CMPT 307: Shortest Paths & Maximum Matchings

Week 12-2

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All-pairs Shortest Path Problem

- given: digraph $G = (V, E)$ and cost function $c : E \to \mathbb{R}$
- goal: find shortest s-t paths, for all $(s, t) \in V \times V$

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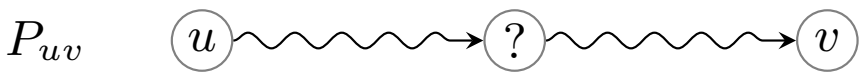
- can we do better?
- dynamic programming

Optimal Substructure

- consider a shortest (u, v)-path P_{uv}
- with internal nodes $\subseteq V_k := \{1, 2, \ldots, k\}$
- $\delta_k(u, v) =$ cost of path P_{uv}



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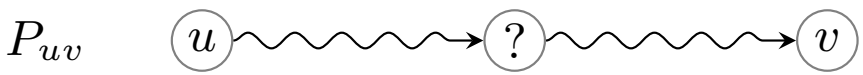

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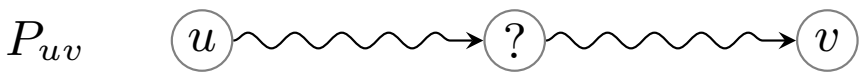

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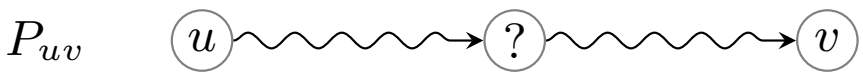

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- $\delta_k(u, v) = \delta_{k-1}(u, k) + \delta_{k-1}(k, v)$

Recursive Formula

$$\delta_k(u,v) = \min\left\{\delta_{k-1}(u,v), \delta_{k-1}(u,k) + \delta_{k-1}(k,v)\right\}, \quad k \geq 1$$

boundary conditions

$$\delta_0(u,v) = \begin{cases} 0, & \text{if } u = v \\ c(u,v) & \text{if } (u,v) \in E \\ \infty & \text{otherwise} \end{cases}$$

