Logical Agents: Propositional Logic

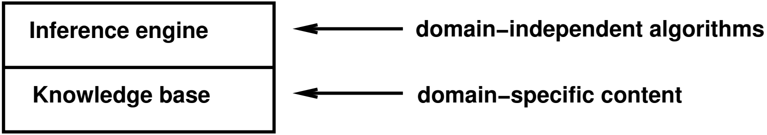
Chapter 7

Outline

Topics:

- Knowledge-based agents
- Example domain: The Wumpus World
- Logic in general
  - models and entailment
- Propositional (Boolean) logic
- Equivalence, validity, satisfiability
- Inference rules and theorem proving
  - forward chaining
  - backward chaining
  - resolution

Knowledge bases



- *Knowledge base* = set of *sentences* in a *formal* language
- *Declarative* approach to building an agent (or other system).
  - Declarative: Sentences express assertions about the domain
- Knowledge base operations:
  - *Tell* it what it needs to know
  - *Ask* (itself?) what to do – *query*
    - ☞ Answers should follow from the contents of the KB

Knowledge bases

Agents can be viewed:

- at the *knowledge level*
  - i.e., *what they know*, regardless of how implemented
- at the *implementation level* (also called the *symbol level*)
  - i.e., data structures and algorithms that manipulate them

☞ Compare: abstract data type vs. data structure used to implement an ADT.

A simple knowledge-based agent

Function KB-Agent(percept) returns an action
   static: KB, a knowledge base
           t, a counter, initially 0, indicating time

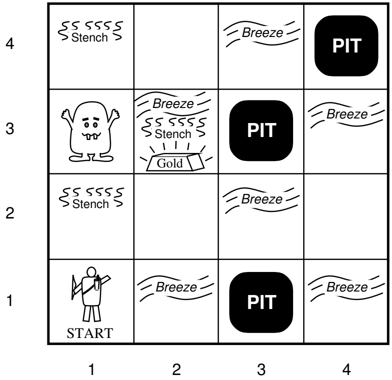
```
Tell(KB, Make-Percept-Sentence(percept, t))
action ← Ask(KB, Make-Action-Query(t))
Tell(KB, Make-Action-Sentence(action, t))
t ← t + 1
return action
```

A simple knowledge-based agent

In the most general case, the agent must be able to:

- Represent states, actions, etc.
- Incorporate new percepts
- Update internal representations of the world
- Deduce hidden/implicit properties of the world
- Deduce appropriate actions

The Wumpus World



Wumpus World PEAS description

Performance measure: gold: $+1000$; death: -1000; -1 per step; -10 for using the arrow

Environment:

- Squares adjacent to wumpus are smelly
- Squares adjacent to pit are breezy
- Glitter iff gold is in the same square
- Shooting kills wumpus if you are facing it
- Shooting uses up the only arrow
- Grabbing picks up gold if in same square
- Releasing drops the gold in same square

Actuators: Left turn, Right turn, Forward, Grab, Release, Shoot

Sensors: Breeze, Glitter, Smell, Bump, Scream

Wumpus world characterisation

Observable: ??

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Observable: No – only *local* perception

Deterministic: ??

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Static: Yes – Wumpus and pits do not move

Discrete: ??

Wumpus world characterisation

Observable: No – only *local* perception

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Static: Yes – Wumpus and pits do not move

Discrete: Yes

Single-agent: ??

Wumpus world characterisation

Observable: No – only *local* perception

Deterministic: Yes – outcomes exactly specified

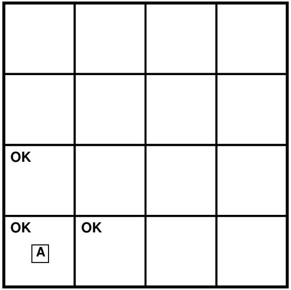
Episodic: No – sequential at the level of actions

Static: Yes – Wumpus and pits do not move

Discrete: Yes

Single-agent: Yes – Wumpus is essentially a natural feature

Exploring a wumpus world



Percept:

[Stench: No, Breeze: No, Glitter: No, Bump: No, Scream: No]

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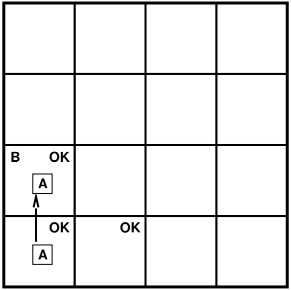

