ENSC327/328
Communications Systems
15: Correlation and Spectral Density of Deterministic Signals

School of Engineering Science
Simon Fraser University

Outline

- Motivation: To study the noise performance of communication systems (analog and digital) we need to learn about “Random processes” (aka Stochastic Processes).
- First, we start from “deterministic” signals.
- Required Background: Probability Theory, Concepts of Energy signals & Power signals
- Energy spectral density and autocorrelation for deterministic signals(2.8)
- Power spectral density and autocorrelation for deterministic power signals
- In chapter 8, we extend these definitions to random processes.

Background: Energy Signals vs Power Signals

- Definition of Power and energy for an arbitrary signal x(t):

- When is a signal an Energy signal?

- when is a signal a Power Signal?

- Periodic signals: Energy signal or Power signal?

FT Property: Correlation

- Correlation Property: $\int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau) dt \quad \leftrightarrow \quad G_1(f) G_2^*(f)$

- Proof:

- Correlation measures the similarity between $g_1(t)$ and $g_2(t)$ shifted by τ seconds.

Rayleigh's Energy Theorem (Parseval's Theorem)

$$E = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 \, df$$

$\Rightarrow$ $|G(f)|^2$ can be viewed as energy density in freq domain.

Note that $|G(f)|^2$ is a real valued function.

Autocorrelation of an Energy Signal (2.8)

- Autocorrelation function of a deterministic energy signal:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t).x^*(t-\tau)dt$$

- This is a measure of the similarity between $x(t)$ and its shifted version $x(t-\tau)$.

- High autocorrelation ➔ high similarity between the signal and its shifted version ➔ the signal is changing ……………………..?

- Let's compare the autocorrelation with energy:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \qquad E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Example: Find the autocorrelation of $\boldsymbol{rect(t)}$

Example: Find the autocorrelation of $\boldsymbol{e^{-\alpha t}u(t)}$

Energy Spectral Density for Energy Signals

- As we already know, if we apply Perseval's theorem to the energy of a signal in time domain, we get:

- $|X(f)|^2$ is the "Energy Spectral Density" or the "Energy Density Spectrum" of the energy signal $x(t)$ and is shown with:

$$\psi_x(f) = |X(f)|^2$$, Unit :

Wiener-Khintchine Theorem

- Theorem: For energy signals, the **auto correlation function** and the **energy spectral density function** are Fourier Transform pairs:

$$R_x(\tau) \xleftrightarrow{\mathcal{F}} \psi_x(f)$$

Proof:

- We already saw the correlation theorem:

$$\int_{-\infty}^{\infty} g_1(t) g_2^*(t-\tau)dt \;\leftrightarrow\; G_1(f)G_2^*(f)$$

- Now replace both $g_1(t)$ and $g_2(t)$ with the same signal, $x(t)$:

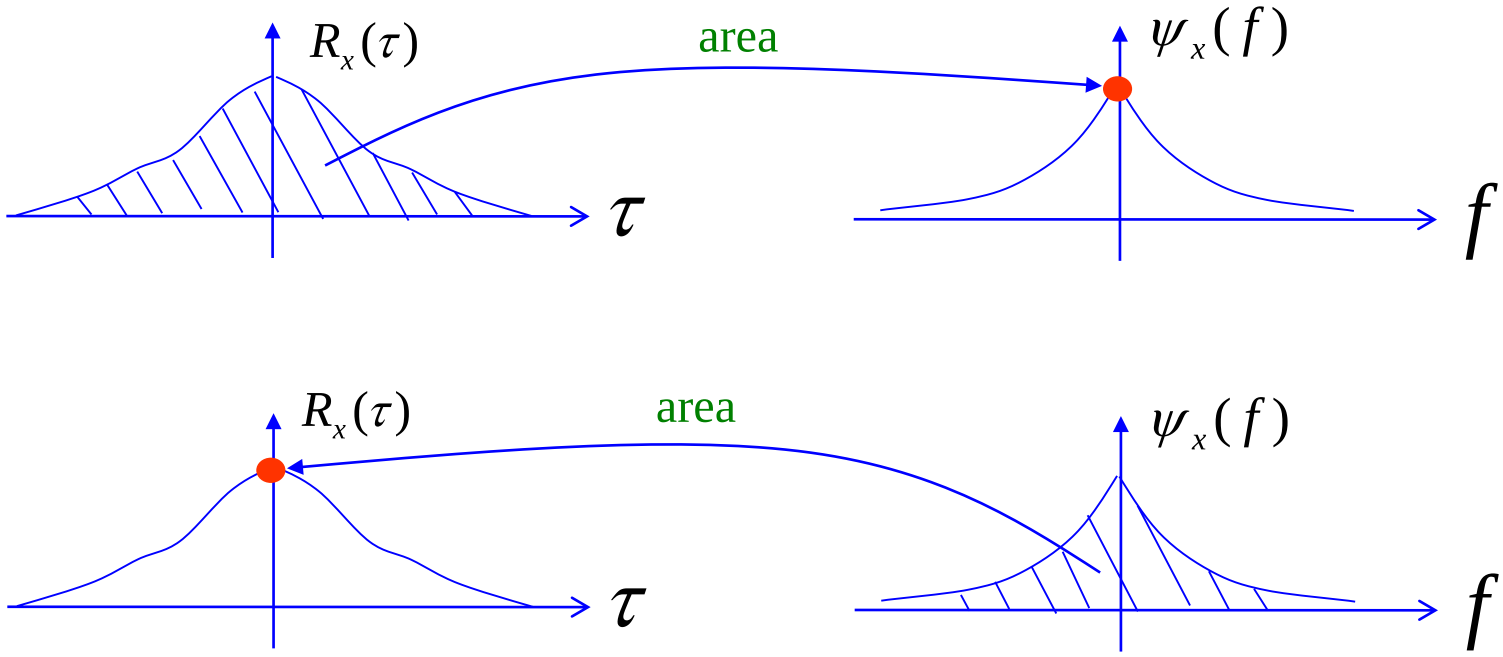
- **Important results from the W-K theorem:**

$$\psi_x(0) = \int_{-\infty}^{\infty} R_x(\tau)d\tau \qquad\qquad R_x(0) = \int_{-\infty}^{\infty} \psi_x(f)df$$

  - Proof: Home work!

