# ENSC-327 Formula sheet for Final Test

## Trigonometric Identities:

$$\cos(u) = \frac{e^{ju} + e^{-ju}}{2}$$

$$\sin(u) = \frac{e^{ju} - e^{-ju}}{2j}$$

$$\cos^2(u) - \sin^2(u) = \cos(2u)$$

$$2\sin(u)\cos(u) = \sin(2u)$$

$$\cos(u)\cos(v) = \frac{1}{2}\cos(u-v) + \frac{1}{2}\cos(u+v)$$

$$\sin(u)\cos(v) = \frac{1}{2}\sin(u-v) + \frac{1}{2}\sin(u+v)$$

$$\sin(u)\sin(v) = \frac{1}{2}\cos(u-v) - \frac{1}{2}\cos(u+v)$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

## Fourier Transform and Inverse Fourier Transform:

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt, \qquad g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df$$

## Fourier Series Expansion for periodic signals:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}, \quad \text{when } X_n = \int_{-\infty}^{\infty} x(t) e^{-j2\pi n f_0 t} dt$$

## Properties of Fourier Transform:

| Name | Time-domain operation (signals assumed real) | Frequency-domain operation |
|---|---|---|
| Superposition | $a_1 x_1(t) + a_2 x_2(t)$ | $a_1 X_1(f) + a_2 X_2(f)$ |
| Time delay | $x(t - t_0)$ | $X(f)\exp(-j2\pi t_0 f)$ |
| Scale change | $x(at)$ | $\lvert a\rvert^{-1} X(f/a)$ |
| Time reversal | $x(-t)$ | $X(-f) = X^*(f)$ |
| Duality | $X(t)$ | $x(-f)$ |
| Frequency translation | $x(t)\exp(j2\pi f_0 t)$ | $X(f - f_0)$ |
| Modulation | $x(t)\cos(2\pi f_0 t)$ | $\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$ |
| Convolution[4] | $x_1(t) * x_2(t)$ | $X_1(f)X_2(f)$ |
| Multiplication | $x_1(t)x_2(t)$ | $X_1(f) * X_2(f)$ |
| Differentiation | $\frac{d^n x(t)}{dt^n}$ | $(j2\pi f)^n X(f)$ |
| Integration | $\int_{-\infty}^{t} x(\lambda)\, d\lambda$ | $X(f)/(j2\pi f) + \frac{1}{2}X(0)\delta(f)$ |

## Parseval's theorem:

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

## Conjugation rule:

$$g(t) \leftrightarrow G(f) \quad \rightarrow \quad g^*(t) \leftrightarrow G^*(-f)$$

Hilbert Transform:

$$\hat{x}(t) \overset{\Delta}{=} x(t) * \frac{1}{\pi t}$$

$$\hat{X}(f) = -j\,\mathrm{sgn}(f)X(f)$$

Single Sideband Modulation:

USB

$$s_{ssb}(t) = \frac{A_c}{2}\left(m(t)\cos(2\pi f_c t) \mp \hat{m}(t)\sin(2\pi f_c t)\right)$$

LSB

**PM**

$$s(t) = A_c \cos(2\pi f_c t + k_p m(t))$$

**FM**

$$s(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau)$$

Single Tone FM:

$$s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos 2\pi\left(f_c + nf_m\right)t$$

Where:

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)}dx$$

(Bessel Function)

$$\boxed{\beta = k_f A_m / f_m}$$

Carson's rule for BW of FM:

- Single Tone $\boxed{\text{Carson's Rule: } B = 2\,\Delta f + 2\,f_m = 2\,\Delta f\,(1 + \frac{1}{\beta})}$

  Where: $\beta = \Delta f \,/\, f_m$

- General message, m(t) : $\boxed{B = 2\,(D+1)W}$

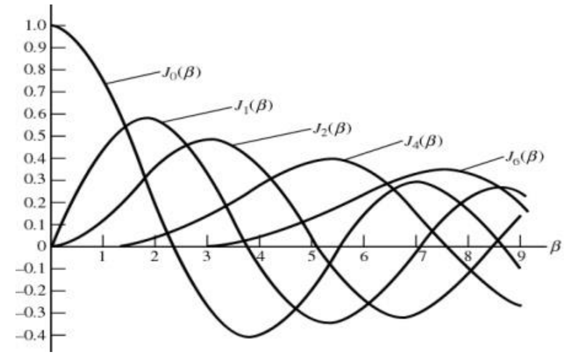
  Where Deviation Ratio: D = $\Delta f$ /W = $k_f$ max$|m(t)|$ $/W$