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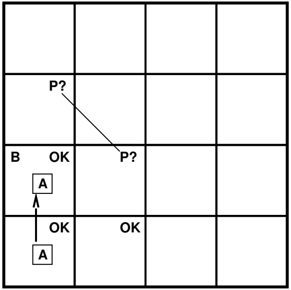

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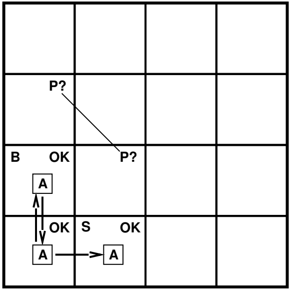

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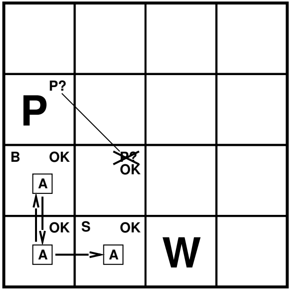

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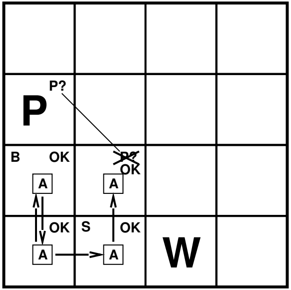

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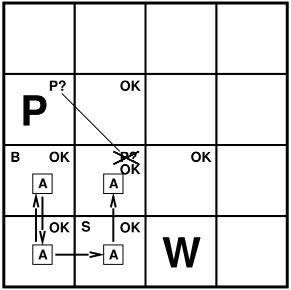

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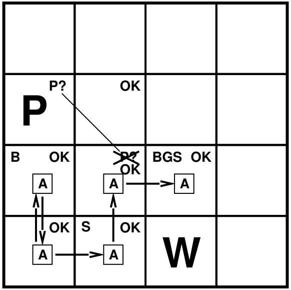

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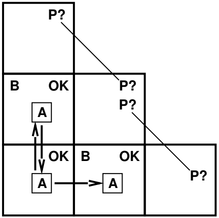
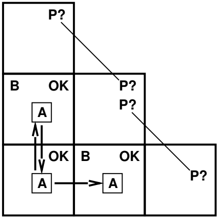
Tight spots



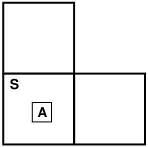
- Breeze in (1,2) and (2,1)
  $\Rightarrow$ no safe actions

Tight spots



- Breeze in (1,2) and (2,1)
  $\Rightarrow$ no safe actions
- If pits are uniformly distributed, (2,2) is more likely to have a pit than (1,3) + (3,1)

Tight spots



- Smell in (1,1)
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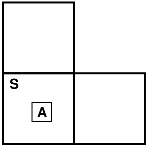


- Smell in (1,1)

  $\Rightarrow$ cannot safely move
- Can use a strategy of *coercion*:
  - shoot straight ahead
  - wumpus was there $\Rightarrow$ dead $\Rightarrow$ safe
  - wumpus wasn't there $\Rightarrow$ safe

Logic in the Wumpus World

- As the agent moves and carries out sensing actions, it performs *logical reasoning*.
  - E.g.: "If (1,3) or (2,2) contains a pit and
    (2,2) doesn't contain a pit
    then (1,3) must contain a pit".
- We'll use logic to represent information about the wumpus world, and to reason about this world.

Logic in general

- A *logic* is a formal language for representing information such that conclusions can be drawn
- The *syntax* defines the sentences in the language
- The *semantics* define the "meaning" of sentences;
  - i.e., define *truth* of a *sentence* in a *world*
- E.g., in the language of arithmetic
  - $x + 2 \geq y$ is a sentence; $x2 + y >$ is not a sentence
  - $x + 2 \geq y$ is true iff the number $x + 2$ is not less than y
  - $x + 2 \geq y$ is true in a world where $x = 7, \; y = 1$
  - $x + 2 \geq y$ is false in a world where $x = 0, \; y = 6$

Semantics: Entailment

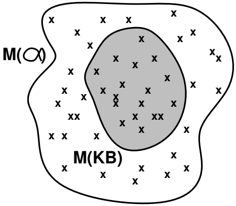
- *Entailment* means that one thing *follows from* another:

  $KB \models \alpha$

- Knowledge base KB *entails* sentence α if and only if:
  - α is true in all worlds where KB is true
  - Or: if KB is true then α must be true.
- E.g., the KB containing "the Canucks won" entails "either the Canucks won or the Leafs won"
- E.g., $x + y = 4$ entails $4 = x + y$
- Entailment is a relationship between sentences (i.e., *syntax*) that is based on *semantics*
- Note: Brains (arguably) process *syntax* (of some sort).

