ENSC327
Communications Systems
10: Wideband FM

School of Engineering Science
Simon Fraser University

Outline

- Required Background
- Spectrum of single tone, Wideband FM (4.5)
- Bessel Function
- BW of FM (4.6)

Required Background

- General form of PM or FM:
- Single tone FM:
- Complex envelope of a bandpass signal, $s(t) = A_c \cos(2\pi f_c t + \phi(t))$
- Fourier Series expansion of a periodic signal:
  - If $x(t)$is periodic with period= T_0, and Frequency= f_0, then $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \qquad X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-j2\pi n f_0 t} dt$$

Complex Envelope of Single-Tone FM

$$s(t) = A_c \cos(2\pi f_c t + \phi(t))$$

- Finding the FT of $s(t)$ is not easy ($\phi(t)$ is inside the cosine).
- To analyze the spectrum, we start with the **complex envelope** of $s(t)$:
- For the single tone FM, we have $s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$
- This is a periodic signal with $T = \frac{1}{f_m} \Longrightarrow$ Can be expanded using **Fourier Series**.

Fourier Series Expansion of $\tilde{s}(t)$

- Fourier Series expansion of $\tilde{s}(t)$:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

where:

$$c_n = f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} e^{j\beta \sin(2\pi f_m t) - j2\pi n f_m t} dt$$

- Let's make the change of variable, $x = 2\pi f_m t$, in the integral. Now we can write:

- The above integral is in the form of a very famous function called the **Bessel Function.**

The Bessel Function

- The n-th order Bessel function of the first kind with argument β, $J_n(\beta)$, is defined as:

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$$

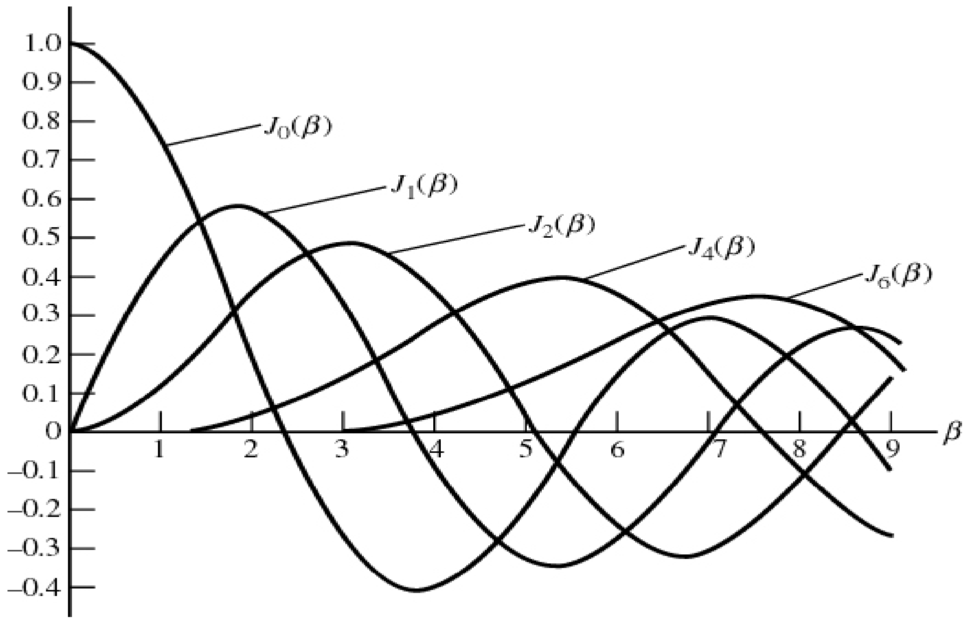
- The Bessel function has many applications including:
  - Understanding and analysis of single tone FM modulation
  - EM wave propagation
  - Heat conduction in cylinders
  - Solving Partial Differential Equations

(See: http://functionspace.com/topic/3564/

Interesting-Applications-of-Bessel-Functions)

Source: http://functionspace.com/topic/3564/Interesting-Applications-of-Bessel-Functions

Bessel Function (Cont.)