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running time: $O(n^3)$

Floyd-Warshall Algorithm

- input: a graph represented by (weighted) adjacency matrix W

FLOYD-WARSHALL(W)

```
1 n = W.rows;
2 D^(0) = W;                                          // boundary conditions
3 for k = 1 to n do
4     let D^k = (d_ij^(k)) be a new n × n matrix;
5     for i = 1 to n do
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- ▷ given $i, j \in V$, how to construct a shortest (i, j)-path?

Matching

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- ▷ a matching is a set $M \subseteq E$ of pairwise non-incident edges

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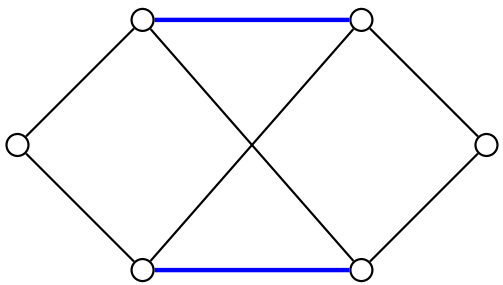

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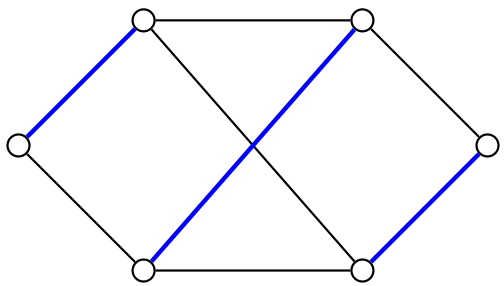