Semantics: Models

- Logicians typically think in terms of *models*, which are complete descriptions of a world, with respect to which truth can be evaluated
- We say m *is a model of* a sentence α if α is true in m
- $M(\alpha)$ is the set of all models of α
- Thus $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$
- E.g. KB = Canucks won and Leafs won
  α = Canucks won



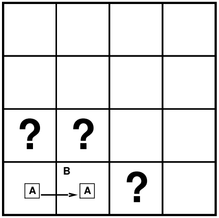
Aside: Semantics

- Logic texts usually distinguish:
  - an *interpretation*, which is some possible world or complete state of affairs, from
  - a *model*, which is an interpretation that makes a specific sentence or set of sentences true.
- The text uses *model* in both senses (so don't be confused if you've seen the terms interpretation/model from earlier courses).
  - And if you haven't, ignore this slide!
- We'll use the text's terminology.

Entailment in the Wumpus World

Consider the situation where the agent detects nothing in [1,1], moves right, detects a breeze in [2,1]

- Consider possible models for just the **?**'s, assuming only pits

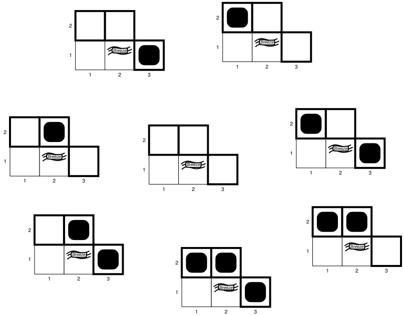


- With no information:

  3 Boolean choices ⇒ 8 possible models

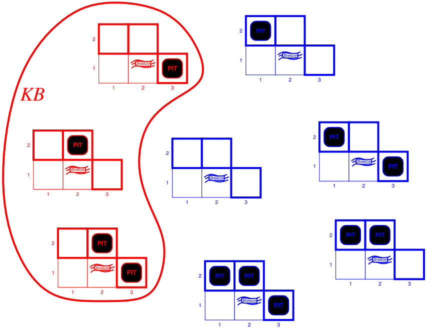
Wumpus Models

Consider possible arrangements of pits in [1,2], [2,2], and [3,1], along with observations:



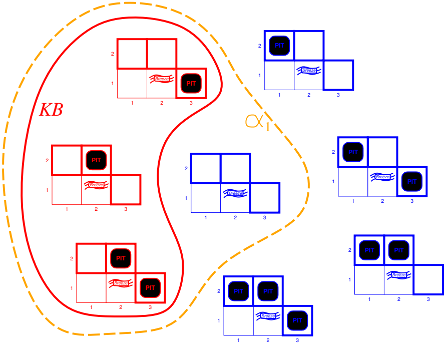
Wumpus Models

Models of the KB:



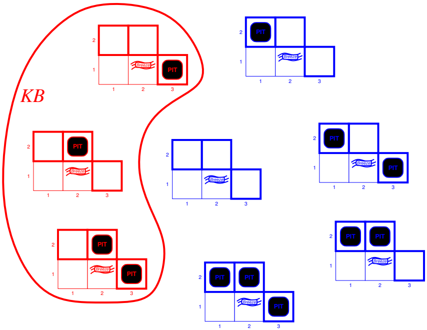
- KB = wumpus-world rules + observations

Wumpus Models



- KB = wumpus-world rules + observations
- α_1 = "[1,2] is safe", $KB \models \alpha_1$, proved by *model checking*

Wumpus Models: Another Example



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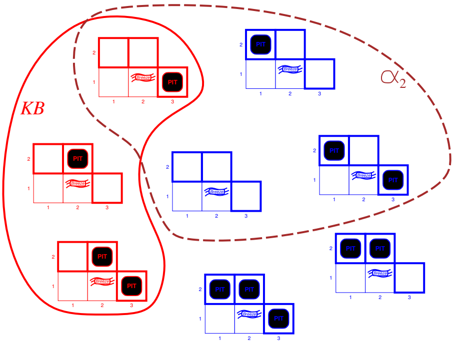

- KB = wumpus-world rules + observations
- α_2 = "[2,2] is safe", $KB \not\models \alpha_2$

Inference

In the case of propositional logic, we can use entailment to derive conclusions by enumerating models.