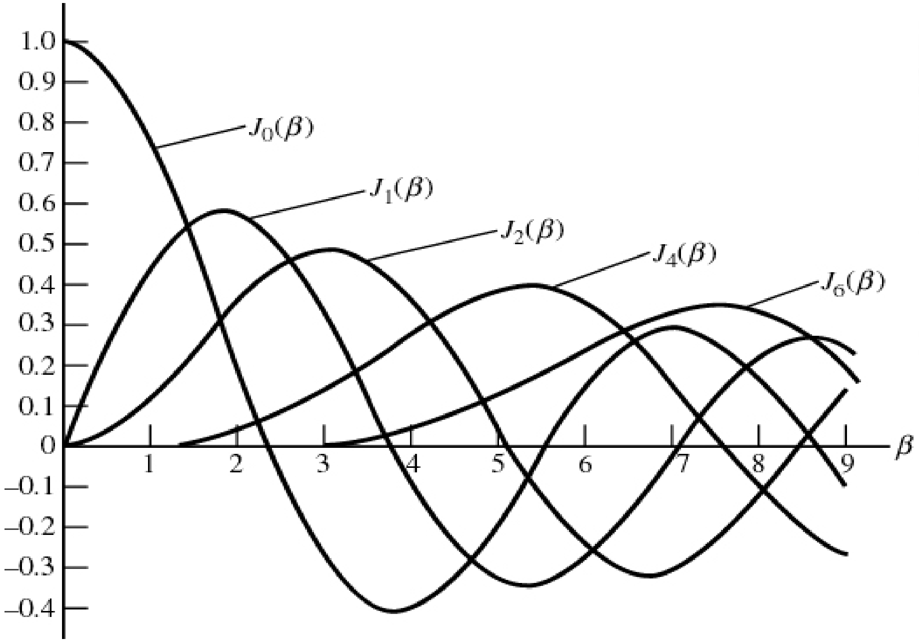
$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$$

- Bessel function table can be found in Appendix 3 of Text. (See notes for a copy)

Properties of Bessel functions (cont.)

- $J_n(\beta) = J_{-n}(\beta)$ for ***n* even**.
- $J_n(\beta) = -J_{-n}(\beta)$ for ***n* odd.**
- When β is very small: $J_0(\beta) \approx 1$, $J_1(\beta) \approx \beta/2$, $J_n(\beta) \approx 0$ n>1
- Power distribution: $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1.$
- Limit of Bessel function for large n: $\lim_{n\to\infty} J_n(\beta) = 0.$

Properties of Bessel functions (cont.)



Values of β for Which $J_n(\beta) = 0$ for $0 \le \beta \le 9$

n		β_{n0}	β_{n1}	β_{n2}
0	$J_0(\beta) = 0$	2.4048	5.5201	8.6537
1	$J_1(\beta) = 0$	0.0000	3.8317	7.0156
2	$J_2(\beta) = 0$	0.0000	5.1356	8.4172
4	$J_4(\beta) = 0$	0.0000	7.5883	—
6	$J_6(\beta) = 0$	0.0000	—	—

- Zeros of the Bessel functions: For each n , what values of β result in $J_n(\beta) = 0$? (Use the above graph or table)

Spectrum of Single Tone FM using the Bessel Function

- Now, back to single-tone FM.
- From slide 5, we have:

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx =$$

- Thus:

$$\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_m t)} =$$

- Finally,

$$s(t) = \mathrm{Re}\left\{ \tilde{s}(t) e^{j2\pi f_c t} \right\} =$$

Single Tone FM Spectrum (Cont.)

