ENSC327 Communications Systems 19: Random Processes

School of Engineering Science

Simon Fraser University

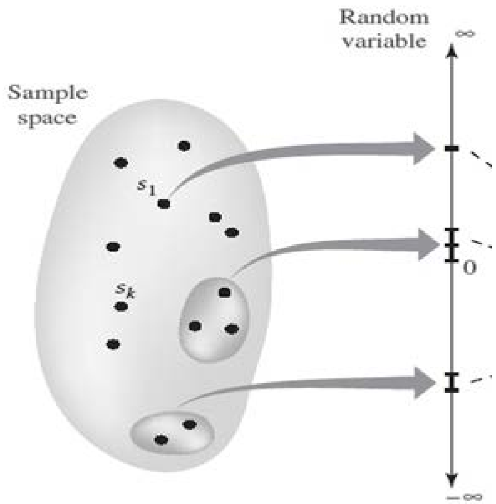
Outline

- Random processes
- “Stationary” and “Wide Sense Stationary” Random Processes
- Autocorrelation of random processes

Random Process

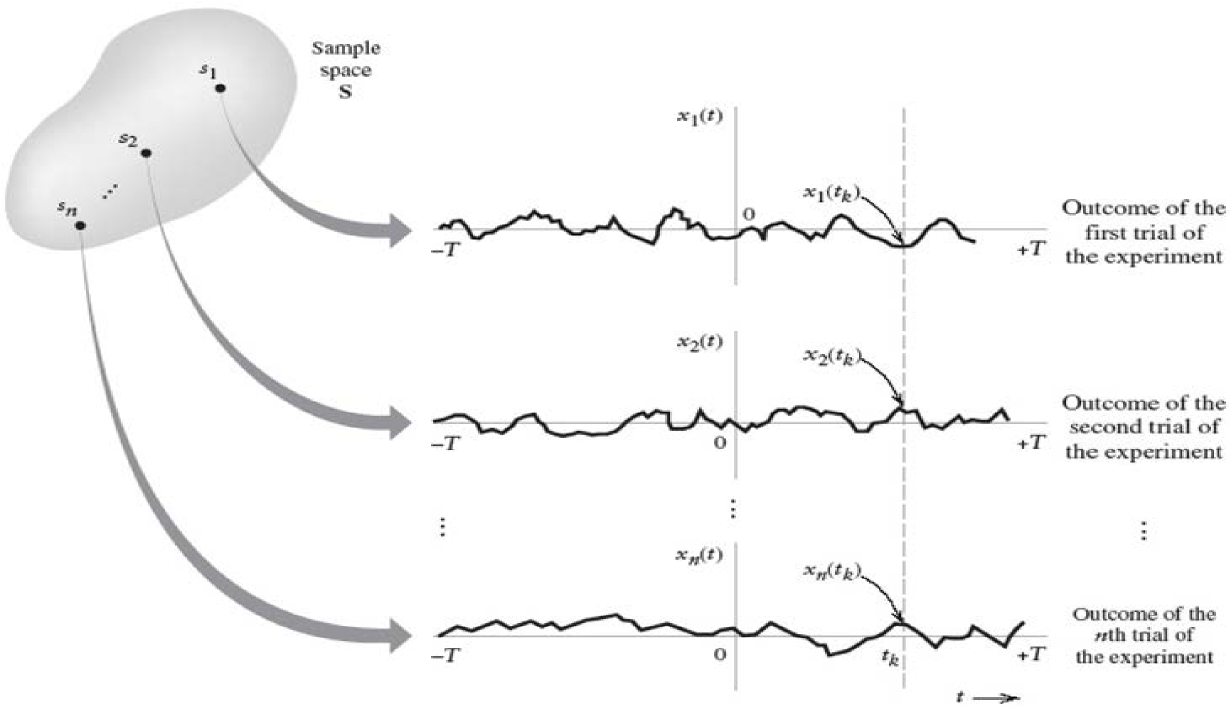
- A deterministic process has only one possible function of how the process evolves under time, i.e. has only one *realization*.
- In a stochastic or random process there could be many realizations, determined by some probability distribution.
- Many time-varying signals are random in nature:
  - Noises
  - Image, audio: usually unknown to the receiver.
  - Stock prices
- Random process represents the mathematical model of these random signals.
- **Definition:** A random process (or stochastic process) is a collection of **random variables** indexed by **time**.
- Notation: $X(t, s)$
  - s: sample point in the sample space (outcome of the random experiment)
  - t: time.
- Simplified notation: $X(t)$