- This is the usual method of computing *truth tables*
- I.e. can use entailment to do *inference*.
- In first order logic we generally can't enumerate all models (since there may be infinitely many of them and they may have an infinite domain).
- An *inference procedure* is a (syntactic) procedure for deriving some formulas from others.

Inference

- Inference is a procedure for computing entailments.
- $KB \vdash \alpha =$ sentence α can be derived from KB by the inference procedure
- Entailment says what things are implicitly true in a KB.
  - Inference is intended to *compute* things that are implicitly true.

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Desiderata:

- *Soundness*: An inference procedure is sound if
  whenever $KB \vdash \alpha$, it is also true that $KB \models \alpha$.
- *Completeness*: An inference procedure is complete if
  whenever $KB \models \alpha$, it is also true that $KB \vdash \alpha$.

Propositional Logic: Syntax

- Propositional logic is a simple logic – illustrates basic ideas
- We first specify the *proposition symbols* or *(atomic) sentences*: P_1, P_2 etc.
- Then we define the language:

  If S_1 and S_2 are sentences then:

  - $\neg S_1$ is a sentence (*negation*)
  - $S_1 \wedge S_2$ is a sentence (*conjunction*)
  - $S_1 \vee S_2$ is a sentence (*disjunction*)
  - $S_1 \Rightarrow S_2$ is a sentence (*implication*)
  - $S_1 \equiv S_2$ is a sentence (*biconditional*)

Propositional Logic: Semantics

- Each model assigns true or false to each proposition symbol
- E.g.: $P_{1,2} \leftarrow \mathit{true}$, $P_{2,2} \leftarrow \mathit{true}$, $P_{3,1} \leftarrow \mathit{false}$
  (With these symbols, 8 possible models, can be enumerated.)
- Rules for evaluating truth with respect to a model m:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true *and*	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true *or*	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false *or*	S_2	is true
$S_1 \equiv S_2$	is true iff	$S_1 \Rightarrow S_2$	is true *and*		
		$S_2 \Rightarrow S_1$	is true		

- Simple recursive process evaluates an arbitrary sentence, e.g.,
  $\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = \mathit{true} \wedge (\mathit{false} \vee \mathit{true}) = \mathit{true} \wedge \mathit{true} = \mathit{true}$

Truth Tables for Connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

Wumpus World Sentences

- Let $P_{i,j}$ be true if there is a pit in $[i, j]$.
- Let $B_{i,j}$ be true if there is a breeze in $[i, j]$.
- Information from sensors: $\neg P_{1,1}$, $\neg B_{1,1}$, $B_{2,1}$
- Also know: "pits cause breezes in adjacent squares"

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- Information from sensors: $\neg P_{1,1}$, $\neg B_{1,1}$, $B_{2,1}$
- "A square is breezy *if and only if* there is an adjacent pit"

  $$B_{1,1} \equiv (P_{1,2} \vee P_{2,1})$$
  $$B_{2,1} \equiv (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

  - Note: $B_{1,1}$ has no "internal structure" – think of it as a string.
  - So must write 1 formula for each square.

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  - Note: $B_{1,1}$ has no "internal structure" – think of it as a string.
  - So must write 1 formula for each square.
- Using logic can conclude $\neg P_{1,2}$ and $\neg P_{2.1}$ from $\neg B_{1,1}$.
- Note, if you wrote the above as:

  $B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})$

  (I.e. "A breeze implies a pit in an adjacent square")

  you could not derive $\neg P_{1,2}$ and $\neg P_{2.1}$ from $\neg B_{1,1}$.