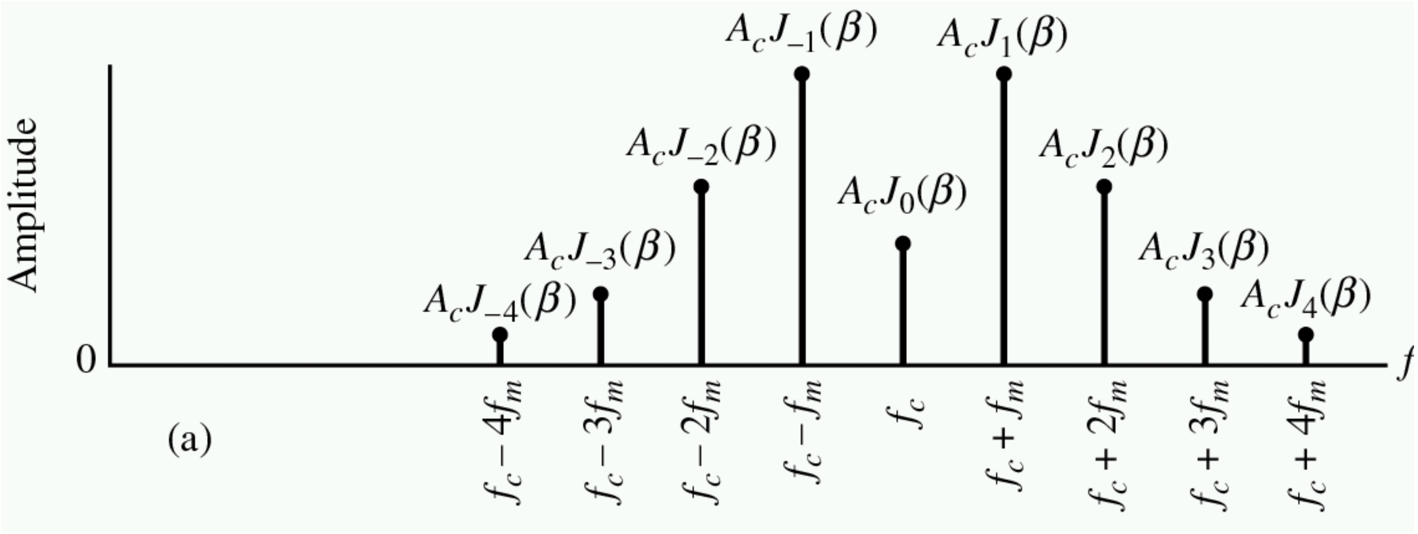
- We just found: $$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos 2\pi\left(f_c + nf_m\right)t$$
- Thus:

$S(f) =$

- Observations:
  - The spectrum of "Single Tone FM" contains components at
  - Theoretically the BW of single-tone FM is
  - However, because of properties of the Bessel func., the BW can be approximated by a limited value.

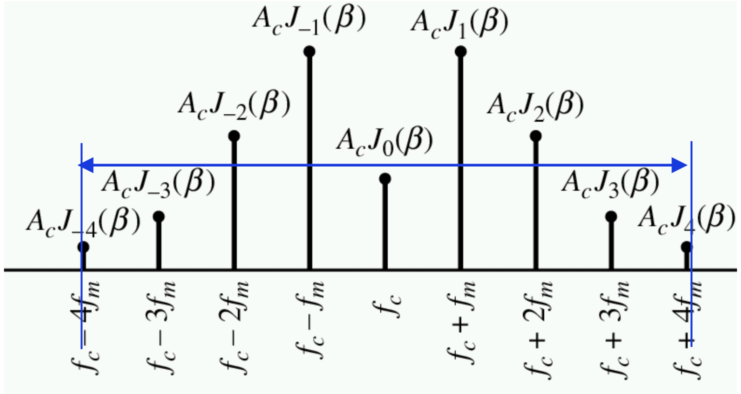
One sided spectrum of Single Tone FM

- Same spectrum as the one we drew on the previous page, only represented as one-sided (Amplitude of each component is doubled)



Bandwidth of Single Tone Wideband FM

- One approximation for limiting the BW of **single tone FM** is defined as the range of frequencies beyond which $J_n(\beta) < 0.01$ for all n.