Graphical Illustration of Previous Results

$$\psi_x(0) = \int_{-\infty}^{\infty} R_x(\tau)d\tau \qquad R_x(0) = \int_{-\infty}^{\infty} \psi_x(f)df$$



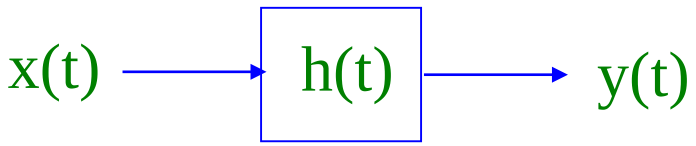
Example

- Find the autocorrelation and energy spectral density of

$$x(t) = A\,\mathrm{sinc}\,(t)$$

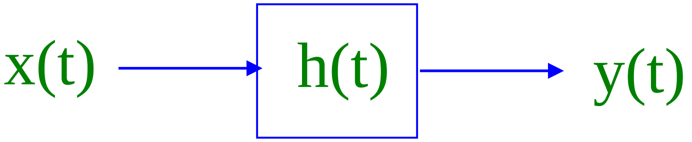
Effect of Filtering

- How does filtering an energy signal affect its Energy spectral density?



Example

$x(t)$ has a power spectral density function, $\psi_x(f)$= rect($\frac{f}{8}$) (J/kHz), (frequency is in kHz). The signal goes through an ideal BPF centered at 3 kHz with BW= 4 kHz. What are the energies contained in the input signal and the output signal, in Jouls?



Cross-correlation of energy signals

- The **cross-correlation** and **cross spectral density** functions, between two deterministic energy signals:

$$R_{xy}(\tau) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} x(t)y^{*}(t-\tau)dt \xleftrightarrow{\mathcal{F}} \psi_{xy}(f) = X_1(f)\, X_2^{*}(f)$$

- Note: Cross spectral density can in general be a complex valued signal.
- What is the cross correlation between $y(t)$ and $x(t)$:

- Orthogonality: x(t) and y(t) are said to be **<u>orthogonal</u>** if their cross-correlation at $\tau = 0$ is zero:

$$R_{xy}(0) = \int_{-\infty}^{\infty} x(t)y^{*}(t)dt = 0.$$

Autocorrelation and Power Spectral Density for Power Signals (2.9)

- Power signals have infinite energy but finite power.
- The "Autocorrelation of a Power signal" is thus defined as:

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x^*(t-\tau)dt$$

- The total power of the signal:

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

- If we truncate $x(t)$ to the interval of [-T,T]:

$$x_T(t) = x(t)rect(\frac{t}{2T}) = \begin{cases} x(t), & \text{-T} < \text{x} < \text{T}. \\ 0, & \text{otherwise.} \end{cases}$$

- Then the power of $x(t)$ can be written as :

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-\infty}^{\infty} |x_T(t)|^2 dt$$

Power Spectral Density (Cont.)

- Use Parseval's theorem (Rayleigh Energy Theorem):

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |X_T(f)|^2 \, df$$

- Exchange the limit and the integral:

$$P = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2 \right) df$$

- Thus, we can define the "Power Spectral Density" of $x(t)$:

$$\boxed{S_x(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2}$$

- Then:

Power Spectral Density for Periodic Signals

- Periodic signals are power signals with a Fourier Series expansion:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}, \quad t_0 \le t < t_0 + T_0$$

$$X_k = \frac{1}{T_o} \int_{T_o} x(t) e^{-jk\omega_o t} dt$$

$$\omega_0 = 2\pi / T_0$$

- The Autocorrelation function of a periodic signal is defined over one period of the signal:

$$R_x(\tau) = \frac{1}{T_0} \int_0^{T_0} x(t)\, x^*(t-\tau)\, dt$$

- It can be shown that the Power Spectral Density function is:

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} = \sum_{k=-\infty}^{\infty} |X_k|^2\, \delta(f - k\, f_0)$$

Summary

- Deterministic Energy signals:

$$R_x(\tau) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

$$\psi_x(f) = |X(f)|^2$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \psi_x(f) df$$

- Wiener-Kintchine Theorem:

$$R_x(\tau) \leftrightarrow \psi_x(f)$$

$$\psi_x(0) = \int_{-\infty}^{\infty} R_x(\tau) d\tau$$

$$R_x(0) = \int_{-\infty}^{\infty} \psi_x(f) df$$

- Deterministic Power signals (Aperiodic):

$$S_x(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(f) df$$

- Deterministic Power signals (Periodic):

$$R_x(\tau) = \frac{1}{T_0} \int_0^{T_0} x(t)\, x^*(t-\tau)\, dt$$

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} = \sum_{k=-\infty}^{\infty} |X_k|^2\, \delta(f - k f_0)$$

- In chapter 8, we will extend the autocorrelation, psd, and the Wiener-Khintchine Theorem to random processes.