrum

- Important Property: If g(t) is real, then G(f) is conjugate symmetric:

$$G^*(f) = G(-f), \quad \text{or} \quad |G(f)| = |G(-f)|, \ \theta(f) = -\theta(-f).$$

where "*" is the complex-conjugate operator.

- Proof:

Properties of FT

- Conjugation rule:

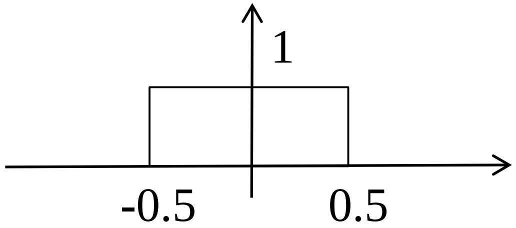
$$g(t) \leftrightarrow G(f) \Rightarrow g^*(t) \leftrightarrow G^*(-f)$$

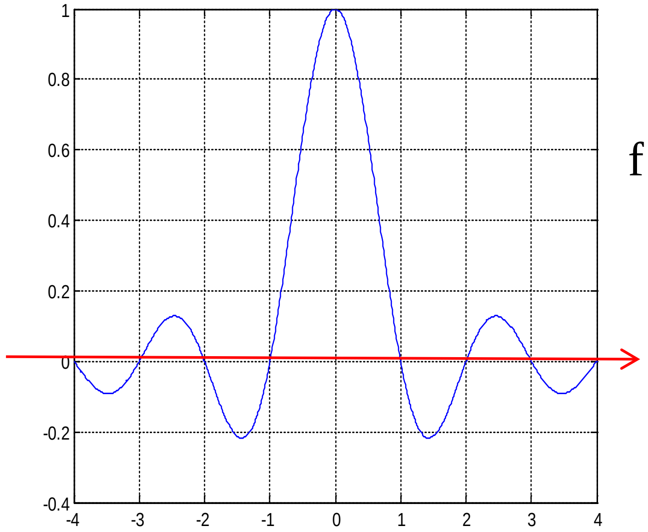
- Symmetry:
  - If g(t) is real and even, then G(f) is real and even.
  - If g(t) is real and odd, then G(f) is imaginary and odd.

Example: Symmetry Rule

- Find the FT of the Unit rectangular function (or gate function):

$$\mathrm{rect}(t) = \begin{cases} 1, & t \in [-0.5,\ 0.5] \\ 0, & \text{otherwise.} \end{cases}$$





$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$ (very useful in this course)

Properties of Fourier Transform (Cont.)

- Dilation (or similarity or time scaling/frequency scaling) :

$$g(at) \leftrightarrow \frac{1}{|a|} G(\frac{f}{a})$$

(a is a real number)

- Proof:

For $a > 0$, $\int_{-\infty}^{\infty} g(at) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} g(\tau) e^{-j2\pi f \frac{\tau}{a}} d\frac{\tau}{a} = \frac{1}{a} G(\frac{f}{a})$.

For a < 0:

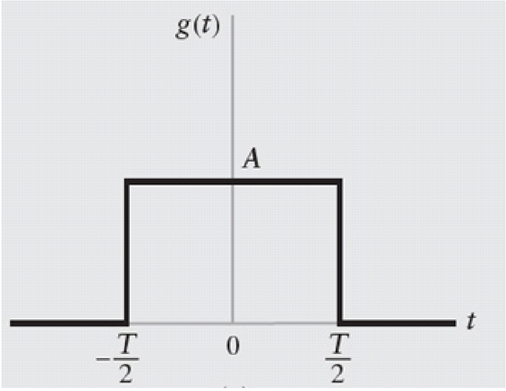
$$\int_{-\infty}^{\infty} g(at) e^{-j2\pi ft} dt = \int_{\infty}^{-\infty} g(\tau) e^{-j2\pi f \frac{\tau}{a}} d\frac{\tau}{a} = -\frac{1}{a} G(\frac{f}{a}) = \frac{1}{|a|} G(\frac{f}{a})$$

Note the change of integral range when a<0.

