CMPT307: Matroids

Week 9-1, 9-2

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Definition

Let S be a finite set and let $\mathcal{I} \subseteq 2^S$. $(S, \mathcal{I})$ is an independent set system if

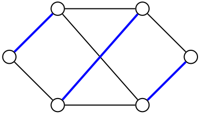
(C1) $\emptyset \in \mathcal{I}$; non-emptiness

(C2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$. heredity

$I \in \mathcal{I}$ is an independent set, otherwise called a dependent set

examples

- $S = \{1, 2, \ldots, n\}$ and $\mathcal{I} = \{|I| \leq 10\}$
- $\langle E, \mathcal{M} \rangle$: $E =$ edge set of a graph and $\mathcal{M} = \{$matchings$\}$



MWIS Problems

Maximum Weight Independent Set (MWIS) problem

- given $(S, \mathcal{I})$ and weight function $w : S \to \mathbb{R}$
- find $I \in \mathcal{I}$, maximizing $w(I) = \sum_{x \in I} w(x)$

Maximum Weight Matching

- given graph $G = (V, E)$ and weight w_e for all $e \in E$
- find a maximum weight matching

- $\langle E, \mathcal{M} \rangle$ is an independent set system
- $\arg\max_{I \in \mathcal{I}} w(I) \Leftrightarrow$ max weight matching

MWIS Problems

Shortest path

- given digraph $G = (V, E)$ and $s, t \in V$ and distances c_e for all $e \in E$
- find a shortest s-t path

- assume $(u, v) \in E, \forall u, v \in V$, otherwise add an edge (u, v) and set $c_{(u,v)} = \infty$
- $\langle E, \mathcal{P} \rangle$: $\mathcal{P} = \{P \mid P$ induces an s-t path$\}$

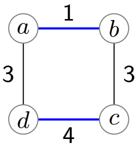
Maximum weight forest

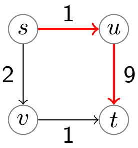
- given $G = (V, E)$ and weight w_e for all $e \in E$
- find a maximum weight forest

- $\langle E, \mathcal{F} \rangle$: $\mathcal{F} = \{F \mid F$ induces a forest in $G\}$

Greedy for maximum weight matching

while $E \neq \emptyset$ **do**
- $e = \arg\max_{e \in E} w_e$;
- **if** $M + e$ *is matching* **then**
  - $M = M + e$; // M is empty in initial
- $E = E - e$;





Greedy for shortest path

starting from s, walk through shortest edge at every step until reaching t;