  ☞ Crucial to express *all* information

Wumpus World KB

For the part of the Wumpus world we're looking at, let

$$KB = \{R_1, R_2, R_3, R_4, R_5\}$$

where

$$\begin{array}{lll} R_1 & \text{is} & \neg P_{1,1} \\ R_2 & \text{is} & B_{1,1} \equiv (P_{1,2} \vee P_{2,1}) \\ R_3 & \text{is} & B_{2,1} \equiv (P_{1,1} \vee P_{2,2} \vee P_{3,1}) \\ R_4 & \text{is} & \neg B_{1,1} \\ R_5 & \text{is} & B_{2,1} \end{array}$$

Truth Tables for Inference

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
f	f	f	f	f	f	f	t	t	t	t	f	f
f	f	f	f	f	f	t	t	t	f	t	f	f
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
f	t	f	f	f	f	f	t	t	f	t	t	f
f	t	f	f	f	f	t	t	t	t	t	t	$\underline{t}$
f	t	f	f	f	t	f	t	t	t	t	t	$\underline{t}$
f	t	f	f	f	t	t	t	t	t	t	t	$\underline{t}$
f	t	f	f	t	f	f	t	f	f	t	t	f
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	t	t	t	t	t	t	f	t	t	f	t	f

- Enumerate rows (different assignments to symbols),
- For $KB \models \alpha$, if KB is true in row, check that α is too

Inference by Enumeration

Function TT-Entails?(KB, α) returns true or false
    inputs: KB, the knowledge base, a sentence in propositional logic
        α the query, a sentence in propositional logic
    symbols $\leftarrow$a list of the proposition symbols in KB and α
    return TT-Check-All(KB, α, symbols, [])

Inference by Enumeration

```
Function TT-Check-All(KB, α, symbols, model) returns true or false
    if Empty?(symbols) then
        if PL-True?(KB, model) then return PL-True?(α, model)
            else return true
    else do
        P ←First(symbols); rest ←Rest(symbols)
        return TT-Check-All(KB, α, rest, Extend(P, true, model)) and
                  TT-Check-All(KB, α, rest, Extend(P, false, model))
```

- Depth-first enumeration of all models
  - Hence, sound and complete
- Algorithm is $O(2^n)$ for n symbols; problem is *co-NP-complete*

Logical Equivalence

- Two sentences are *logically equivalent* iff true in same models:
  $\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$
- The following should be familiar:

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) \\
\neg(\neg\alpha) &\equiv \alpha \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) \\
(\alpha \equiv \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))
\end{aligned}
$$

Validity and Satisfiability

- A sentence is *valid* if it is true in *all* models,
  e.g., $A \lor \neg A, \quad A \Rightarrow A, \quad (A \land (A \Rightarrow B)) \Rightarrow B$

Validity and Satisfiability

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  $KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid

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  $KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid
- A sentence is *satisfiable* if it is true in *some* model
  e.g., $A \vee B$, C
- A sentence is *unsatisfiable* if it is true in *no* models
  e.g., $A \wedge \neg A$

Validity and Satisfiability

- A sentence is *valid* if it is true in *all* models,
  e.g., $A \vee \neg A$, $\quad A \Rightarrow A$, $\quad (A \wedge (A \Rightarrow B)) \Rightarrow B$
- Validity is connected to inference via the *Deduction Theorem*:
  $KB \models \alpha$ if and only if $(KB \Rightarrow \alpha)$ is valid
- A sentence is *satisfiable* if it is true in *some* model
  e.g., $A \vee B$, $\quad C$
- A sentence is *unsatisfiable* if it is true in *no* models
  e.g., $A \wedge \neg A$
- Satisfiability is connected to inference via the following:
  $KB \models \alpha$ if and only if $(KB \wedge \neg\alpha)$ is unsatisfiable
  - I.e., prove α by *reductio ad absurdum*

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  e.g., $A \wedge \neg A$
- Satisfiability is connected to inference via the following:

  $KB \models \alpha$ if and only if $(KB \wedge \neg\alpha)$ is unsatisfiable
  - I.e., prove α by *reductio ad absurdum*
- What often proves better for determining $KB \models \alpha$ is to show that $KB \wedge \neg\alpha$ is unsatisfiable.

Proof Methods

Proof methods divide into (roughly) two kinds:

1. *Application of inference rules:*
   - Legitimate (sound) generation of new sentences from old
   - *Proof* = a sequence of inference rule applications.
   - Can use inference rules as operators in a standard search algorithm.
   - Typically require translation of sentences into a *normal form*
2. *Model checking:*

Possibilities:

- Truth table enumeration (always exponential in n)
- Improved backtracking, e.g., DPLL
- Heuristic search in model space (sound but incomplete)
  e.g., min-conflicts hill-climbing algorithms

Specialised Inference: Forward and Backward Chaining

- We consider a very useful, restricted case: *Horn Form*
  - KB = *conjunction* of *Horn clauses*
- Horn clause =
  - proposition symbol; or
  - (conjunction of symbols) $\Rightarrow$ symbol
- E.g., C, $(B \Rightarrow A)$, $(C \wedge D \Rightarrow B)$
  Not: $(\neg B \Rightarrow A)$, $(B \vee A)$

Horn clauses

Technically a Horn clause is a *clause* or disjunction of literals, with *at most* one positive literal.