- Compress (expand) in time ➔ in frequency

Example of Dilation Property

- Find the FT of the Rectangular Pulse of width T and compare with the FT of rect(t) :



$$g(t) = A\,\mathrm{rect}(\frac{t}{T})$$

Uncertainty Principle of the FT

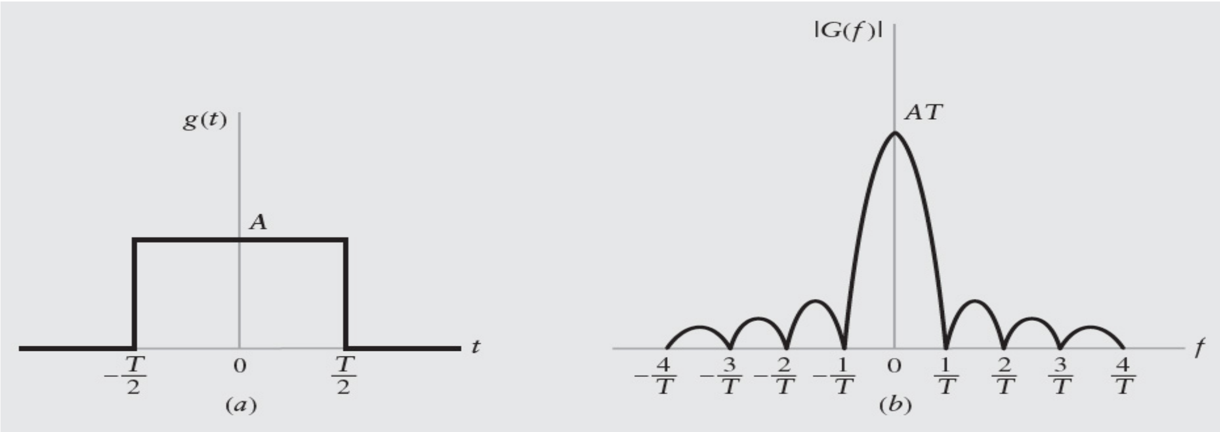


FIGURE 2.2 (*a*) Rectangular pulse. (*b*) Amplitude spectrum.

Narrow in time
Wide in frequency

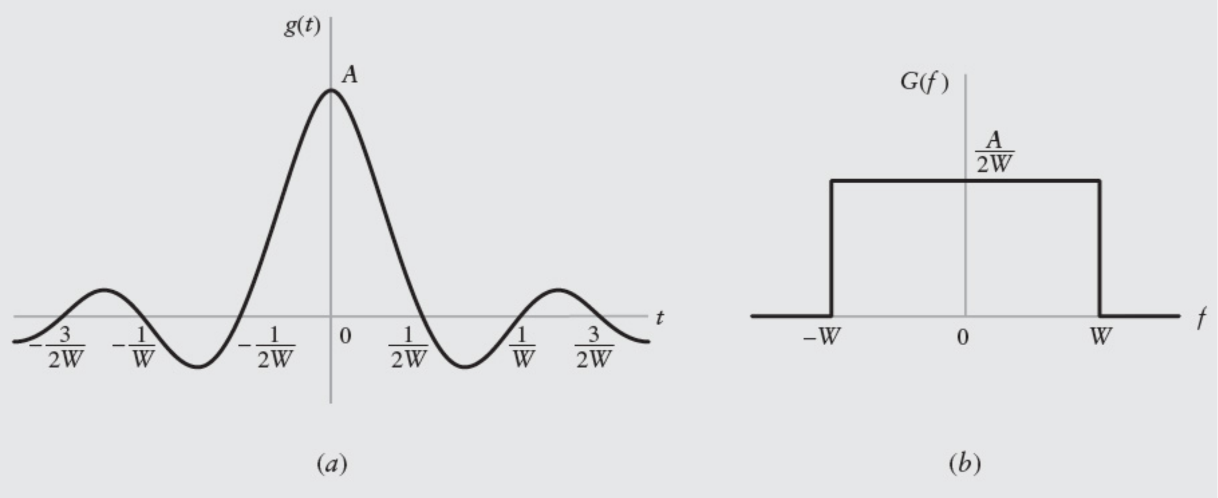


FIGURE 2.8 (*a*) Sinc pulse $g(t)$. (*b*) Fourier transform $G(f)$.