- Using this approximation we do the following:

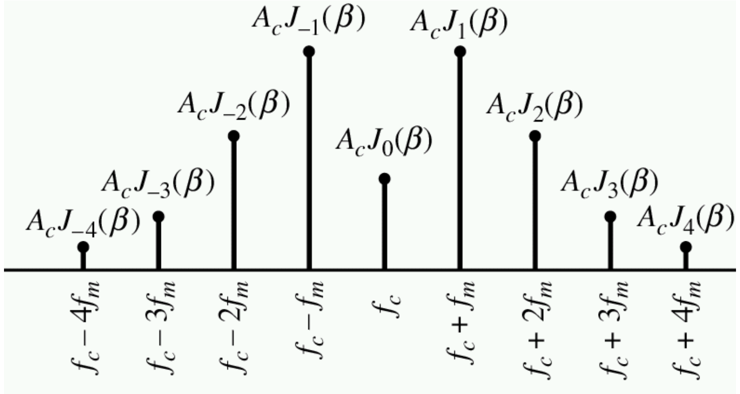
  1- Use the Bessel function table to find the value of n_{max} that satisfies the above requirement.

  2- The corresponding bandwidth is then:

Bandwidth of Single Tone FM (Cont.)

- Table 4.2 of text lists $2\,n_{max}$ for different values of β :

β	$2\,n_{max}$
0.1	2
0.3	4
0.5	4
1.0	6
2	8
5	16



- Relationship between bandwidth (B) and frequency deviation (Δf):

Example

Find the bandwidth of a single tone FM modulated signal with $\beta = 5$ and frequency of message (single tone) = 15 kHz. What is the bandwidth with respect to Δf. **Draw the spectrum.**

Normalized Bandwidth with respect to β

- Fig 4.9 of text shows the "normalized bandwidth" (B / Δf) w.r.t. β
- Observation: B/Δf approaches 2 as β increases.

(Recall that the range of the instantaneous frequency is 2Δf.)

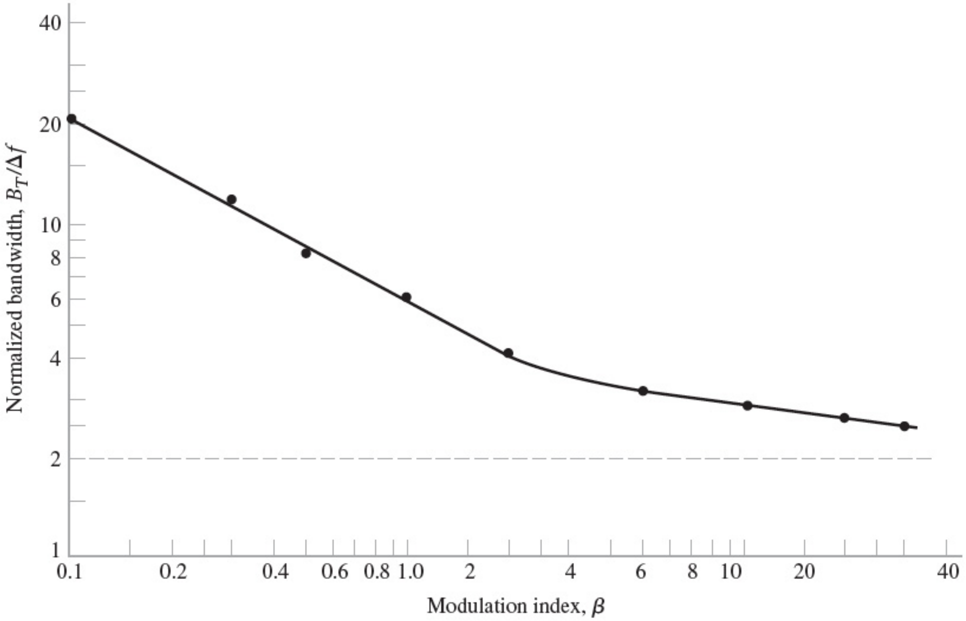
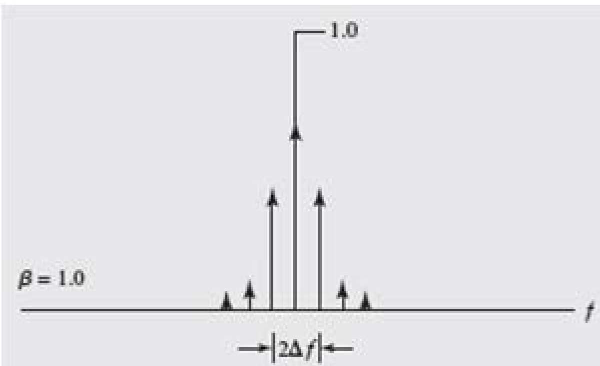