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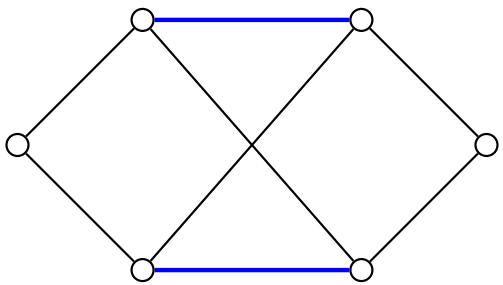

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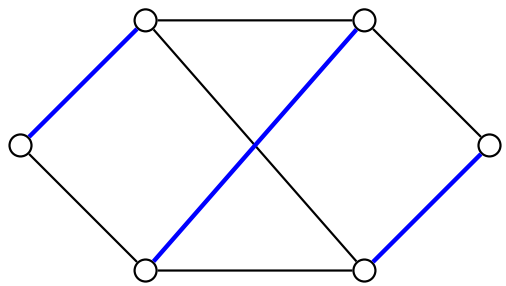

maximum

Maximum (cardinality) matching

- ▷ given: undirected graph $G = (V, E)$
- ▷ goal: compute a maximum matching M

Basic Idea

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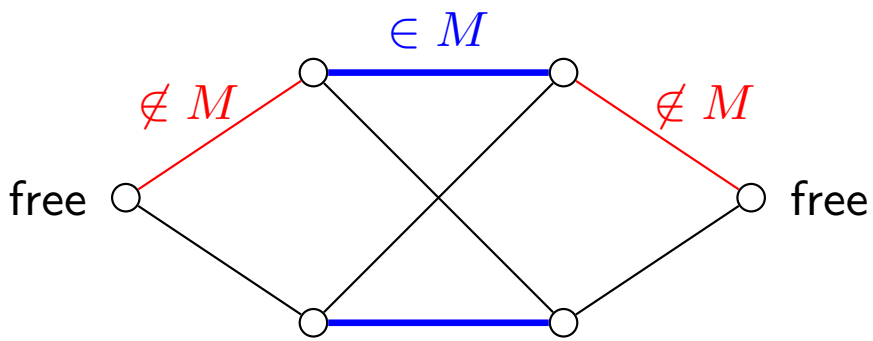
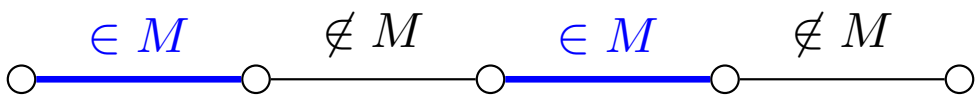


Augmenting Paths

M-alternating path



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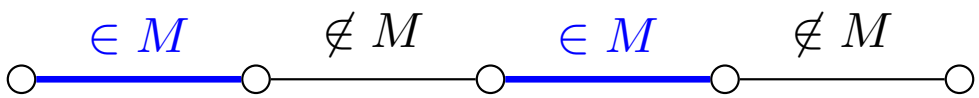

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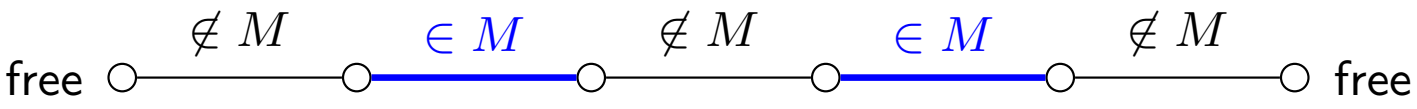

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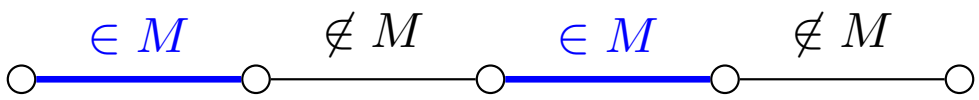

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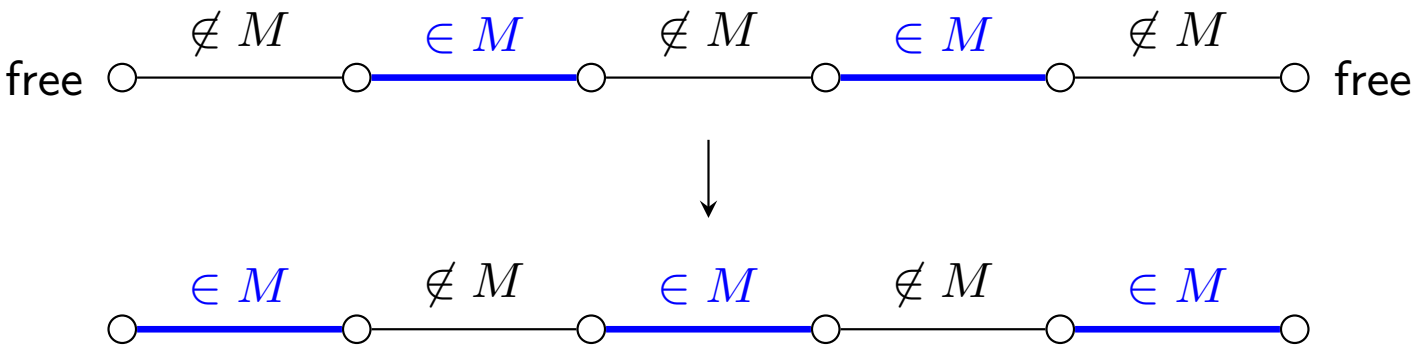