Wide in time
Narrow in frequency

Properties of Fourier Transform (Cont.)

- Reflection Property:

Apply the scaling property : $g(at) \leftrightarrow \frac{1}{|a|} G(\frac{f}{a})$ to find the FT of g(-t):

If in addition g(t) is real, G(f) is conjugate symmetric,

$$\Rightarrow \quad g(-t) \leftrightarrow G(-f) = G^*(f).$$

Properties of Fourier Transform (Cont.)

- Duality:

$g(t) \leftrightarrow G(f)$ ➔ $G(t) \leftrightarrow g(-f)$

- Proof:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \longrightarrow g(-t) = \int_{-\infty}^{\infty} G(f) e^{-j2\pi f t} df$$

Since t and f are independent variables, by ignoring their physical meanings, we can interchange t and f:

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi ft} dt$$

This means that the FT of G(t) is g(-f).

Example of Duality Property

Apply the duality property to find the FT of $g(t) = A\,\mathrm{sinc}(2Wt)$

Solution:

The sign or "signum" function (sgn(t))

(Note: This example is useful later when we study single sideband (SSB) communications and Hilbert transform)

Given $\text{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases} \quad \leftrightarrow \quad \frac{1}{j\pi f}$

What's the FT of $\frac{1}{j\pi t}$?

Properties of Fourier Transform (Cont.)

- Time shifting (delay):

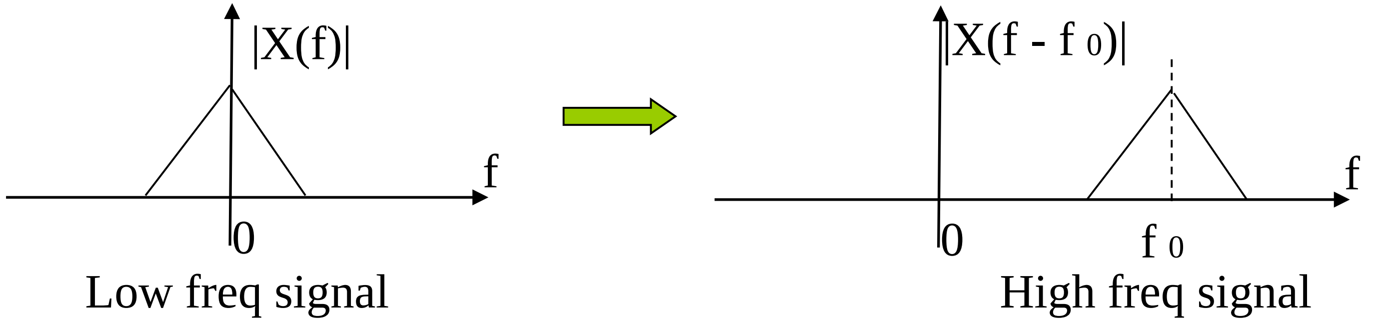
$$g(t) \leftrightarrow G(f) \quad \Longrightarrow \quad g(t-t_0) \leftrightarrow G(f)e^{-j2\pi f t_0}$$

(Time delay only affects the phase spectrum.)

- Frequency shifting:

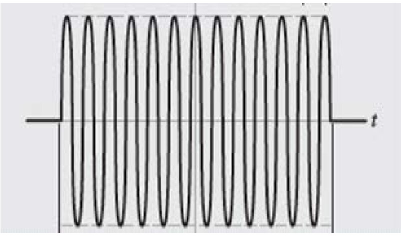
$$g(t) \leftrightarrow G(f) \quad \Longrightarrow \quad g(t)e^{j2\pi f_0 t} \leftrightarrow G(f-f_0)$$

- Very useful in the study of communication systems (modulation):



Example of Frequency Shifting

What is the FT of $g(t) = rect(\frac{t}{T})\cos(2\pi f_c t)$?