Random Process vs RV



- Random variable: an outcome is mapped to a number.

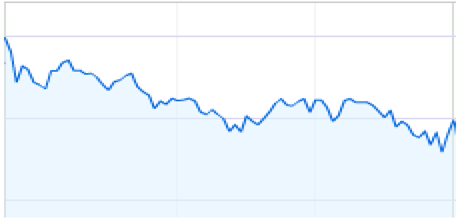
- Random process: If the outcome of a random experiment is a random **function of time**, then we have a random process.



Random Processes (Cont.)

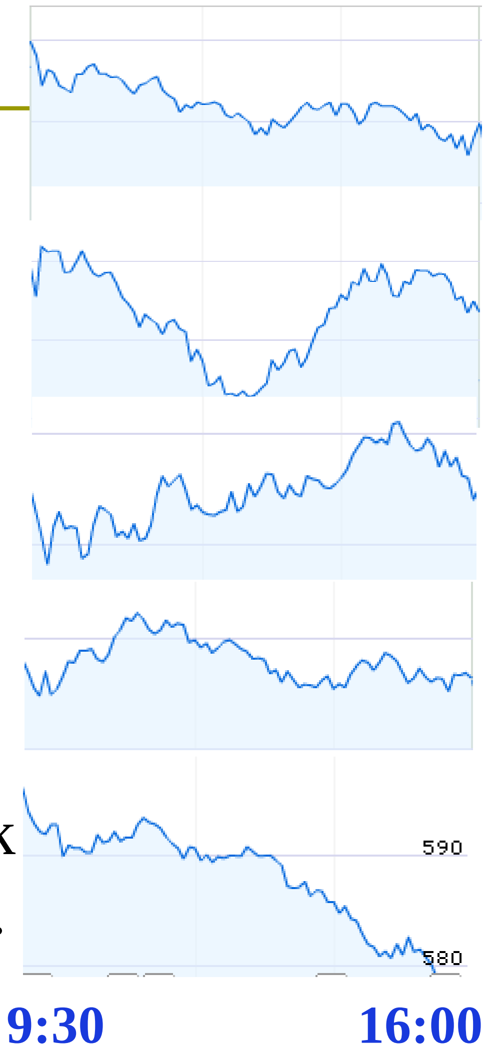
- For a fixed sample point s_j, $X(t, s_j)$ is a realization or a sample function of the random process.
- Notation:

$$X(t, s_j) \text{ or simply } x_j(t).$$



- The set of all possible sample functions is called the "ensemble".
- Example: The song played on a specific radio station at 7:30 AM can be considered as a random process.
  - Sample Space: Set of all possible songs
  - Sample functions: waveforms representing song number j, $x_j\ (t)$.

Random Processes



- For a fixed time t_0, the outcome of a Random experiment is a **random variable**. denoted by $X(t_0)$.
Example: The price of Apple's stock at 12pm each day.

- For a fixed sample point, s_j, the outcome is a **function of time**, $x_j\ (t)$. Example: Apple's stock price from 9:30 am to 4:00 pm on a specific day.

- For a fixed s_j and t_0 , $X(t_0, s_j) = \ x_j\ (t_0)$ is a **number**.

Example

$X(t) = A\ \cos(\omega_0\, t)$, where ω_0 is fixed and $A\ \sim U[0,1]$(Uniform distribution).

Cumulative Distribution of a Random Process

- For a RV to be fully characterized we need to have its cdf or pdf.