- I.e. of form $A_0 \vee \neg A_1 \vee \cdots \vee \neg A_n$ or
  $\neg A_1 \vee \cdots \vee \neg A_n$
- These can be written: $A_1 \wedge \cdots \wedge A_n \Rightarrow A_0$ or
  $A_1 \wedge \cdots \wedge A_n \Rightarrow \bot$
- We won't bother with rules of the form $A_1 \wedge \cdots \wedge A_n \Rightarrow \bot$
  - Rules of this form are called *integrity constraints*.
  - They don't allow new facts to be derived, but rather rule out certain combinations of facts.

Reasoning with Horn clauses

- *Modus Ponens* (for Horn form): Complete for Horn KBs

$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

- Can be used with *forward chaining* or *backward chaining*.
- These algorithms are very natural; forward chaining runs in *linear* time

Example

KB:

$$P \Rightarrow Q,$$
$$L \wedge M \Rightarrow P,$$
$$B \wedge L \Rightarrow M,$$
$$A \wedge P \Rightarrow L,$$
$$A \wedge B \Rightarrow L,$$
$$A,$$
$$B$$

Forward chaining

Idea:

- Fire any rule whose premises are satisfied in the KB,
- Add its conclusion to the KB, until query is found

Forward chaining algorithm

Procedure:

$C := \{\}$;

repeat

choose $r \in A$ *such that*

r *is* '$b_1 \wedge \cdots \wedge b_m \Rightarrow h$'

$b_i \in C$ *for all* i, *and*

$h \notin C$;

$C := C \cup \{h\}$

until no more choices

Forward chaining algorithm (from text)

```
Function PL-FC-Entails?(KB,q) returns true or false
  inputs: KB the knowledge base, a set of propositional Horn clauses
        q the query, a proposition symbol
  local variables: count a table, indexed by clause, initially # of premises
        inferred a table, indexed by symbol, each entry initially false
        agenda a list of symbols, initially symbols known true in KB
  while agenda is not empty do
     p ←Pop(agenda)
     unless inferred[p] do
        inferred[p] ←true
        for each Horn clause c in whose premise p appears do
           decrement count[c]
           if count[c] = 0 then do
              if Head[c] = q then return true; Push(Head[c], agenda)
  return false
```

Forward chaining example

KB:

$$P \Rightarrow Q,$$
$$L \wedge M \Rightarrow P,$$
$$B \wedge L \Rightarrow M,$$
$$A \wedge P \Rightarrow L,$$
$$A \wedge B \Rightarrow L,$$
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Query Q:

Forward chaining example

KB:

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Query Q:

- From A and B, conclude L

Forward chaining example

KB:

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$A \wedge P \Rightarrow L,$
$A \wedge B \Rightarrow L,$
$A,$
B

Query Q:

- From A and B, conclude L
- From L and B, conclude M

Forward chaining example

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$P \Rightarrow Q,$
$L \wedge M \Rightarrow P,$
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$A \wedge P \Rightarrow L,$
$A \wedge B \Rightarrow L,$
$A,$
B

Query Q:

- From A and B, conclude L
- From L and B, conclude M
- From L and M, conclude P

Forward chaining example

KB:

$$
\begin{aligned}
&P \Rightarrow Q,\\
&L \wedge M \Rightarrow P,\\
&B \wedge L \Rightarrow M,\\
&A \wedge P \Rightarrow L,\\
&A \wedge B \Rightarrow L,\\
&A,\\
&B
\end{aligned}
$$

Query Q:

- From A and B, conclude L
- From L and B, conclude M
- From L and M, conclude P
- From P conclude Q

Completeness

FC, when run to completion, derives every atomic sentence entailed by KB

1. FC reaches a *fixed point* where no new atomic sentences are derived
2. Can consider the final state as a model m, assigning true/false to symbols
3. *Claim:* Every clause in the original KB is true in m
   Proof: Suppose a clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in m
   Then $a_1 \wedge \ldots \wedge a_k$ is true in m and b is false in m
   Therefore the algorithm has not reached a fixed point!
4. Hence m is a model of KB
5. If $KB \models q$, q is true in *every* model of KB, including m