This is a modulation technique called amplitude modulation. The effect is that the spectrum (Fourier Transform) of rect(T/t) gets shifted to $\pm f_c$.

Properties of Fourier Transform (Cont.)

- Differentiation:

$$\frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f)$$

- Proof:

Start from $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$

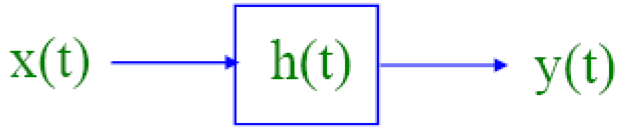
$$\frac{dx(t)}{dt} = \int_{-\infty}^{\infty} X(f) \frac{de^{j2\pi ft}}{dt} df = \int_{-\infty}^{\infty} \left(j2\pi f \; X(f)\right) e^{j2\pi ft} df$$

➔ $dx(t)/dt \leftrightarrow (j2\pi f) X(f)$

➔ Each derivative operation creates one more term of $j2\pi f$

This property is used in FM demodulation

Convolution



- Convolution: (Recall)The convolution operation describes the input-output relationship of a linear time-invariant (LTI) system (ENSC 380, 383)
- The convolution of two signals is defined as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- The formula is related to the properties of LTI system and impulse response.
- Note: it is very easy to make mistake about this formula. Please be very careful, as it may appear in the exam.
- More on this later.

Properties of Fourier Transform (Cont.)

- Convolution property (one of the most useful properties of FT)

$$\text{Let } g_1(t) \leftrightarrow G_1(f), \quad g_2(t) \leftrightarrow G_2(f),$$

$$\text{then } g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \quad \leftrightarrow \quad G_1(f) G_2(f)$$

Time domain convolution ➔ frequency domain product

Proof: $\text{Let } \mathrm{x(t)} = \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau,$

$$\mathrm{X(f)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \, e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} g_1(\tau) e^{-j2\pi f \tau} \int_{-\infty}^{\infty} g_2(t-\tau) \, e^{-j2\pi f (t-\tau)} dt d\tau = G_1(f) G_2(f)$$

Properties of Fourier Transform (Cont.)

- Modulation:

$$g_1(t) \leftrightarrow G_1(f), \quad g_2(t) \leftrightarrow G_2(f),$$

$$g_1(t)g_2(t) \leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda = G_1(f) * G_2(f)$$

Time domain product ➔ frequency domain convolution

Rayleigh's Energy Theorem (Parseval's Theorem)

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Proof:

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} G^*(f) e^{-j2\pi ft} df\, dt$$

$$= \int_{-\infty}^{\infty} G^*(f) \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt df = \int_{-\infty}^{\infty} G^*(f) G(f) df = \int_{-\infty}^{\infty} |G(f)|^2 df$$

We can calculate the total energy of a signal in either domain.

Energy (Cont.)

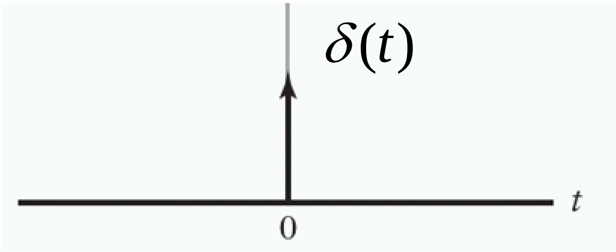
Can we find the energy of the signal $g(t)$ with the time period $[t_1, t_2]$ directly from it's FT?