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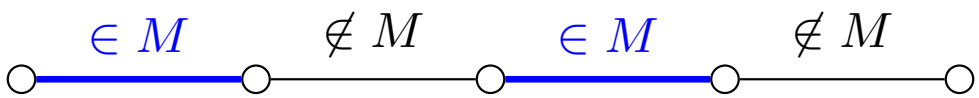

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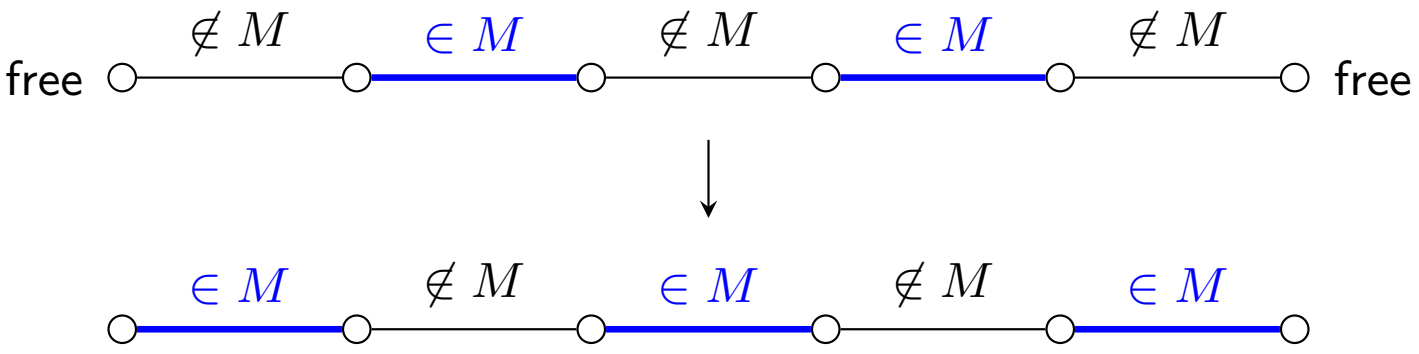

symmetric difference: $M \Delta N = (M \backslash N) \cup (N \backslash M),\ M, N \subseteq E$

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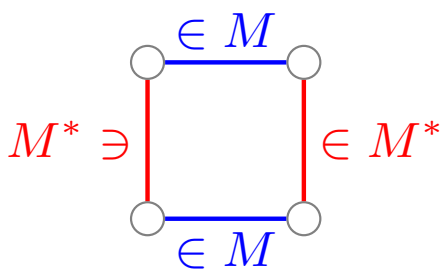
- $M \Delta M^*$ contains node disjoint cycles and paths



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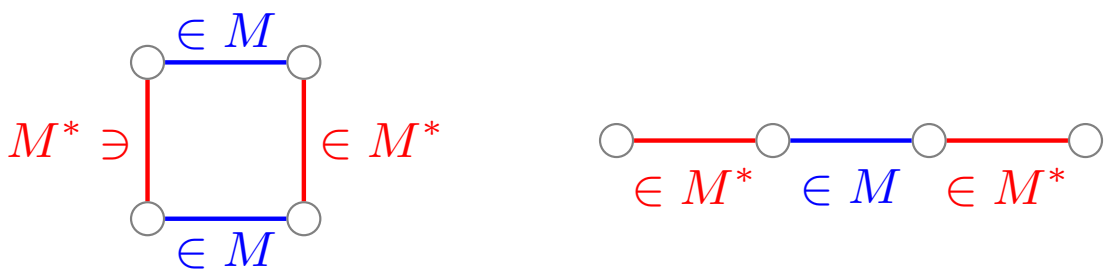

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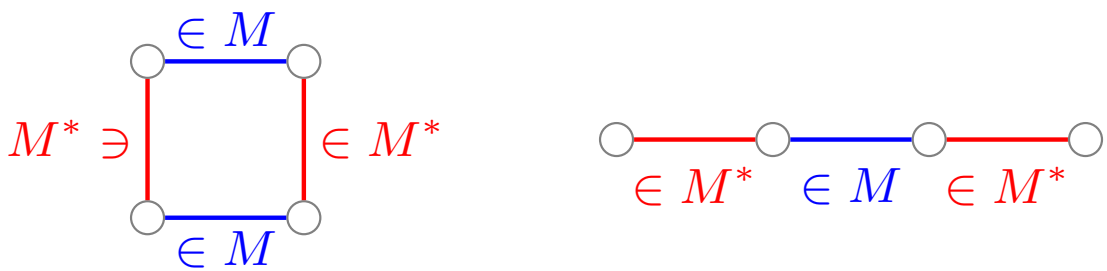

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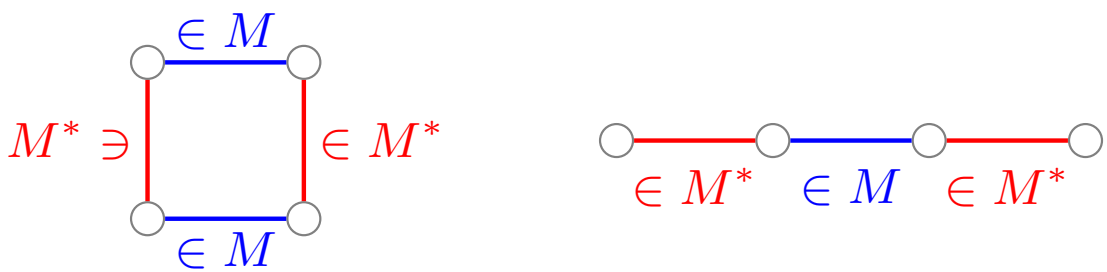

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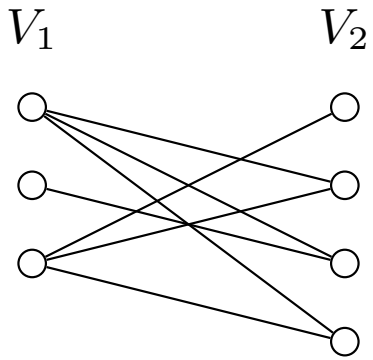
