CMPT307: Probabilistic Analysis

Week 4-2

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A Hiring Problem

- a company needs to hire new office assistants
- interview a candidate, and hire if better than current
- costs incurred by interviewing (c_1) and hiring (c_2)

HIRE-ASSISTANT(n)

```
best = 0 ;                    // 0 is a least-qualified dummy candidate
for i = 1 to n do
    interview candidate i;
    if candidate i is better than current best then
        best = i;
        hire candidate i;
```

- $m = \#$ hired people, then total cost $= O(c_1 n + c_2 m)$
- assume $c_2 > c_1$, in the worst-case it is $O(c_2 n)$

Probabilistic Analysis

- make assumptions about the distribution of the input
- analyze in average-case, *e.g.* running time, solution

the hiring problem

- assume applicants arrive in random order
- what is the expected total cost?

$$\mathrm{I}\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A \text{ does not occur} \end{cases}$$

Example

Flipping a fair coin n times. Determine the expected number of heads.

$$X_i = \mathrm{I}\{\text{head at } i\text{th toss}\}$$

$$H := \#\text{heads} = \sum_{i=1}^{n} X_i$$

$$\mathbb{E}[H] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \Pr(X_i = 1) = \frac{n}{2}$$

Analysis

$$X_i = \mathrm{I}\{i \text{ is hired}\} := \begin{cases} 1, & i \text{ is hired} \\ 0, & i \text{ is not hired} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i = \# \text{ hired people}$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \Pr(X_i = 1) = \sum_{i=1}^{n} \frac{1}{i} =: H_n$$

- harmonic number: $H_n = \ln n + O(1)$
- average hiring cost $= O(c_2 \ln n)$
- what if we have no knowledge about distribution?

Randomized Algorithm

- output non-deterministic results
- use random number generator
- analyze the expected solution value

RANDOMIZED-HIRE-ASSISTANT(n)

```
randomly permute the list of candidates;
HIRE-ASSISTANT(n);
```

- the above algorithm imposes a distribution to the input
- expected hiring cost $= O(c_2 \ln n)$

Random Permutations

PERMUTE-BY-SORTING(A)

```
n = A.length;
for i = 1 to n do
    P[i] = RANDOM(1, n^3);                    // priority of i
sort A, using P as sort keys;
```

Theorem

If all $P[i]$s are unique, the algorithm yields a uniform random permutation.

- show that any permutation occurs with equal chance $\frac{1}{n!}$
- what is the probability of "$P[i]$'s are unique"?

Proof of the Theorem

assume $\textsc{Permute-By-Sorting}(A)$ yields $A[\sigma(1)..\sigma(n)]$, where $(\sigma(1), \ldots, \sigma(n))$ is a permutation of $(1, \ldots, n)$

$E_i :=$ $A[i]$ receives the $\sigma(i)$th smallest priority

to show: $\Pr\{E_1 \cap E_2 \cap \cdots \cap E_n\} = \frac{1}{n!}$

$$\begin{aligned}
& \Pr\{E_1 \cap E_2 \cap \cdots \cap E_n\} \\
= & \Pr\{E_1\} \cdot \Pr\{E_2 \mid E_1\} \cdots \Pr\{E_i \mid E_{i-1} \cap \cdots \cap E_1\} \cdots \\
& \Pr\{E_n \mid E_{n-1} \cap \cdots \cap E_1\} \\
= & \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{n-i+1} \cdots \frac{1}{1} = \frac{1}{n!}
\end{aligned}$$

Random Permutations

RANDOMIZE-IN-PLACE(A)

```
n = A.length;
for i = 1 to n do
    swap A[i] with A[RANDOM(i, n)];
```

Theorem

RANDOMIZE-IN-PLACE computes a uniform random permutation.

- k-permutation of A: a set containing k elements of A, with no repetitions
- after the ith loop: we have i-permutation of A
- to show: $A[1..i]$ contains this permutation with probability $(n-i)!/n!$

Proof of the Theorem

after loop i, assume we get $A[1..i] := \langle x_1, \ldots, x_i \rangle$

$$E_1 := \text{the first } (i-1) \text{ loops yield } \langle x_1, \ldots, x_{i-1} \rangle$$
$$E_2 := \text{the } i\text{th loop puts } x_i \text{ into } A[i]$$

to show $\Pr\{\langle x_1, \ldots, x_n \rangle\} = \Pr\{E_1 \cap E_2\} = \frac{(n-i)!}{n!}$

- $i = 1$: $\Pr\{\langle x_1 \rangle\} = \frac{1}{n}$
- assume true for $i-1$: $\Pr\{E_1\} = \frac{(n-i+1)!}{n!}$

$$\begin{aligned}\Pr\{E_1 \cap E_2\} &= \Pr\{E_1\} \cdot \Pr\{E_2 \mid E_1\} \\ &= \frac{(n-i+1)!}{n!} \cdot \frac{1}{n-i+1} = \frac{(n-i)!}{n!}\end{aligned}$$