FIGURE 4.9 Universal curve for evaluating the one percent bandwidth of an FM wave.

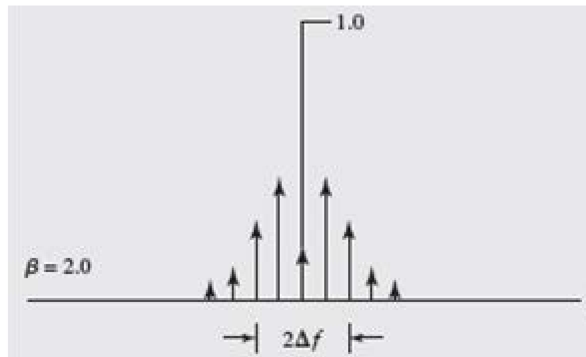
Effect of Message Amplitude on Spectrum

- For a fixed message frequency, f_m, let's look at the effect of changing the message amplitude, A_m.

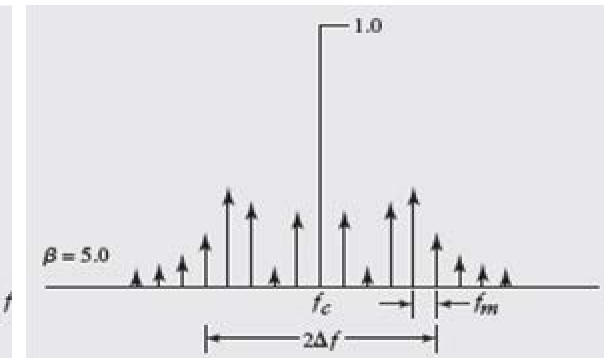
$\beta = 1$ or $\Delta f = f_m$



$\beta = 2$ or $\Delta f = 2 f_m$



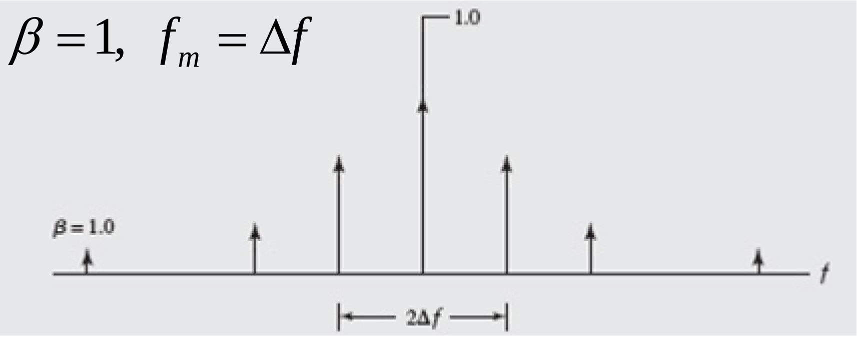
$\beta = 5$ or $\Delta f = 5 f_m$

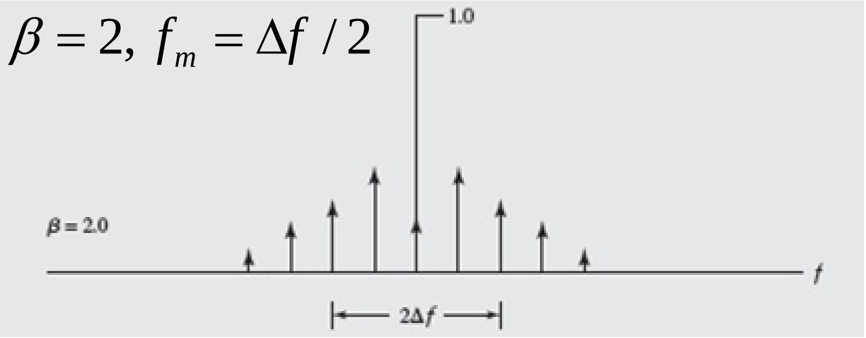


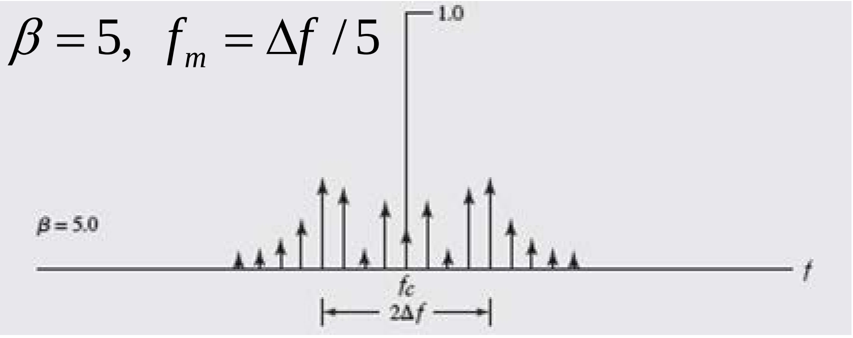
- Observations:

Effect of Message Frequency on Spectrum

- Now let's fix the message amplitude, A_m , and change message frequency, f_m.
- Observations:







Carson's Rule for Single Tone FM

- Carson's rule is an approximation to FM's BW, which applies to both single tone FM and FM with an arbitrary modulating message.
- For **single tone message**:

Carson's Rule: $B = 2\,\Delta f + 2\,f_m = 2\,\Delta f\,(1 + \frac{1}{\beta})$

  - For very small β:
  - For very large β:
  - For $1 \leq \beta \leq 20$, Carson's rule is an under-estimation of the BW.

Carson's Rule for General FM (Message=m(t))

- For FM with arbitrary message $m(t)$ with bandwidth W:

Carson's Rule: B= 2 (Δf + W)

- New definition:

Deviation Ratio: D = Δf /W = $k_f \max|m(t)|\ /W$

(This is a generalization of $\beta = \Delta f\ /\ f_m$)

- Carson's rule can be written as:

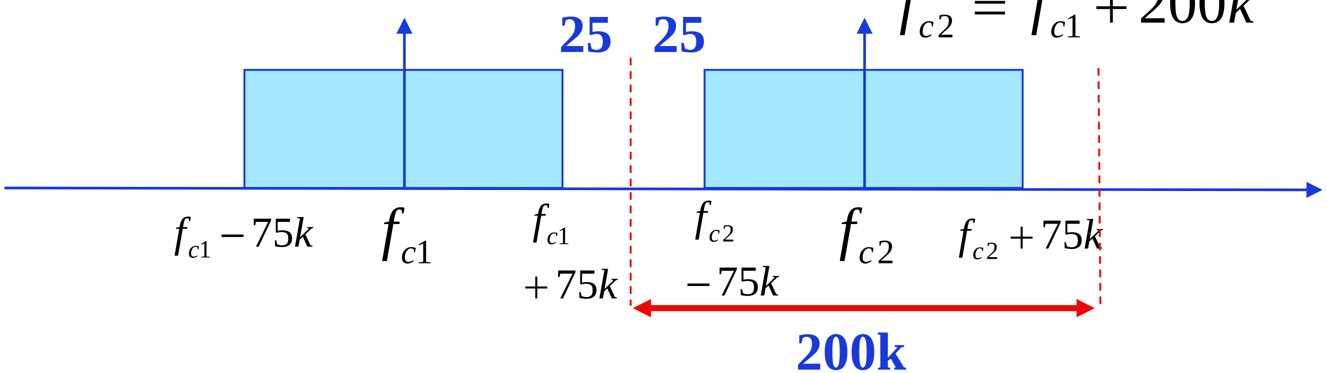
$$B = 2\,(D + 1)W$$

Example

- In FM radio, the max message bandwidth is W = 15kHz, and the allowed max frequency deviation is Δf = 75 KHz. What is the BW of FM radio channels as per Carson's rule? Compare with the previous example which used single tone.

In practice, 200 kHz is allocated. This leaves a 25 kHz guard region above and below the carrier freq to reduce the interference with other FM channels.

$$f_{c2} = f_{c1} + 200k$$



APPENDIX 3

BESSEL FUNCTIONS

A3.1 *Series Solution of Bessel's Equation*

In its most basic form, *Bessel's equation of order n* is written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{A3.1}$$

which is one of the most important of all variable-coefficient differential equations. For each order n, a solution of this equation is defined by the power series

$$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{2}x\right)^{n+2m}}{m!(n+m)!} \tag{A3.2}$$

The function $J_n(x)$ is called a *Bessel function of the first kind of order n*. Equation (A3.1) has two coefficient functions—namely, $1/x$ and $(1 - n^2/x^2)$. Hence, it has no finite singular points except the origin. It follows therefore that the series expansion of Eq. (A3.2) converges for all $x > 0$. Equation (A3.2) may thus be used to numerically calculate $J_n(x)$ for $n = 0, 1, 2, \ldots$. Table A3.1 presents values of $J_n(x)$ for different orders n and varying x. It is of interest to note that the graphs of $J_0(x)$ and $J_1(x)$ resemble the graphs of cos x and sin x, respectively; see the graphs of Fig. 4.6 in Chapter 4.

TABLE A3.1 *Table of Bessel Functions*[a]

				$J_n(x)$					
$n\backslash x$	0.5	1	2	3	4	6	8	10	12
0	0.9385	0.7652	0.2239	−0.2601	−0.3971	0.1506	0.1717	−0.2459	0.0477
1	0.2423	0.4401	0.5767	0.3391	−0.0660	−0.2767	0.2346	0.0435	−0.2234
2	0.0306	0.1149	0.3528	0.4861	0.3641	−0.2429	−0.1130	0.2546	−0.0849
3	0.0026	0.0196	0.1289	0.3091	0.4302	0.1148	−0.2911	0.0584	0.1951
4	0.0002	0.0025	0.0340	0.1320	0.2811	0.3576	−0.1054	−0.2196	0.1825
5	—	0.0002	0.0070	0.0430	0.1321	0.3621	0.1858	−0.2341	−0.0735
6		—	0.0012	0.0114	0.0491	0.2458	0.3376	−0.0145	−0.2437
7			0.0002	0.0025	0.0152	0.1296	0.3206	0.2167	−0.1703
8			—	0.0005	0.0040	0.0565	0.2235	0.3179	0.0451
9				0.0001	0.0009	0.0212	0.1263	0.2919	0.2304
10				—	0.0002	0.0070	0.0608	0.2075	0.3005
11					—	0.0020	0.0256	0.1231	0.2704
12						0.0005	0.0096	0.0634	0.1953
13						0.0001	0.0033	0.0290	0.1201
14						—	0.0010	0.0120	0.0650

[a]For more extensive tables of Bessel functions, see Abramowitz and Stegun (1965, pp. 358–406).