- P is an M-augmenting path

Maximum Matching on Bipartite Graphs

find maximum matching for bipartite graph $G = (V_1, V_2, E)$



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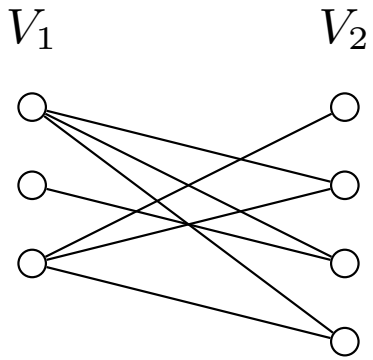

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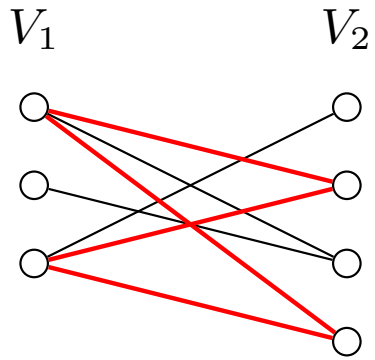

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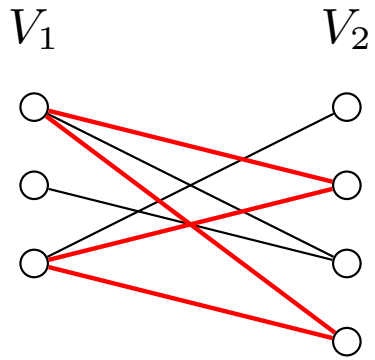
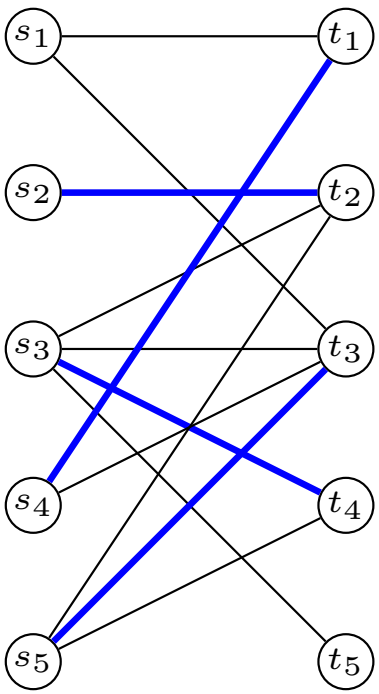


how to find M-augmenting paths?

- ▷ bipartite graphs do not have odd cycles (augmenting paths have odd number of edges)
- ▷ start with a free node $s \in V_1$ and search alternating path until finding another free node $t \in V_2$

Finding Augmenting Paths



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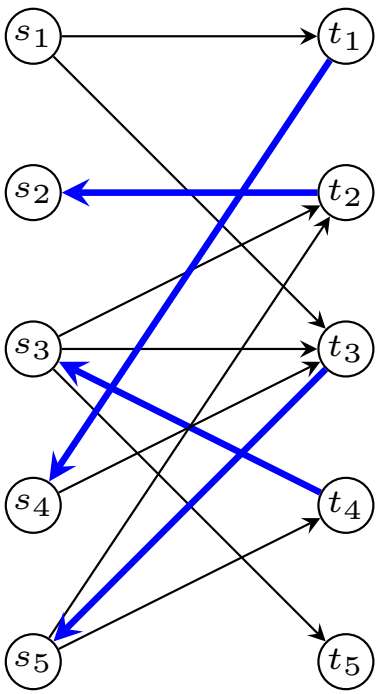