Dirac Delta Function (Unit Impulse)

- The Dirac delta function δ(t) is defined as a function satisfying two conditions:

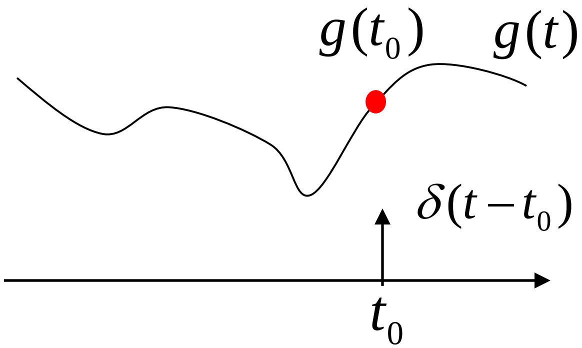
$$\delta(t) = 0 \quad \text{for } t \neq 0.$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



- ➔ δ(t) is an even function: δ(-t) = δ(t) .
- The definition implies the sifting property of Delta function:

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt =$$



Dirac Delta Function (Cont.)

- Since δ(t) is even function, we can rewrite this as

$$g(t_0) = \int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = \int_{-\infty}^{\infty} g(t)\delta(t_0 - t)dt.$$

- Changing the variables, we get the convolution:

$$\int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau = g(t), \quad \text{or} \quad g(t) * \delta(t) = g(t).$$

- ➔ The convolution of δ(t) with any function is that function itself.

- This is called the replication property of the delta function.

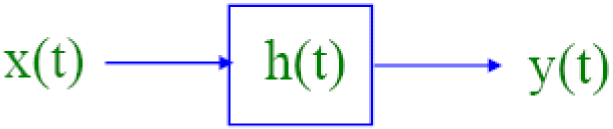
Linear and time-invariant system

- **Linearity**: a system is linear if the input $a_1x_1(t)+a_2x_2(t)$ leads to the output $a_1y_1(t)+a_2y_2(t)$, where y1, y2 are the output of x1 and y2 respectively.
- **Time-invariant system**: a system is time-invariant if the **delayed** input $x(t-t_0)$ has the output $y(t-t_0)$, where y(t) is the output of x(t).

A system is LTI if it's both linear and time-invariant.

LTI System (Cont.)

- A linear and time-invariant system is fully characterized by its output to the unit impulse, which is called impulse response, denoted by h(t).
- The output to any input is the convolution of the input with the impulse response:



LTI System (Cont.)

- Proof of the convolution expression:

  We start from the sifting property:

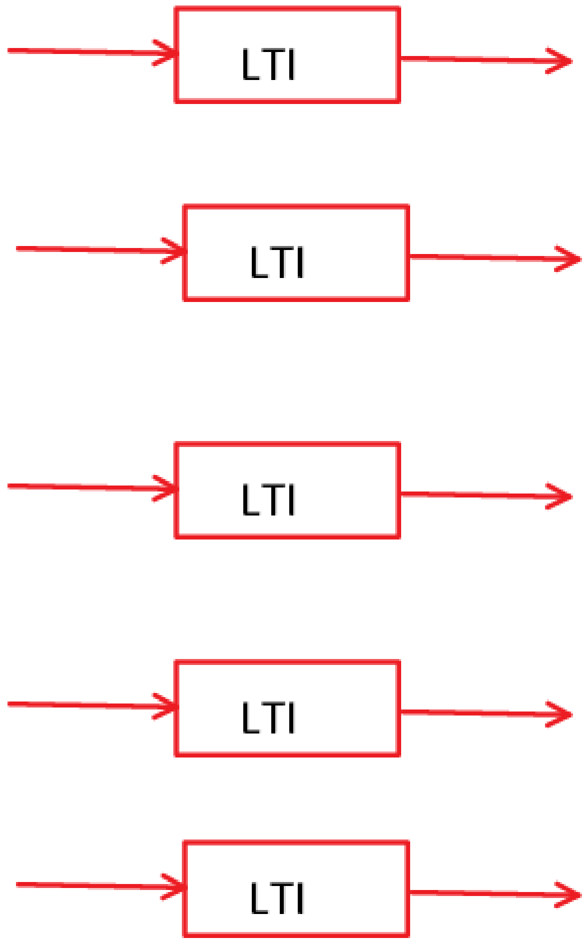
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- This can be viewed as the linear combination of delayed unit impulses.
- By the properties of LTI, $\delta(t-\tau) \to h(t-\tau)$

  the output of x(t) will be the linear combination of delayed impulse responses:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

LTI System (Cont.)