Backward chaining

- We won't develop an algorithm for backward chaining here, but will just consider it informally.
- Idea with backward chaining:
  Start from query q and work backwards.
- To prove q by BC:
  - check if q is known already;
  - otherwise prove (by BC) all premises of some rule concluding q
- Avoid loops: Check if new subgoal is already on the goal stack
- Avoid repeated work: Check if new subgoal
  1. has already been proved true, or
  2. has already failed

Backward chaining example

KB:

$$P \Rightarrow Q, \quad L \wedge M \Rightarrow P, \quad B \wedge L \Rightarrow M, \quad A \wedge P \Rightarrow L,$$
$$A \wedge B \Rightarrow L, \quad A, \quad B$$

Query Q:

Backward chaining example

KB:

$$P \Rightarrow Q, \quad L \land M \Rightarrow P, \quad B \land L \Rightarrow M, \quad A \land P \Rightarrow L,$$
$$A \land B \Rightarrow L, \quad A, \quad B$$

Query Q:

- Establish P as a subgoal.

Backward chaining example

KB:

$$P \Rightarrow Q, \quad L \wedge M \Rightarrow P, \quad B \wedge L \Rightarrow M, \quad A \wedge P \Rightarrow L,$$
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Query Q:

- Establish P as a subgoal.
- Can prove P by proving L and M

Backward chaining example

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$$A \wedge B \Rightarrow L, \quad A, \quad B$$

Query Q:

- Establish P as a subgoal.
- Can prove P by proving L and M
- For M:
  - Can prove M if we can prove B and L

Backward chaining example

KB:

$$P \Rightarrow Q, \quad L \wedge M \Rightarrow P, \quad B \wedge L \Rightarrow M, \quad A \wedge P \Rightarrow L,$$
$$A \wedge B \Rightarrow L, \quad A, \quad B$$

Query Q:

- Establish P as a subgoal.
- Can prove P by proving L and M
- For M:
  - Can prove M if we can prove B and L
  - B is known to be true

Backward chaining example

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$$P \Rightarrow Q, \quad L \wedge M \Rightarrow P, \quad B \wedge L \Rightarrow M, \quad A \wedge P \Rightarrow L,$$
$$A \wedge B \Rightarrow L, \quad A, \quad B$$

Query Q:

- Establish P as a subgoal.
- Can prove P by proving L and M
- For M:
  - Can prove M if we can prove B and L
  - B is known to be true
  - L can be proven by proving A and B.

Backward chaining example

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$$P \Rightarrow Q, \quad L \wedge M \Rightarrow P, \quad B \wedge L \Rightarrow M, \quad A \wedge P \Rightarrow L,$$
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  - Can prove M if we can prove B and L
  - B is known to be true
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  - A and B are known to be true
- For L:
  - L can be proven by proving A and B.
  - A and B are known to be true

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$$P \Rightarrow Q, \quad L \wedge M \Rightarrow P, \quad B \wedge L \Rightarrow M, \quad A \wedge P \Rightarrow L,$$
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Query Q:

- Establish P as a subgoal.
- Can prove P by proving L and M
- For M:
  - Can prove M if we can prove B and L
  - B is known to be true
  - L can be proven by proving A and B.
  - A and B are known to be true
- For L:
  - L can be proven by proving A and B.
  - A and B are known to be true
- L and M are true, thus P is true, thus Q is true

Forward vs. backward chaining

- FC is *data-driven*, cf. automatic, unconscious processing,
  - E.g., object recognition, routine decisions
  - May do lots of work that is irrelevant to the goal
  - Good for reactive agents

Forward vs. backward chaining

- FC is *data-driven*, cf. automatic, unconscious processing,
  - E.g., object recognition, routine decisions
  - May do lots of work that is irrelevant to the goal
  - Good for reactive agents
- BC is *goal-driven*, appropriate for problem-solving,
  - E.g., Where are my keys? How do I get a job?
  - Complexity of BC can be *much less* than linear in size of KB
  - Can also sometimes be exponential in size of KB
  - Good for question-answering and explanation

General Propositional Inference: Resolution

Resolution is a rule of inference defined for *Conjunctive Normal Form* (CNF)

- *CNF: conjunction* of *disjunctions* of *literals*
- A *clause* is a *disjunctions* of *literals*.
- E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$.
  - ☞ Write as: $(A \vee \neg B)$, $(B \vee \neg C \vee \neg D)$

Resolution

- *Resolution* inference rule:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

  where ℓ_i and m_j are complementary literals. (I.e. $\ell_i \equiv \neg m_j$.)