Greedy Algorithms

Greedy for maximum weight forest

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while E ≠ ∅ do
    e = arg max_{e∈E} w_e;
    if F + e is forest then
        F = F + e;                  // F is empty in initial
    E = E − e;
```

- we can not find a counterexample (this one could be correct?)

Question

For any problem that is in the class of MWIS problem, under what conditions the greedy algorithm can solve it?

Matroid

Definition

An independent set system $(S, \mathcal{I})$ is a matroid if

(C3) If $I, J \in \mathcal{I}$, $|I| < |J|$, then $I + x \in \mathcal{I}$ for some $x \in J \backslash I$. exchange

Uniform matroid

$S =$ any countable set, $\mathcal{I} = \{I \subseteq S \mid |I| \leq k\}$.

Partition matroid

S is partitioned into m sets $S_1, \cdots, S_m$. Given m integers $k_1, \cdots, k_m$, define $\mathcal{I} = \{I \subseteq S \mid |I \cap S_i| \leq k_i, i = 1, \cdots, m\}$.

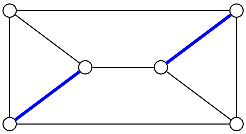
- ▷ for $|I| < |J|$, exists i s.t. $|J \cap S_i| > |I \cap S_i|$
- ▷ let $x \in S_i \cap (J \backslash I)$, hence $I + x \in \mathcal{I}$

Basis

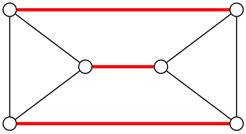
Definition

$B \subseteq S$ is called a basis if B is inclusion-wise maximal independent set of S, *i.e.*, $B \in \mathcal{I}$ there is no $I \in \mathcal{I}$ with $B \subset I \subseteq S$.

- basis B cannot be extended by adding elements from $S \backslash B$
- consider independent set system $\langle E, \mathcal{M} \rangle$



M



M^*

both M and M^* are bases of E

Basis

(C3) If $I, J \in \mathcal{I}$, $|I| < |J|$, then $I + x \in \mathcal{I}$ for some $x \in J \backslash I$

(C4) $\forall U \subseteq S$, all bases of U have the same size

Lemma. (C3) and (C4) are equivalent.

Proof

(C3 $\Rightarrow$ C4) let B_1, B_2 be two bases and assume $|B_1| < |B_2|$

- by (C3), $\exists x \in B_2 \backslash B_1$ s.t. $B_1 + x \in \mathcal{I}$ contradiction

(C4 $\Rightarrow$ C3) consider $I, J \in \mathcal{I}$ with $|I| < |J|$

- assume $I + x \notin \mathcal{I}$ for all $x \in J \backslash I$
- then I must be a basis of $I \cup J$
- since $J \subseteq I \cup J$, by (C4) $|I| \geq |J|$, a contradiction

- rank of matroid $(S, \mathcal{I})$: $r(S) = |B|$, where B is a basis of S
- $\langle E, \mathcal{M} \rangle$ (*cf.* s.3) is not a matroid

Graphic matroid

Given $G = (V, E)$, let $S = E$ and $\mathcal{I} = \{F \subseteq E \mid F$ induces a forest in $G\}$.

(C4): $\forall U \subseteq E$, let G' be induced by U, with k components

- let $B \subseteq U$ be any basis
- B contains k spanning trees
- $|B| = |V| - k$

MWIS problem

Given an independent set system $(S, \mathcal{I})$ and a nonnegative weight function $w : S \to \mathbb{R}^+$, find a maximum weight independent set $I \in \mathcal{I}$.

Greedy

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initialize: I = ∅;
while ∃x ∈ S\I with I + x ∈ 𝓘 do
    choose x with w(x) maximum;
    I = I + x;
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Theorem

Greedy computes an independent set of maximum weight if and only if $(S, \mathcal{I})$ is a matroid.

$\Leftarrow$ Proof of Sufficiency

assumptions

- X = greedy solution and Y = optimal solution
  $\Rightarrow$ X and Y are bases of $S \Rightarrow |X| = |Y|$
- $X = \{x_1, \cdots, x_m\}, \quad Y = \{y_1, \cdots, y_m\}$
  $w(x_1) \geq \cdots \geq w(x_m), \quad w(y_1) \geq \cdots \geq w(y_m)$

to show: $w(x_i) \geq w(y_i)$ for all i

- let $k+1$ be smallest index s.t. $w(x_{k+1}) < w(y_{k+1})$
- define $I = \{x_1, \cdots, x_k\}$ and $J = \{y_1, \cdots, y_{k+1}\}$
- as $|I| < |J|$, by (C3) $\Rightarrow \exists y_i \in J \backslash I$ s.t. $I + y_i \in \mathcal{I}$
- note $w(y_i) \geq w(y_{k+1}) > w(x_{k+1})$
- thus in step $k+1$, greedy will choose y_i instead of x_{k+1}, a contradiction!

$\Rightarrow$ Proof of Necessity

know: greedy computes a max-weight independent set $I \in \mathcal{I}$

to show: $(S, \mathcal{I})$ is a matroid

- (C3) by contradiction: assume $I, J \in \mathcal{I}$, $|I| < |J|$ and $I + x \notin \mathcal{I}$ for every $x \in J \backslash I$
- let $k = |I|$ and define a weight function w:

$$w(x) = \begin{cases} k+2, & \text{if } x \in I \\ k+1, & \text{if } x \in J \backslash I \\ 0, & \text{otherwise} \end{cases}$$

- greedy picks all $x \in I$ in the first k steps and terminates (as $I + x \notin \mathcal{I}$), yielding $w(I) = k(k+2)$
- but $w(J) \geq (k+1)\,|J| \geq (k+1)^2 > w(I)$, a contradiction!