Fourier Transform of the delta function

Applications of the Unit Impulse Function

- FT of DC signal: $g(t) = 1 \leftrightarrow G(f) = \delta(f).$

  (i.e., DC signal only has 0 frequency component).

- Proof:
  - We know:

$$g(t) = \delta(t) \leftrightarrow G(f) = 1$$

  - apply the duality property the above

Applications of the Unit Impulse Function (Cont.)

- FT of $e^{j2\pi f_0 t}$: $e^{j2\pi f_0 t} \leftrightarrow \delta(f - f_0).$
- (Intuitively: A pure complex exponential signal only has one frequency component)
- Proof:

Applications of the Unit Impulse Function (Cont.)

Find the FTs of $\cos 2\pi f_0 t$ and $\sin 2\pi f_0 t$

Fourier Series

- Suppose x(t) is periodic with period T0:

  Let $\omega_0 = 2\pi / T_0$:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}, \quad t_0 \le t < t_0 + T_0$$

$$X_k = \frac{1}{T_o} \int_{T_o} x(t) e^{-jk\omega_o t} dt$$

- Represent x(t) as the linear combination of fundamental signal and harmonic signals (or basis functions)
- X_k: Fourier coefficients.

Fourier Series (cont.)

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

- Example: Find the Fourier series expansion of

$$x(t) = \cos(\omega_0 t) + \sin^2(2\omega_0 t)$$

- Method 1: Use the definition

$$X_k = \frac{1}{T_o}\int_{T_o} x(t) e^{-jk\omega_o t} dt$$

- Method 2: use trigonometric identity and Euler's theorem:

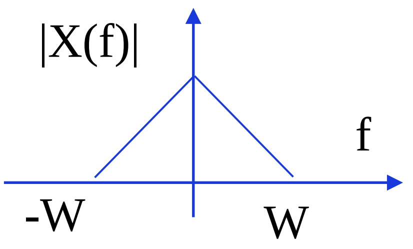
$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\sin^2(2\omega_0 t) = \frac{1}{2}\left(1 - \cos 4\omega_0 t\right) = \frac{1}{2} - \frac{1}{4}\left(e^{j4\omega_0 t} + e^{-j4\omega_0 t}\right)$$

Definitions of Bandwidth (Chap 2.3)

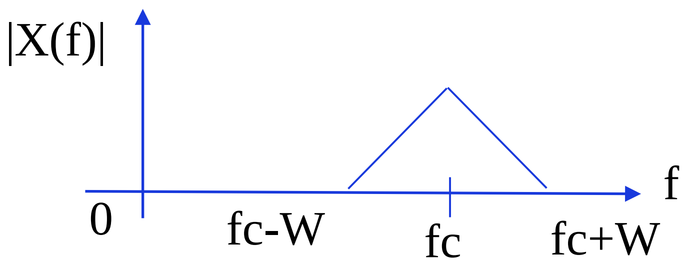
- Bandwidth: A measure of the extent of **significant** spectral content of the signal in **positive** frequencies.
  - The definition is not rigorous, because the word "significant" can have different meanings.
- For band-limited signal, the bandwidth is well-defined:

Low-pass Signals:



Bandwidth is W.

Bandpass signals

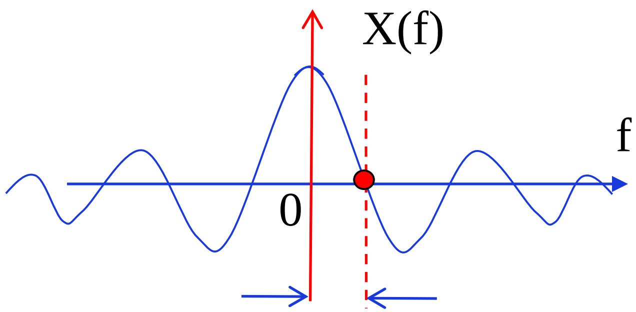


Bandwidth is 2W.