- E.g., $\dfrac{P_{1,3} \vee P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$
- Resolution is sound and complete for propositional logic

Using resolution to compute entailments

To show whether $KB \models \alpha$, show instead that $KB \wedge \neg\alpha$ is unsatisfiable:

1. Convert $KB \wedge \neg\alpha$ into conjunctive normal form.
2. Use resolution to determine whether $KB \wedge \neg\alpha$ is unsatisfiable.
3. If so then $KB \models \alpha$; otherwise $KB \not\models \alpha$.

Conversion to CNF

E.g.: $B_{1,1} \equiv (P_{1,2} \vee P_{2,1})$

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E.g.: $B_{1,1} \equiv (P_{1,2} \vee P_{2,1})$

1. Eliminate $\equiv$, replacing $\alpha \equiv \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

$$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$

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   $$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$$

Conversion to CNF

E.g.: $B_{1,1} \equiv (P_{1,2} \vee P_{2,1})$

1. Eliminate $\equiv$, replacing $\alpha \equiv \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
   $$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$$
3. Move $\neg$ inwards using de Morgan's rules and double-negation:
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$$

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4. Apply distributivity law ($\vee$ over $\wedge$) and flatten:
   $$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$$

Conversion to CNF

E.g.: $B_{1,1} \equiv (P_{1,2} \vee P_{2,1})$

1. Eliminate $\equiv$, replacing $\alpha \equiv \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
   $$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.
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For resolution, then write as

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}),\ (\neg P_{1,2} \vee B_{1,1}),\ (\neg P_{2,1} \vee B_{1,1})$$

Resolution Algorithm

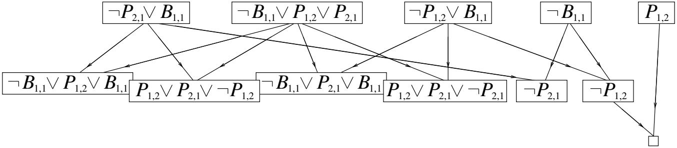
Function PL-Resolution(KB, α) returns true or false
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

```
clauses ←the set of clauses in CNF(KB ∧ ¬α)
new ←{ }
loop do
   if clauses contains the empty clause then return true
   if Ci, Cj are resolvable clauses where
           PL-Resolve(Ci, Cj) ∉ clauses
      then clauses ←clauses ∪ PL-Resolve(Ci, Cj)
      else return false
```

☞ Note that the algorithm in the text is buggy

Resolution Example

- E.g.: $KB = (B_{1,1} \equiv (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$,
  $\alpha = \neg P_{1,2}$
- Show $KB \models \alpha$ by showing that $KB \wedge \neg\alpha$ is unsatisfiable:



Resolution: Another Example

Show:

$$\{r \Rightarrow u, u \Rightarrow \neg w, \neg r \Rightarrow \neg w\} \models \neg w$$

Resolution: Continued

There is a great deal that can be done to improve the basic algorithm:

- Unit resolution: propagate unit clauses (e.g. $\neg B_{1,1}$) as much as possible.
  - Note that this correspoinds to the *minimum remaining values* heuristic in constraint satisfaction!
- Eliminate tautologies
- Eliminate redundant clauses
- Eliminate clauses with literal ℓ where the complement of ℓ doesn't appear elsewhere.
- Set of support: Do resolutions on clauses with ancestor in $\neg\alpha$.
  - Similar to backward chaining – keep a focus on the goal.

Summary

- Logical agents apply *inference* to a *knowledge base* to derive new information and make decisions
- Basic concepts of logic:
  - *syntax*: formal structure of *sentences*
  - *semantics*: *truth* of sentences wrt *models*
  - *entailment*: necessary truth of one sentence given another
  - *inference*: deriving sentences from other sentences
  - *soundness*: derivations produce only entailed sentences
  - *completeness*: derivations can produce all entailed sentences

Summary (Continued)

- Wumpus world requires the ability to represent partial and negated information, reason by cases, etc.
- Forward, backward chaining are complete for Horn clauses.
- Forward chaining is linear-time for Horn clauses.
- Resolution is complete for propositional logic.
- Propositional logic lacks expressive power