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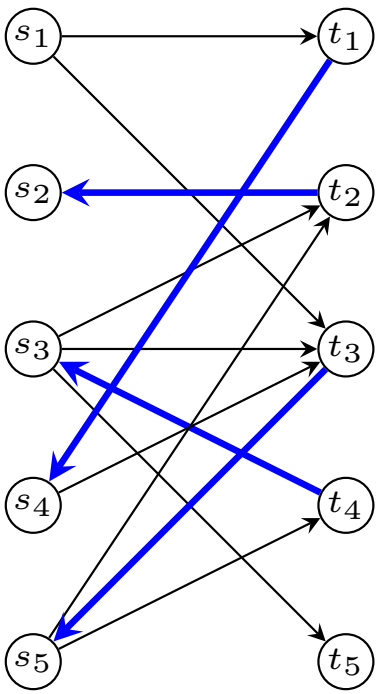

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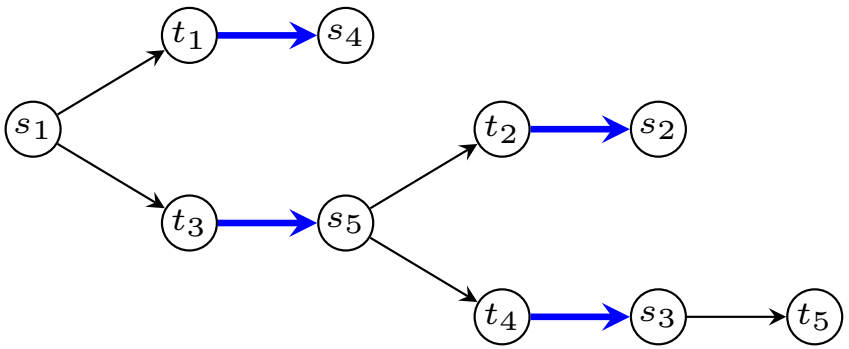

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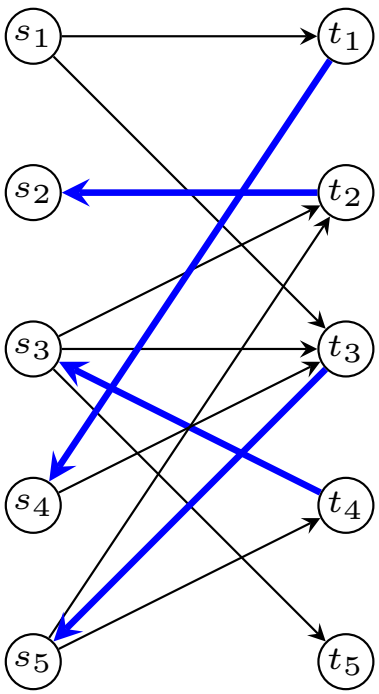

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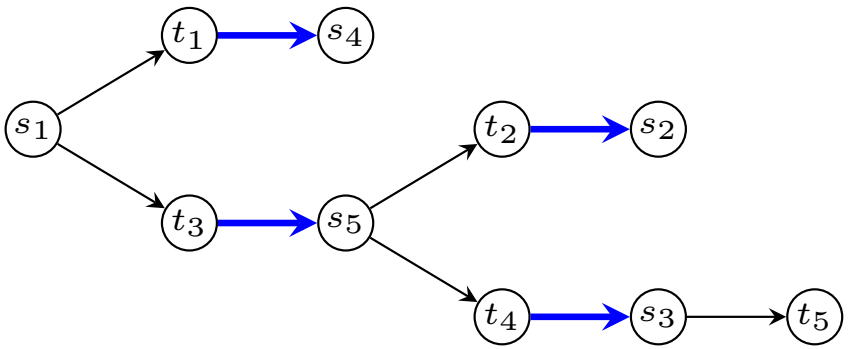

3. an M-augmenting path is found if exists

Augmenting Path Algorithm

BIPARTITE-MATHCING(G)

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3     construct the auxiliary digraph D;                          // O(m)
4     apply BFS(D, s) to find an augmenting path P ;              // O(m)
5     if P ≠ nil then
6         M := MΔE(P);                                  // augment along P  O(m)
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- ▷ running time $O(nm)$