$J_n(x)$

| $n \backslash x$ | 0.5 | 1 | 2 | 3 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.9385 | 0.7652 | 0.2239 | −0.2601 | −0.3971 | 0.1506 | 0.1717 | −0.245 |
| 1 | 0.2423 | 0.4401 | 0.5767 | 0.3391 | −0.0660 | −0.2767 | 0.2346 | 0.043 |
| 2 | 0.0306 | 0.1149 | 0.3528 | 0.4861 | 0.3641 | −0.2429 | −0.1130 | 0.254 |
| 3 | 0.0026 | 0.0196 | 0.1289 | 0.3091 | 0.4302 | 0.1148 | −0.2911 | 0.058 |
| 4 | 0.0002 | 0.0025 | 0.0340 | 0.1320 | 0.2811 | 0.3576 | −0.1054 | −0.219 |
| 5 | — | 0.0002 | 0.0070 | 0.0430 | 0.1321 | 0.3621 | 0.1858 | −0.234 |
| 6 | | — | 0.0012 | 0.0114 | 0.0491 | 0.2458 | 0.3376 | −0.014 |
| 7 | | | 0.0002 | 0.0025 | 0.0152 | 0.1296 | 0.3206 | 0.216 |

Value of $\beta$ for which $J_n(\beta) = 0$ for $0 \le \beta \le 9$.

| n | | $\beta_{n0}$ | $\beta_{n1}$ | $\beta_{n2}$ |
|---|---|---|---|---|
| 0 | $J_0(\beta) = 0$ | 2.4 | 5.5 | 8.7 |
| 1 | $J_1(\beta) = 0$ | 0 | 3.8 | 7.0 |
| 2 | $J_2(\beta) = 0$ | 0 | 5.1 | 8.4 |
| 4 | $J_4(\beta) = 0$ | 0 | 7.6 | - |




## TABLE A6.2 *Fourier-Transform Pairs*

| *Time Function* | *Fourier Transform* |
|---|---|
| $\text{rect}\left(\frac{t}{T}\right)$ | $T \operatorname{sinc}(fT)$ |
| $\operatorname{sinc}(2Wt)$ | $\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$ |
| $\exp(-at)u(t), \quad a > 0$ | $\frac{1}{a + j2\pi f}$ |
| $\exp(-a\lvert t\rvert), \quad a > 0$ | $\frac{2a}{a^2 + (2\pi f)^2}$ |
| $\exp(-\pi t^2)$ | $\exp(-\pi f^2)$ |
| $\begin{cases} 1 - \frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \ge T \end{cases}$ | $T \operatorname{sinc}^2(fT)$ |
| $\delta(t)$ | $1$ |
| $1$ | $\delta(f)$ |
| $\delta(t - t_0)$ | $\exp(-j2\pi f t_0)$ |
| $\exp(j2\pi f_c t)$ | $\delta(f - f_c)$ |
| $\cos(2\pi f_c t)$ | $\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$ |
| $\sin(2\pi f_c t)$ | $\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$ |
| $\operatorname{sgn}(t)$ | $\frac{1}{j\pi f}$ |
| $\frac{1}{\pi t}$ | $-j \operatorname{sgn}(f)$ |
| $u(t)$ | $\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$ |
| $\sum_{i=-\infty}^{\infty} \delta(t - iT_0)$ | $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$ |

Autocorrelation and power spectral density deterministic signals

**Energy Signals:**

$$R_x(\tau) \overset{\Delta}{=} \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

$$\psi_x(f) = |X(f)|^2$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \psi_x(f)df$$

Theorem: For energy signals, the auto correlation function and power spectral density function are Fourier Transform pairs.

**Power signals (Aperiodic):**

$$S_x(f) = \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$$

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_x(f)df$$

**Power signals(Periodic):**

□ Deterministic Power signals (periodic):

$$R_x(\tau) = \sum_{n=-\infty}^{\infty} |X_n|^2 e^{j2\pi n f_0 t} \qquad S_x(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0)$$

where $X_n$'s are the Fourier Series coefficients of $x(t)$.

Conditional probability and Baye's Rule:

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Autocorrelation of a R.P.:

$$R_X(t_1, t_2) = E\{X(t_1)X^*(t_2)\}$$

Theorem: <u>For WSS random processes, the autocorrelation function and the power spectral density, $S_X(f)$, are Fourier transform pairs.</u>

Distribution of a Gaussian (Normal) R.V. :

$$X \sim \mathcal{N}(\mu, \sigma^2). \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X \sim \mathcal{N}(0,1)$. then Q-function: (Table values will be given if needed):

$$Q(u) = \int_u^{\infty} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(s)ds.$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau).$$

$$R_{YX}(\tau) = R_X(\tau) * h(\tau).$$

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau).$$

Narrow band noise:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

Where:

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \le f \le B. \\ 0, & \text{otherwise.} \end{cases}$$

Various formulas for SNR and BER (digital baseband) will either be given to you as needed, or you may be asked <u>to derive them</u> or <u>prove them</u>.