Definitions of Bandwidth (Cont.)

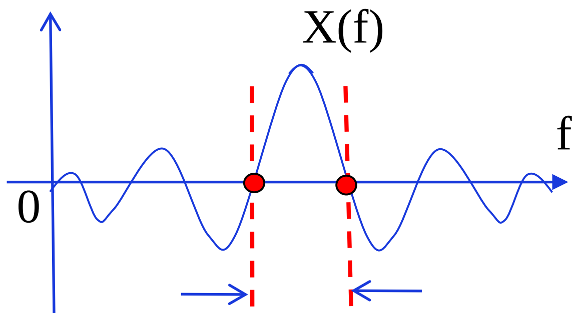
- When the signal is not band-limited different definitions exist:
  - Def. 1: Null-to-null bandwidth
  - Null: A frequency at which the spectrum is zero.

For low-pass signals:



Bandwidth is half of main lobe width
(recall: only pos freq is counted in bandwidth)

For Bandpass signals:

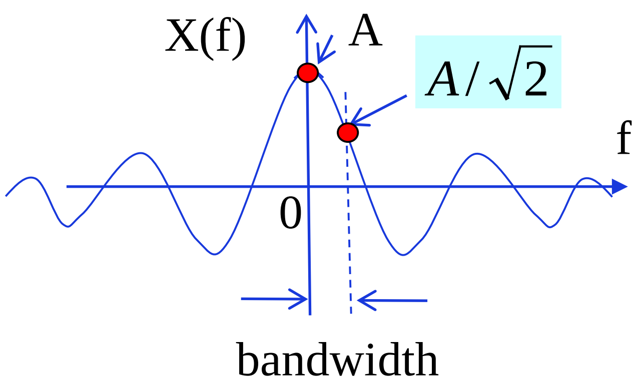


Bandwidth = main lobe width

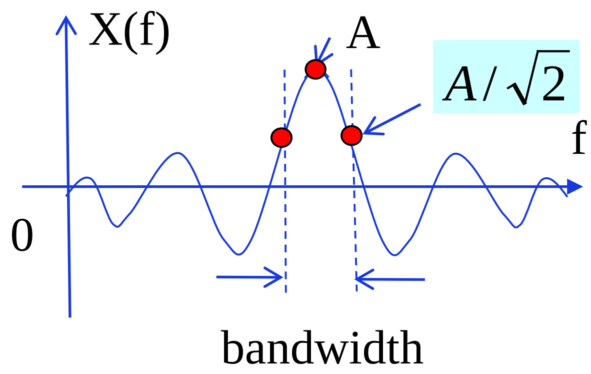
Definitions of Bandwidth (Cont.)

- Def. 2: 3dB bandwidth

Low-pass Signals



Bandpass signals



$|X(f)|^2$ drops to 1/2 of the peak value, which corresponds to 3dB difference in the log scale.

$$10\log_{10} 0.5 = -3dB$$

Definitions of Bandwidth (Cont.)

- Def. 3: Root Mean-Square (RMS) bandwidth

$$W_{rms} = \left(\frac{\int_{-\infty}^{\infty} (f - f_c)^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df} \right)^{1/2}$$

f_c : center freq.

$$\overline{G}(f) = \frac{|G(f)|^2}{\int_{-\infty}^{\infty} |G(f)|^2 df}$$: Normalized squared spectrum.

since $\int_{-\infty}^{\infty} \overline{G}(f) df = 1.$

The RMS bandwidth is the standard deviation of the normalized squared spectrum.

Final Note on Bandwidth

- Radio spectrum is a scarce and expensive resource:
  - US license fee: ~ $77 billions / year

- Communication system companies try to provide the desired quality of service with minimum bandwidth.