- For two RVs to be fully characterized we need to have their joint cdf or pdf:

- A random process consists of one RV at each instant of time, t. => In general consists of infinite RVs. => to be fully characterized we need to have the k-fold joint cdf (or pdf)

$$F_{X(t_1),\ldots,X(t_k)}\left(x_1,\ldots,x_k\right)=P\left(X(t_1)\le x_1,\ldots,X(t_k)\le x_k\right)$$

for any $t_1,\ldots,t_k$ and any real numbers $x_1, \ldots, x_k$.

- This makes the characterization of random processes impossible in general.
- Fortunately most random processes of interest can be modeled with certain features that makes them easier to characterize.

Stationarity

- For a general RP, the k-fold joint pdf is time-dependent.
- However if a RP is **“first-order stationary”,** then:

$$F_{X(t_1+\tau)}(x) = F_{X(t_1)}(x) \quad \text{for all } t_1 \text{ and } \tau.$$

- This means that the cdf (and pdf) of the RP X(t) at a fixed time t, is independent of time.
- Consequently the mean and variance of the RP X(t) at each fixed time t, are also independent of time.
- A random process is called **second-order stationary** if the 2nd order CDF is independent of time origin:

$$F_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2) = F_{X(t_1),X(t_2)}(x_1,x_2) \text{ for all } t_1, t_2 \text{ and } \tau.$$

- A random process is called **strict-sense stationary** if :

$$F_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_k+\tau)}(x_1,x_2,\ldots,x_k) = F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_2,\ldots,x_k).$$

for all k, $\tau, and\ t_1, t_2, \ldots, t_k,$ and for any k.

Example

$X(t) = A\ \cos(\omega_0\, t)$, where ω_0 is fixed and $A\ \sim U[0,1]$(Uniform distribution). Is $X(t)$ first-order stationary?

IID Random Processes

- An example of a strictly stationary process is one in which all X(t_i)'s are mutually **Independent and Identically Distributed**.
- Such a random process is called **IID random process**.
- In this case,

$$F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_2,\ldots,x_k)=$$

- Since the joint pdf above does not depend on the times {ti}, the process is strictly stationary.
- An example of IID process is white noise (studied later)
  - Widely used in communications theory

Covariance of Random Processes

- Recall: Covariance of two random variables:

$$Cov(X,Y) = E\{[X-\mu_X][Y-\mu_Y]\} = E\{XY\} - \mu_X \mu_Y$$

- Consider $X(t_1)$ and $X(t_2)$: samples of X(t) at t_1 and t_2.

  $X(t_1)$ and $X(t_2)$ are both random variables, so we can define their covariance:

$$Cov(X(t_1), X(t_2)) = E\{X(t_1)X(t_2)\} - \mu_{X(t_1)}\mu_{X(t_2)}$$

- Now, if $X(t)$ is second order (or higher) stationary, then:

Wide-Sense Stationarity (WSS)

- In many cases we do not require a random process to have all of the properties of the 2^{nd} order stationarity.

- A random process is said to be **wide-sense stationary** or **weakly stationary** if and only if
  - Its mean is independent of time
  - Its covariance depends only on the time difference.

- Note: SSS $\Rightarrow$ WSS, 2^{nd} order Stationary $\Rightarrow$ WSS. But the inverse of these statements is not necessarily true.

Autocorrelation of a Random Processes

- The autocorrelation function of a random process at t1 and t2 is defined as:

$$R_X(t_1,t_2) = E\left\{X(t_1)X^*(t_2)\right\} =$$

- If X(t) is stationary to the 2nd or higher order, $R_X(t_1, t_2)$ only depends on the time difference t_1-t_2, so it can be written as a single variable function:

Properties of Autocorrelation Function

For real-valued wide-sense stationary X(t), we have:

1. $R_X(0) = E\left\{X^2(t)\right\}.$

2. $R_X(\tau)$ has even symmetry: $R_X(-\tau) = R_X(\tau)$.

Proof:

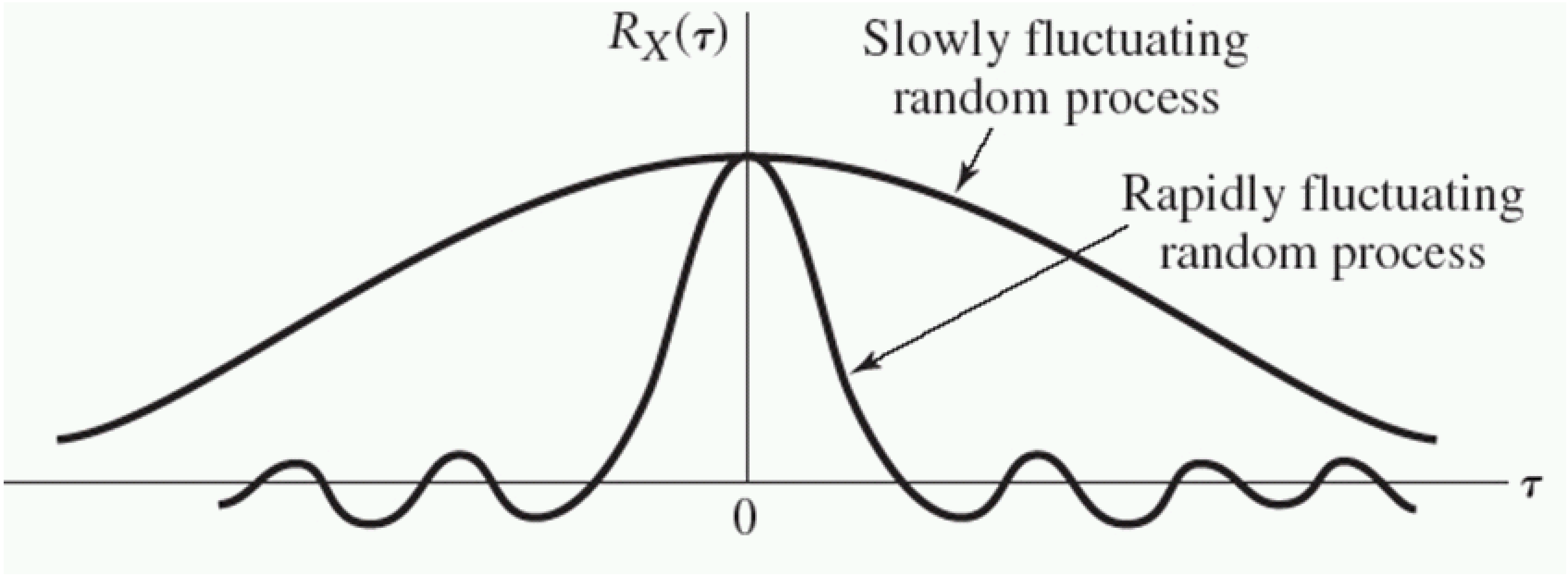
3. $R_X(\tau)$ is maximum at the $\tau = 0$.

- Proof:

Autocorrelation (Cont.)

$$R_X(t,s) = E\left\{X(t)X^*(s)\right\} = R_X(t-s).$$

- Autocorrelation is a measure of how fast a RP fluctuates:



Example 1

$$X(t) = A\cos(2\pi f t + \Theta)$$

A and f: constant. Θ: uniform random variable in $[0, 2\pi]$.

Find the autocorrelation of X(t). Is X(t) wide-sense stationary?

Example 2

$$X(t) = A\cos(2\pi f t)$$

A: uniform random variable in [0, 1], f: Constant

Find the autocorrelation of X(t). Is X(t) WSS?

Example 3

- A random process $X(t)$ consists of three possible sample functions: $x_1(t)=1$, $x_2(t)=3$, and $x_3(t)=\sin(t)$. Each occurs with equal probability. Find its mean and auto-correlation. Is it wide-sense stationary?
- **Solution:**